Comprehensive MATHEMATICS

JEE Advanced 2020

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Mc Graw Hill

Contents

A Word to the Reader	*v*
Syllabus	*vii*
Format of Questions in this Book	*xi*
1. Complex Numbers	1.1–1.67
2. Theory of Equations	2.1–2.56
3. Progressions	3.1–3.55
4. Logarithm	4.1–4.21
5. Permutations and Combinations	5.1–5.40
6. Binomial Theorem	6.1–6.45
7. Matrices	7.1–7.34
8. Determinants	8.1–8.54
9. Probability	9.1–9.55
10. Trigonometry	10.1–10.48
11. Trigonometric Equations	11.1–11.18
12. Solution of Triangles and Applications of Trigonometry	12.1–12.46
13. Inverse Trigonometric Functions	13.1–13.24
14. Cartesian System of Coordinates and Locus	14.1–14.20
15. Straight Lines	15.1–15.47
16. Circles	16.1–16.52
17. Conic Section (Parabola, Ellipse, Hyperbola)	17.1–17.41
18. Three Dimensional Geometry	18.1–18.34
19. Functions	19.1–19.33
20. Limit and Continuity	20.1–20.44
21. Differentiation	21.1–21.48
22. Applications of Derivatives	22.1–22.47
23. Indefinite Integration	23.1–23.63
24. Definite Integrals	24.1–24.55
25. Differential Equations	25.1–25.38
26. Vector Algebra	26.1–26.40
IIT-JEE 2008 Mathematics (Solved Paper-I)	P1.1–P1.10
IIT-JEE 2008 Mathematics (Solved Paper-II)	P2.1–P2.11
IIT-JEE 2009 Mathematics (Solved Paper-I)	P1.1–P1.10
IIT-JEE 2009 Mathematics (Solved Paper-II)	P2.1–P2.10
IIT-JEE 2010 Mathematics (Solved Paper-I)	P1.1–P1.11
IIT-JEE 2010 Mathematics (Solved Paper-II)	P2.1–P2.9

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Format of Questions in this Book

SECTION A

STRAIGHT OBJECTIVE TYPE

This section contains **multiple choice questions**. Each question has four choices (a), (b), (c) and (d), out of which **only one answer is correct**.

SECTION B

MULTIPLE CORRECT ANSWERS TYPE

This section contains **multiple correct answers type questions**. Each question has four choices (a), (b), (c) and (d), out of which **one or more answers are correct**.

SECTION C

LINKED COMPREHENSION TYPE

Each set in this section contains a paragraph followed by questions. Each question has four choices (a), (b), (c) and (d), out of which **one or more answers are correct**.

SECTION D

WRITE THE ANSWER TYPE

Each question in this section has an answer which is a non-negative integer.

(Contd.)

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SECTION E

MATRIX MATCH TYPE

Each question in this section contains statements given in two columns, which have to be matched. Statements in **Column I** are labelled as a, b, c and d whereas statements in **Column II** are labelled as p, q, r and s. The answers to these questions have to be appropriately bubbled as illustrated in the following example.

If the correct matches are a – q, a – r, b – p, b – s, c – s and d – q, then the correctly bubbled matrix will look like the following.

	p	q	r	s
a	p	**q**	**r**	s
b	**p**	q	r	**s**
c	p	q	**r**	**s**
d	p	**q**	r	s

SECTION F

REASONING TYPE

Each question in section has four choices (a), (b), (c) and (d) out of which only one is correct. Mark your choices as follows:

(a) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 **is** a correct explanation for STATEMENT-1

(b) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 **is NOT** a correct explanation for STATEMENT-1

(c) STATEMENT-1 is True, STATEMENT-2 is False

(d) STATEMENT-1 is False, STATEMENT-2 is True

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CHAPTER 1

COMPLEX NUMBERS

1.1 DEFINITIONS

A number of the form $a + ib$ where $a, b \in \mathbf{R}$, the set of real numbers, and $i = \sqrt{-1}$, is called a *complex number*. A complex number can also be defined as an *ordered pair* of real numbers a and b, and may be written as (a, b), where the first number denotes the *real part* and the second number denotes the *imaginary part*. If $z = a + ib$, then the real part of z is denoted by $\mathrm{Re}(z)$ and the imaginary part of z is denoted by $\mathrm{Im}(z)$.

A complex number z is said to be *purely real* if $\mathrm{Im}(z) = 0$ and is said to be *purely imaginary* if $\mathrm{Re}(z) = 0$. Note that the complex number $0 = 0 + i0$ is both purely real and purely imaginary. It is the only complex number with this property.

We denote the set of all complex numbers by $\mathbf{C}$. That is, $\mathbf{C} = \{a + ib \mid a, b \in \mathbf{R}\}$. Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are said to be equal if $a_1 = a_2$ and $b_1 = b_2$.

1.2 ALGEBRAIC OPERATIONS WITH COMPLEX NUMBERS

1. *Addition:* $(a + ib) + (c + id) = (a + c) + i(b + d)$
2. *Subtraction:* $(a + ib) - (c + id) = (a - c) + i(b - d)$
3. *Multiplication:* $(a + ib)(c + id)$
   $= (ac - bd) + i(ad + bc)$
4. *Reciprocal:* If at least one of a, b is non-zero then the reciprocal of $a + bi$ is given by

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

5. *Quotient:* If at least one of c, d is non-zero, then quotient of $a + bi$ and $c + di$ is given by

$$\frac{a+bi}{c+di} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

1.3 CONJUGATE OF COMPLEX NUMBER

Let $z = a + ib$ be a complex number. We define conjugate of z, denoted by $\bar{z}$ to be the complex number $a - ib$. That is, if $z = a + ib$, then $\bar{z} = a - ib$.

Properties of Conjugate of a Complex Number

(i) $z_1 = z_2 \Leftrightarrow \bar{z}_1 = \bar{z}_2$

(ii) $\overline{(\bar{z})} = z$

(iii) $z + \bar{z} = 2\,\mathrm{Re}(z)$

(iv) $z - \bar{z} = 2i\,\mathrm{Im}(z)$

(v) $z = \bar{z} \Leftrightarrow z$ is purely real

(vi) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary.

(vii) $z\bar{z} = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2$

(viii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(ix) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(x) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(xi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$

(xii) If $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ where $a_0, a_1, \ldots a_n$ and z are complex number, then

$$\overline{P(z)} = \bar{a}_0 + \bar{a}_1(\bar{z}) + \bar{a}_2(\bar{z})^2 + \cdots + \bar{a}_n(\bar{z})^n$$

$$= \bar{P}(\bar{z})$$

where $\bar{P}(z) = \bar{a}_0 + \bar{a}_1 z + \bar{a}_2 z^2 + \cdots + \bar{a}_n z^n$

(xiii) If $R(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials in z, and $Q(z) \neq 0$, then

$$\overline{R(z)} = \frac{\bar{P}(\bar{z})}{\bar{Q}(\bar{z})}$$

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For instance

$$\overline{\left(\frac{z+3z^2}{z-1}\right)} = \frac{\bar{z}+3\bar{z}^2}{\bar{z}-1}$$

(xiv) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$

where a_i, b_i, c_i $(i = 1, 2, 3)$ are complex numbers.

1.4 MODULUS OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number. We define the *modulus* or the *absolute value* of z to be the real number $\sqrt{a^2+b^2}$ and denote it by $|z|$.
Note that $|z| \geq 0 \ \forall \ z \in \mathbf{C}$

Properties of Modulus

If z is a complex number, then

(i) $|z| = 0 \Leftrightarrow z = 0$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iii) $-|z| \leq Re(z) \leq |z|$

(iv) $-|z| \leq Im(z) \leq |z|$

(v) $z\bar{z} = |z|^2$

If z_1, z_2 are two complex numbers, then

(vi) $|z_1 z_2| = |z_1|\,|z_2|$

(vii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, if $z_2 \neq 0$

(viii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1\bar{z}_2$
$= |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2)$

(ix) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1\bar{z}_2$
$= |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1\bar{z}_2)$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(xi) If a and b are real numbers and z_1, z_2 are complex numbers, then
$|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

(xii) If $z_1, z_2 \neq 0$, then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$
$\Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

(xiii) *Triangle Inequality.* If z_1 and z_2 are two complex numbers, then $|z_1 + z_2| \leq |z_1| + |z_2|$.
The equality holds if and only if $z_1\bar{z}_2 \geq 0$.
In general, $|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$ and the sign equality sign holds if and only if the ratio of any two non-zero terms is positive.

(xiv) $|z_1 - z_2| \leq |z_1| + |z_2|$

(xv) $||z_1| - |z_2|| \leq |z_1| + |z_2|$

(xvi) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(xvii) If a_1, a_2, a_3, are four complex numbers, then $|z - a_1| + |z - a_2| + |z - a_3| + |z - a_4| \geq \max\{|a_1 - a_l| + |a_m - a_n|: l, m, n$ are distinct integers lying in $\{2, 3, 4\}$ and $m < n\}$.

1.5 GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

A complex number $z = x + iy$ can be represented by a point P whose Cartesian co-ordinates are (x, y) referred to rectangular axes Ox and Oy, usually called the *real* and *imaginary* axes respectively. The plane is called the *Argand plane, complex plane* or *Gaussian plane*. The point $P(x, y)$ is called the *image* of the complex number z and z is said to be the *affix* or *complex co-ordinate* of point P.

Note
All purely real numbers lie on the real axis and all purely imaginary numbers lie on the imaginary axis. The complex number $0 = 0 + i0$ lies at the origin O.

We have $OP = \sqrt{x^2 + y^2} = |z|$. Thus, $|z|$ is the length of OP.

1.6 ARGUMENT OF A COMPLEX NUMBER

If z is a non-zero complex numbers represented by point P in the complex plane, then argument of z is the angle which OP makes with the positive direction of the real axis. See Fig. 1.1.

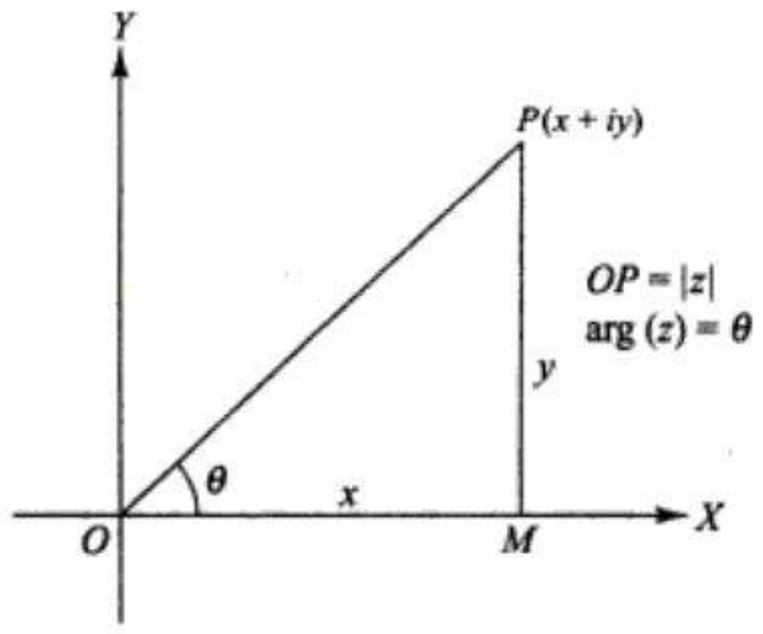

Fig. 1.1

Note
Argument of a complex number is not unique, since, if θ is a value of the argument, then $2n\pi + \theta$ where $n \in \mathbf{I}$, the set of integers, are also values of the argument of z. The value θ of the argument which satisfies the inequality $-\pi < \theta \leq \pi$ is called the *principal value* of the argument or *principal argument*.

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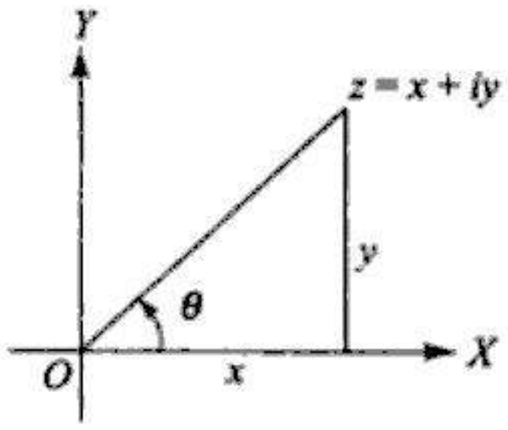
Principal Value of the Argument for Different Positions of z in the Complex Plane

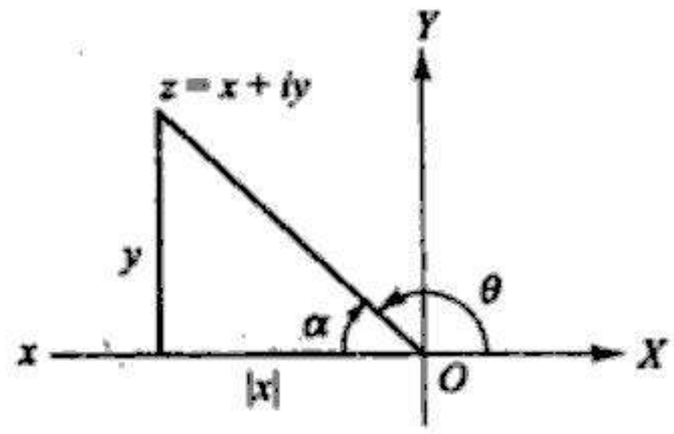
1. When z lies in the first quadrant



$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

Fig. 1.2 (i)

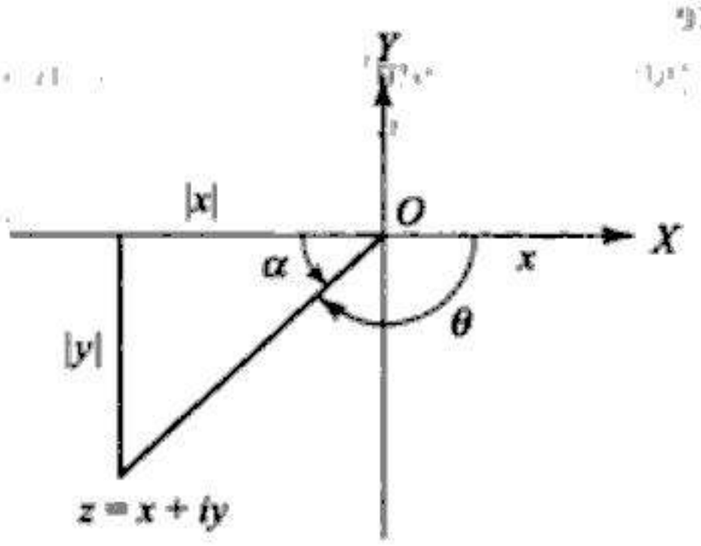
2. When z lies in the second quadrant



$$\arg(z) = \pi - \tan^{-1}\left(\frac{y}{|x|}\right)$$

Fig. 1.2 (ii)

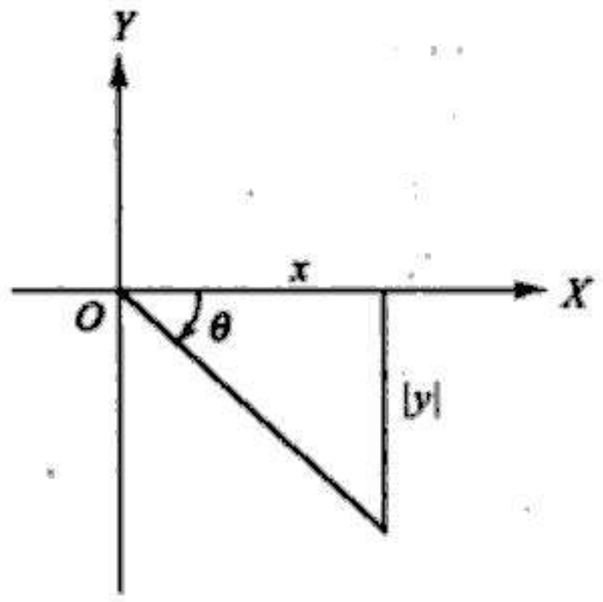
3. When z lies in the third quadrant



$$\arg(z) = -\pi + \tan^{-1}\left(\frac{|y|}{|x|}\right)$$

Fig. 1.2 (iii)

4. When z lies in the fourth quadrant



$$\arg(z) = -\tan^{-1}\left(\frac{|y|}{x}\right)$$

Fig. 1.2 (iv)

Thus, if $z = x + iy$, then

$$\arg(z) = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0, y \geq 0 \\ \tan^{-1}(y/x) - \pi & \text{if } x < 0, y < 0 \\ \pi/2 & \text{if } x = 0, y > 0 \\ -\pi/2 & \text{if } x = 0, y < 0 \end{cases}$$

Caution A usual mistake committed by the students is to take the argument of $z = x + iy$ as $\tan^{-1}(y/x)$ irrespective of the values of x and y. Kindly remember that $\tan^{-1}(y/x)$ lies in the interval $(-\pi/2, \pi/2)$ whereas the principal value of argument of z lies in the interval $(-\pi, \pi]$.

1.7 POLAR FORM OF A COMPLEX NUMBER

Let z be a non-zero complex number, then we can write

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arg(z)$.

For instance,

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

and $$-1 - i = \sqrt{2}\left\{\cos\left(-\frac{3\pi}{4}\right) + \sin\left(-\frac{3\pi}{4}\right)\right\}$$

In fact, if $z = r(\cos\theta + i\sin\theta)$, then z is also given by

$$z = r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$$

where k is any integer.

Euler's Formula

The complex number $\cos\theta + i\sin\theta$ is denoted by $e^{i\theta}$ or $\text{cis}\,\theta$. That is

$$e^{i\theta} = \text{cis}\,\theta = \cos\theta + i\sin\theta$$

Some Important Results Involving Argument

If z, z_1 and z_2 are complex numbers, then

(i) $\arg(\bar{z}) = -\arg(z)$

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(ii) *arg $(z_1 z_2)$ = arg (z_1) + arg (z_2)

In fact,

arg $(z_1 z_2)$ = arg (z_1) + arg (z_2) + $2k\pi$

where

$$k = \begin{cases} 0 \text{ if } -\pi < \arg(z_1) + \arg(z_2) \le \pi \\ 1 \text{ if } -2\pi < \arg(z_1) + \arg(z_2) \le -\pi \\ -1 \text{ if } -\pi < \arg(z_1) + \arg(z_2) \le 2\pi \end{cases}$$

(iii) *arg $(z_1\ \bar{z}_2)$ = arg (z_1) − arg (z_2)

(iv) *arg $\left(\frac{z_1}{z_2}\right)$ = arg (z_1) − arg (z_2)

In fact,

arg $\left(\frac{z_1}{z_2}\right)$ = arg (z_1) − arg (z_2) + $2k\pi$

where k is defined as in (ii) with + sign between arg (z_1) and arg (z_2) replaced by −

(v) $|z_1 + z_2| = |z_1 - z_2|$

$\Leftrightarrow$ arg (z_1) − arg $(z_2) = \pi/2$

(vi) $|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow$ arg (z_1) = arg (z_2)

If $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

where $r_1 = |z_1|$, $r_2 = |z_2|$

and θ_1 = arg (z_1), θ_2 = arg (z_2), then

(vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

$= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$

(viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2| \cos(\theta_1 - \theta_2)$

$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$

1.8 VECTOR REPRESENTATION OF COMPLEX NUMBERS

We can also represent the complex number $z = x + iy$ by using the vector $\overrightarrow{OP}$ joining the origin O of the complex plane to the point $P(x, y)$, instead of using point P itself. The length of the vector $\overrightarrow{OP}$, that is, $|\overrightarrow{OP}|$ is the modulus of z. The angle between the positive real axis and the vector $\overrightarrow{OP}$, more exactly, the angle through which the positive real axis must be rotated to cause it to have the same direction as $\overrightarrow{OP}$ (considered positive if the rotation is counter-clockwise and negative otherwise) is the argument of the complex number z.

* L.H.S. and R.H.S. may differ by a multiple of 2π.

1.9 GEOMETRICAL REPRESENTATION OF ALGEBRAIC OPERATIONS ON COMPLEX NUMBERS

Let z_1 and z_2 be two complex numbers represented by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively.

Sum

By definition, $z_1 + z_2$ should be represented by the point $(x_1 + x_2, y_1 + y_2)$. This point is nothing but the vertex P which completes the parallelogram with the line segments joining the origin with z_1 and z_2 as the adjacent sides. See Fig. 1.3

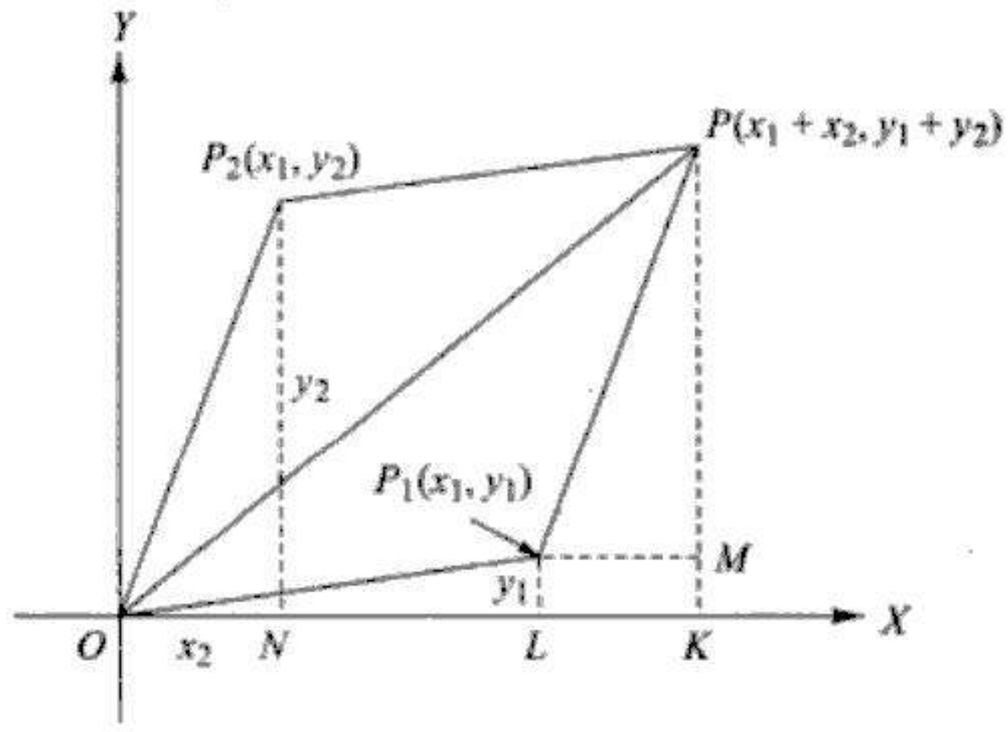

Fig. 1.3

Note that the addition of two complex numbers z_1 and z_2 follows the same law of addition as that of vectors, represented both in magnitude and direction by the line segments joining the origin and the points representing z_1 and z_2, for

$$OP_1 + OP_2 = OP_1 + P_1P = OP$$

Difference We first represent $-z_2$ by P'_2 so that $P_2P'_2$ is bisected at O. Complete the parallelogram $OP_1\ PP'_2$. Then, it can be easily seen that P represents the difference $z_1 - z_2$. See Fig. 1.4.

As $OP_1\ PP'_2$ is a parallelogram so $P_1P = OP'_2$. Using vector notation, we have

$$z_1 - z_2 = OP_1 - OP_2 = OP_1 + P_2O = OP_1 + OP'_2$$
$$= OP_1 + P_1P = OP = P_2P_1$$

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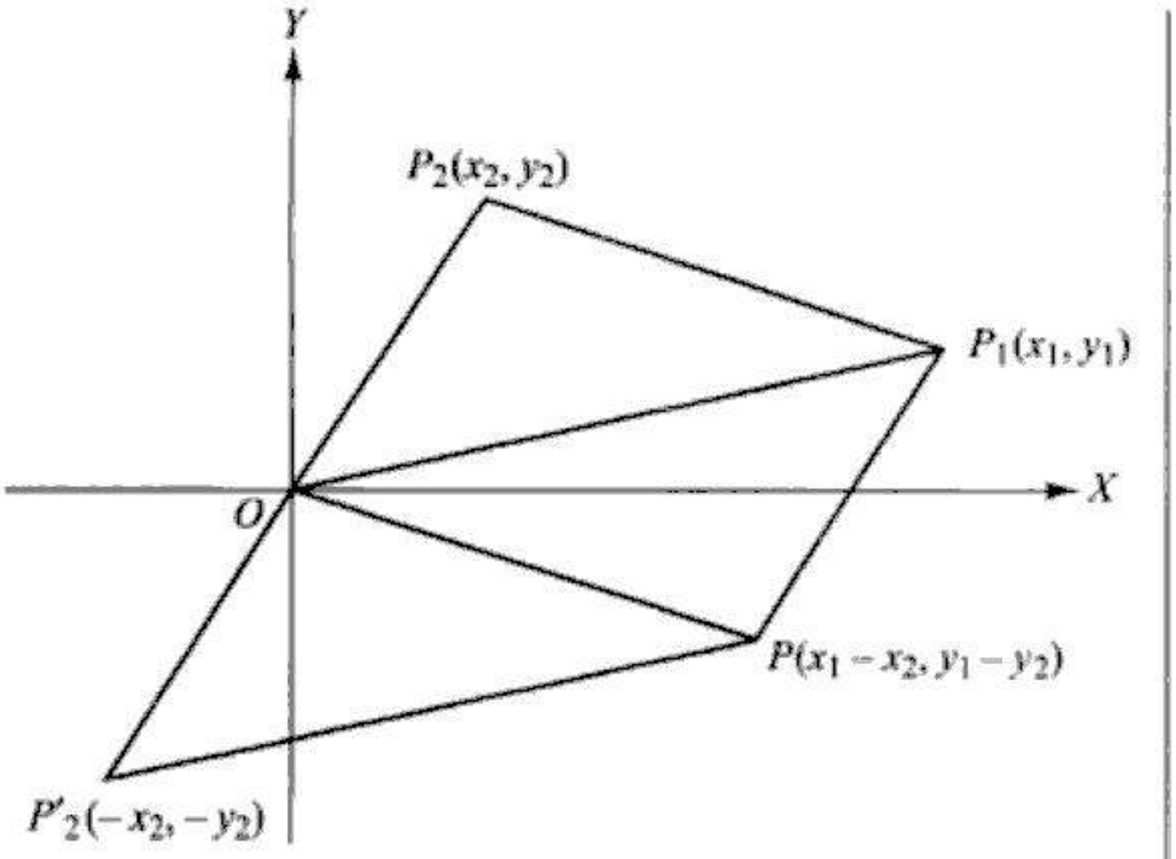

Fig. 1.4

Note the complex number $z_1 - z_2$ is represented by the vector $\boldsymbol{P_2P_1}$, where the points P_2 and P_1 represent the complex number z_1 and z_2 respectively.
Note that $\arg(z_1 - z_2)$ is the angle through which OX must be rotated in the anticlockwise direction so that it becomes parallel to P_2P_1.

Product (Multiplication)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$$

Thus $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$

This shows that modulus of the product of two complex numbers is the product of their moduli, and the argument of the product of any two complex numbers is the sum of their arguments. Using this, we shall derive a geometrical interpretation of the product of two complex numbers.
Let E be a point on the x-axis such that $OE = 1$ unit. (See Fig 1.5.) Complete the triangle OP_1E. Now, taking OP_2 as the base, construct a triangle OPP_2 similar to OP_1E so that

$$OP : OP_1 = OP_2 : OE$$

i.e., $\quad OP = OP_1 \cdot OP_2 \qquad [\because OE = 1]$

Also $\quad \angle P_2OP = \angle EOP_1 = \angle XOP_1 = \theta_1$

Thus $\quad \angle XOP = \theta_1 + \theta_2$

Hence P represents the complex number for which the modulus is $r_1 r_2$ and the argument is $\theta_1 + \theta_2$. That is, it represents the complex number $z_1 z_2$.

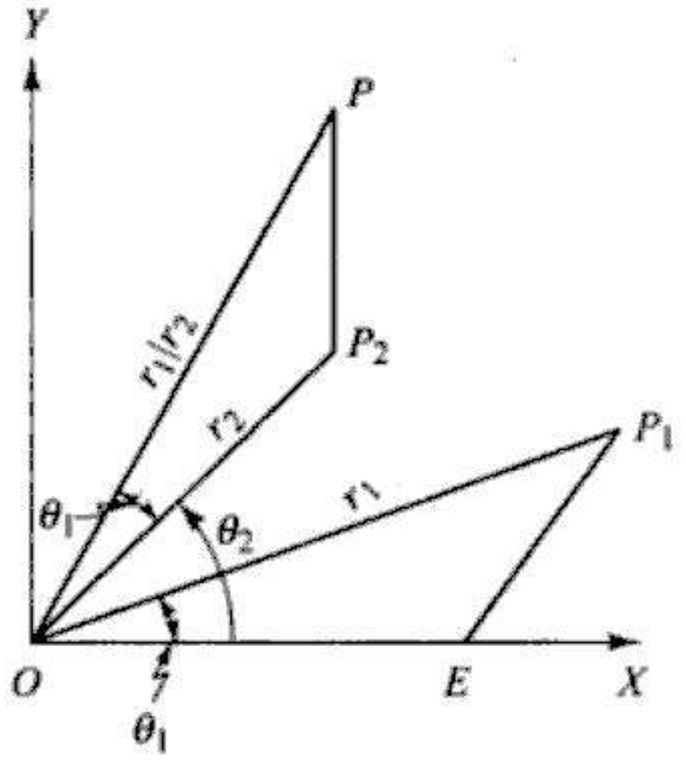

Fig. 1.5

Quotient

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. We take $z_2 \neq 0$, so that $r_2 \neq 0$. Now

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

We shall use this to get a geometrical interpretation of the quotient of a complex number by a non-zero complex number.
Let P_1 and P_2 represent z_1 and z_2 respectively. On OP_1 construct the triangle OPP_1 similar to OEP_2, where E lies on the x-axis and $OE = 1$ unit. (See Fig. 1.6.)

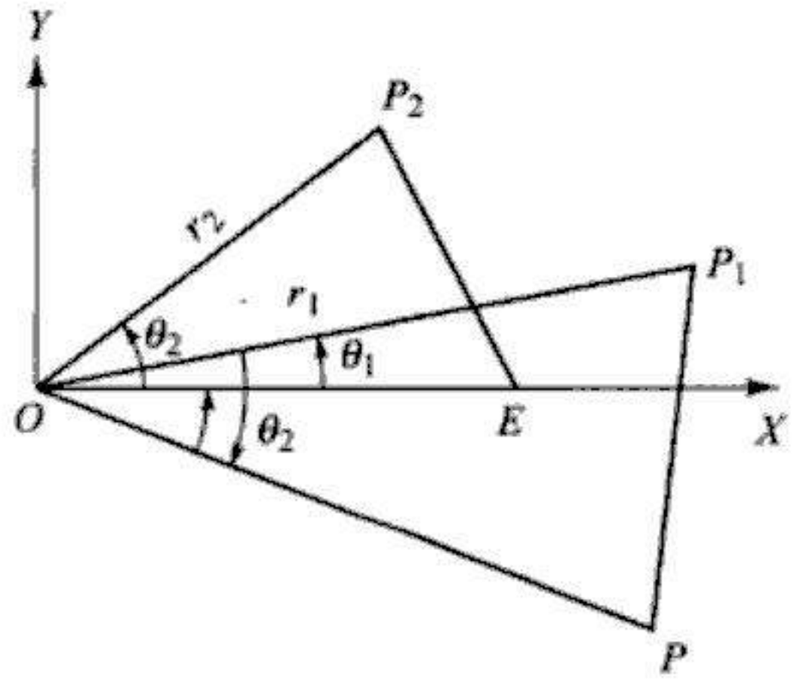

Fig. 1.6

Now, $\quad OP : OE = r_1 : r_2 \Rightarrow OP = \dfrac{r_1}{r_2}$

Also $\quad \angle XOP = \theta_1 - \theta_2$

The point P thus represents the quotient z_1/z_2, since its modulus is r_1/r_2 and its arguments is $\theta_1 - \theta_2$.

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Remark
Note that if θ_1 and θ_2 are the principal values of arg z_1 and arg z_2 then $\theta_1 + \theta_2$ is not necessarily the principal value of arg$(z_1\ z_2)$, nor is $\theta_1 - \theta_2$ necessarily the principal value of arg(z_1/z_2).

Interpretation of $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

If z_1, z_2, z_3 are the vertices of a triangle ABC described in the counter-clockwise sense, then

(i) $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \angle BAC = \alpha$ (say), and

(ii) $\frac{z_3 - z_1}{z_2 - z_1} = \frac{CA}{BA}(\cos\alpha + i\sin\alpha)$

See Fig. 1.7.

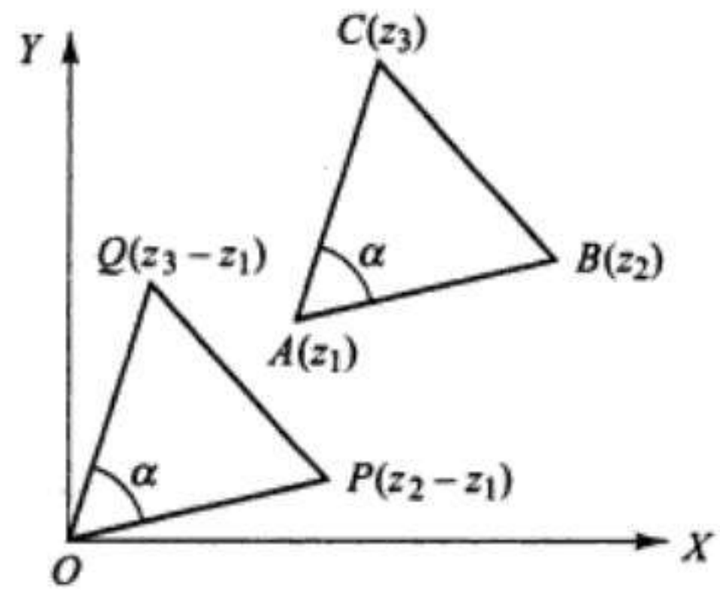

Fig. 1.7

Corollary The points z_1, z_2, z_3 will be collinear if and only if angle $\alpha = 0$ or π, i.e., if and only if $\frac{z_3 - z_1}{z_2 - z_1}$ is purely real.

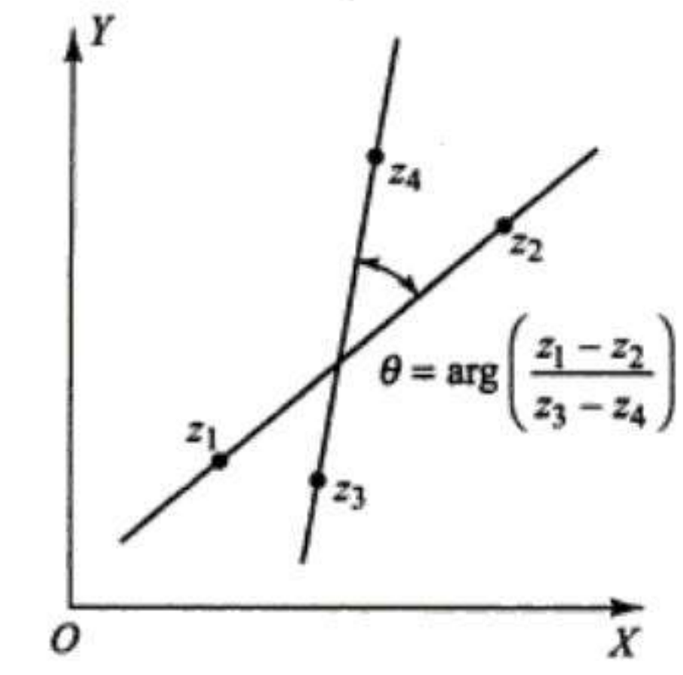

Fig. 1.8

Interpretation of $\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$

Let z_1, z_2, z_3 and z_4 be four complex numbers. Then the line joining z_4 and z_3 is inclined to the line joining z_2 and z_1 at the following angle:

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$$

Corollary The line joining z_4 and z_3 is inclined at 90° to the line joining z_2 and z_1 if

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\frac{\pi}{2}$$

i.e., if $z_1 - z_2 = \pm ik(z_3 - z_4)$, where k is a non-zero real number. (Fig. 1.9).

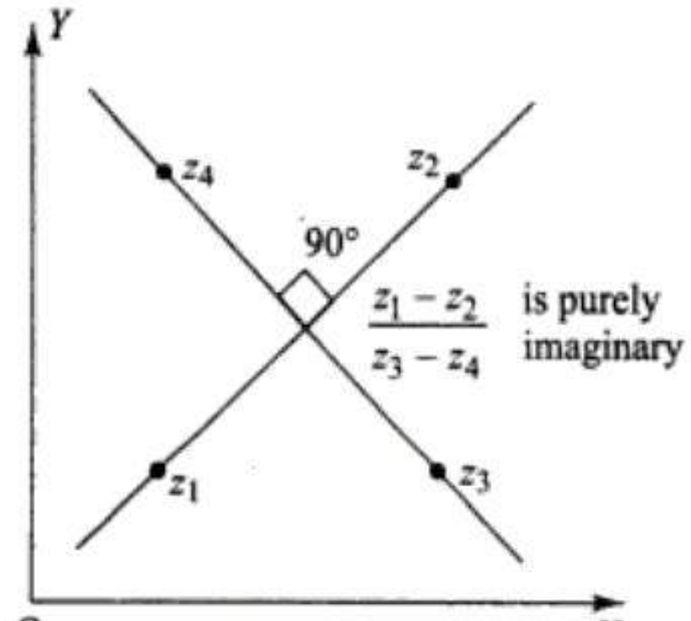

Fig. 1.9

1.10 SOME IMPORTANT GEOMETRICAL RESULTS AND EQUATIONS

1. Distance Formula
Distance between $A(z_1)$ and $B(z_2)$ is given by

$$AB = |z_2 - z_1|$$

Fig. 1.10

2. Section Formula
The point $P(z)$ which divides the join of the segment AB in the ratio $m : n$ is given by

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$$z = \frac{mz_2 + nz_1}{m+n}$$

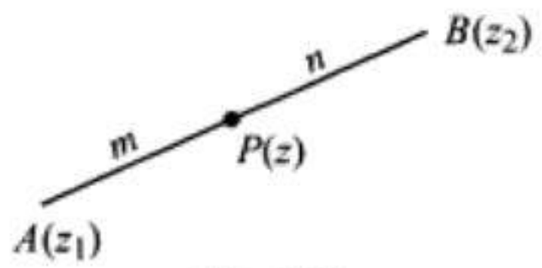


Fig. 1.11

3. Mid-point Formula

Mid-point $M(z)$ of the segment AB is given by

$$z = \frac{1}{2}(z_1 + z_2)$$

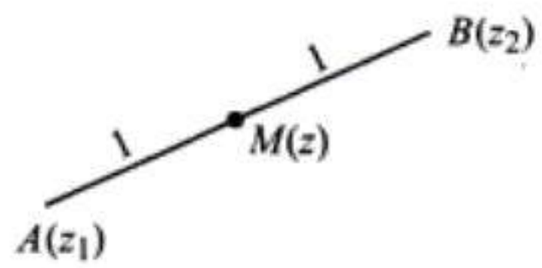


Fig. 1.12

4. Condition(s) *for four non-collinear* $A(z_1)$, $B(z_2)$, $C(z_3)$ *and* $D(z_4)$ *to represent vertices of a*

(i) Parallelogram

The diagonals AC and BD must bisect each other

$\Leftrightarrow$ $\frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4)$

$\Leftrightarrow$ $z_1 + z_3 = z_2 + z_4$

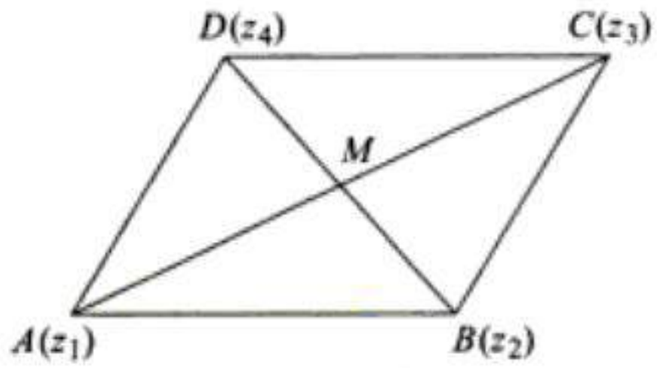


Fig. 1.13

(ii) Rhombus

(a) the diagonals AC and BD bisect each other

$\Leftrightarrow$ $z_1 + z_3 = z_2 + z_4$, and

(b) a pair of two adjacent sides are equal, for instance, $AD = AB$

$\Leftrightarrow$ $|z_4 - z_1| = |z_2 - z_1|$

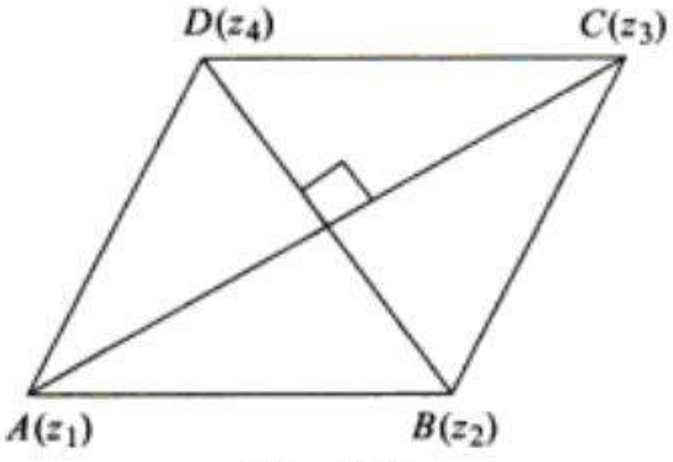


Fig. 1.14

(iii) Square

(a) the diagonals AC and BD bisect each other

$\Leftrightarrow$ $z_1 + z_3 = z_2 + z_4$

(b) a pair of adjacent sides are equal; for instance, $AD = AB$

$\Leftrightarrow$ $|z_4 - z_1| = |z_2 - z_1|$

(c) the two diagonals are equal, that is,

$AC = BD \Leftrightarrow |z_3 - z_1| = |z_4 - z_2|$

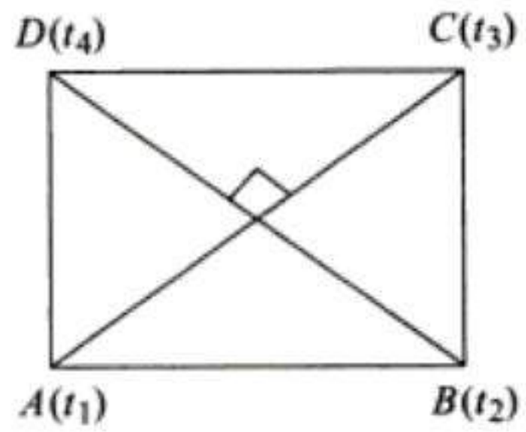


Fig. 1.15

(iv) Rectangle

(a) the diagonals AC and BD bisect each other

$\Leftrightarrow$ $z_1 + z_3 = z_2 + z_4$

(b) the diagonals AC and BD are equal

$\Leftrightarrow$ $|z_3 - z_1| = |z_4 - z_2|$.

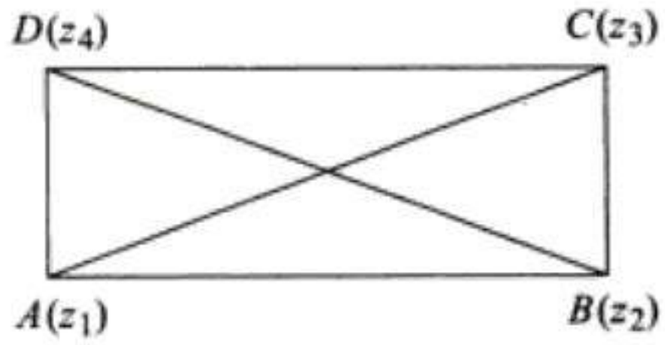
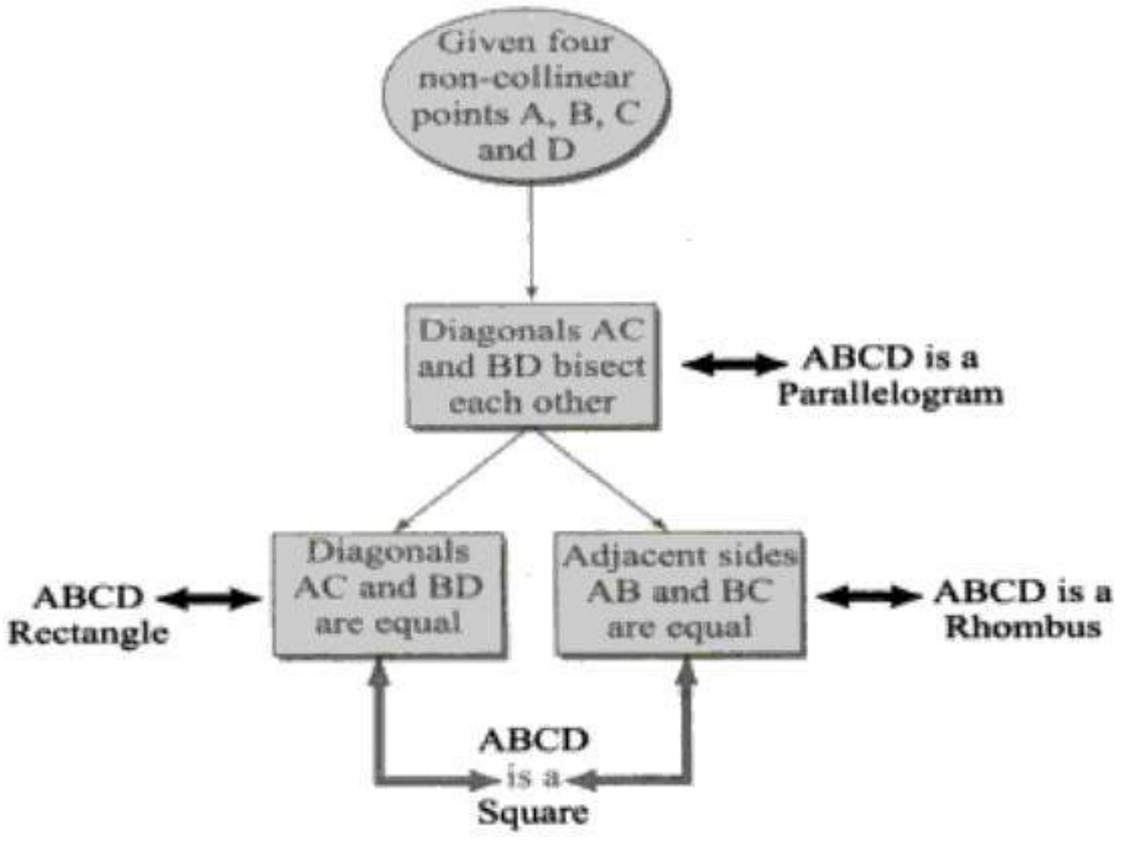


Fig. 1.16

Showing that four non-collinear points form a parallelogram/Rhombus/Square/Rectangle



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5. Centroid, Incentre, Orthocentre and Circumcentre of a Triangle

Let ABC be a triangle with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$,

(i) **Centroid** $G(z)$ of the ΔABC is the point of concurrence of medians of ΔABC and is given by

$$z = \frac{1}{3}(z_1 + z_2 + z_3)$$

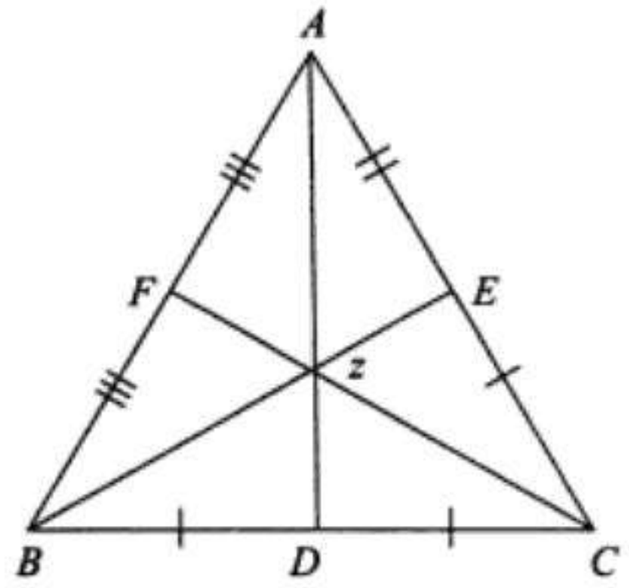

Fig. 1.17

(ii) **Incentre** $I(z)$ of the ΔABC is the point of concurrence of internal bisectors of angles of ΔABC and is given by

$$z = \frac{az_1 + bz_2 + cz_3}{a + b + c}$$

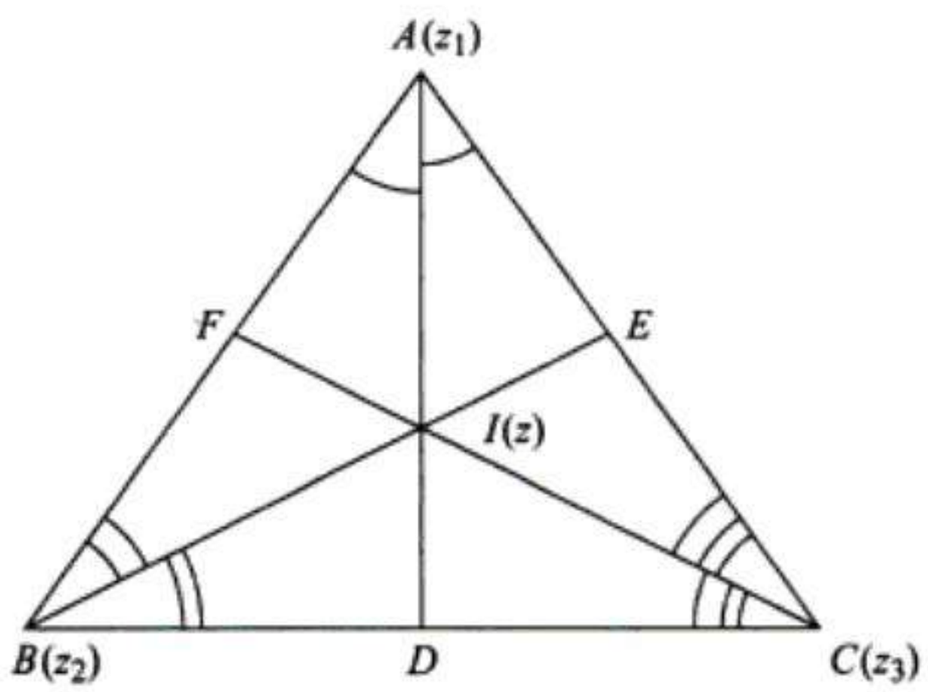

Fig. 1.18

(iii) **Circumcentre** $S(z)$ of the ΔABC is the point of concurrence of perpendicular bisectors of sides of ΔABC and is given by

$$z = \frac{(z_2 - z_3)|z_1|^2 + (z_3 - z_1)|z_2|^2 + (z_1 - z_2)|z_3|^2}{\bar{z}_1(z_2 - z_3) + \bar{z}_2(z_3 - z_1) + \bar{z}_3(z_1 - z_2)}$$

or

$$z = \frac{\begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix}}$$

Also $$z = \frac{z_1(\sin 2A) + z_2(\sin 2B) + z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$$

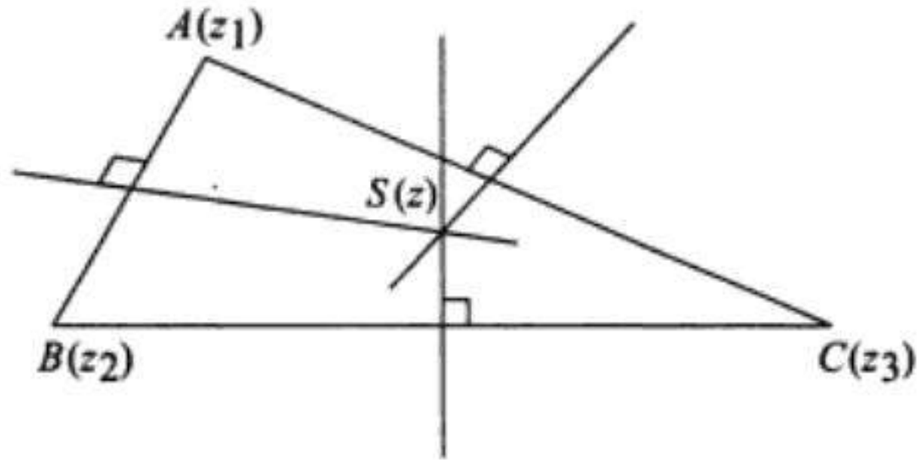

Fig. 1.19

(iv) **Orthocentre** $H(z)$ of the ΔABC is the point of concurrence of altitudes of ΔABC and is given by

$$z = \frac{\begin{vmatrix} z_1^2 & \bar{z}_1 & 1 \\ z_2^2 & \bar{z}_2 & 1 \\ z_3^2 & \bar{z}_3 & 1 \end{vmatrix} + \begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix}}$$

or $$z = \frac{(\tan A)\, z_1 + (\tan B)\, z_2 + (\tan C) z_3}{\tan A + \tan B + \tan C}$$

or $$z = \frac{(a \sec A)\, z_1 + (b \sec B) z_2 + (c \sec C) z_3}{a \sec A + b \sec B + c \sec C}$$

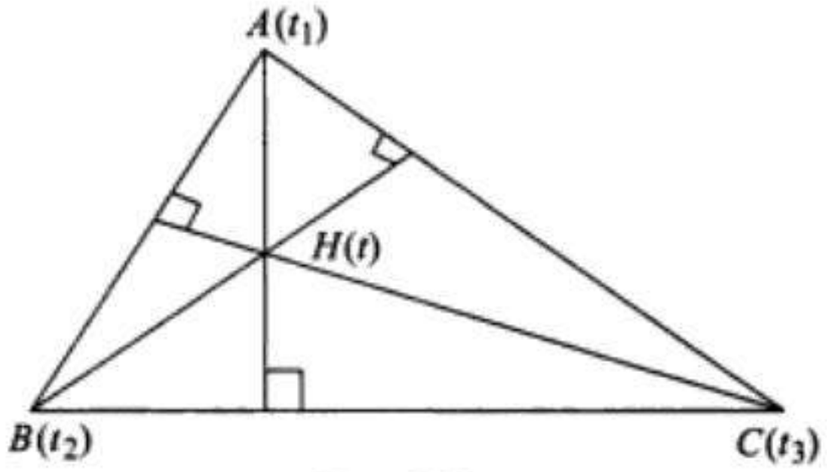

Fig. 1.20

Remark

In case circumcentre of ΔABC is at the origin, then orthocentre of triangle is given by $z_1 + z_2 + z_3$. See Example 22.

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Euler's Line

The centroid G of a triangle lies on the segment joining the orthocentre H and the circumcentre S of the triangle. G divides the join of H and S in the ratio 2 : 1.

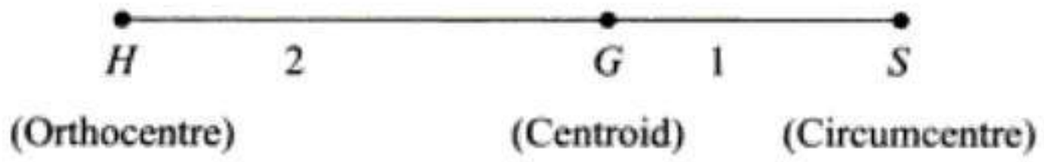

Fig. 1.21

Thus,

$$z_G = \frac{1}{3}(z_H + 2z_S)$$

6. Area of a Triangle

Area of ΔABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \left|\frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$$

$$= \left|\frac{1}{2}\operatorname{Im}(\bar{z}_1 z_2 + \bar{z}_2 z_3 + \bar{z}_3 z_1)\right|$$

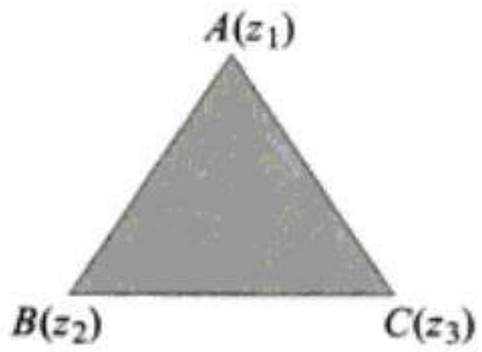

Fig. 1.22

7. Condition for Triangle to be Equilateral

Triangle ABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is equilateral if and only if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

$$\Leftrightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$$

$$\Leftrightarrow \quad z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1$$

$$\Leftrightarrow \quad z_1^2 = z_2 z_3 \text{ and } z_2^2 = z_1 z_3$$

$$\Leftrightarrow \quad \begin{vmatrix} 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \\ 1 & z_1 & z_2 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad \frac{z_2 - z_1}{z_3 - z_2} = \frac{z_2 - z_2}{z_1 - z_2}$$

$$\Leftrightarrow \quad \frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3} = 0$$

where $z = \frac{1}{3}(z_1 + z_2 + z_3)$

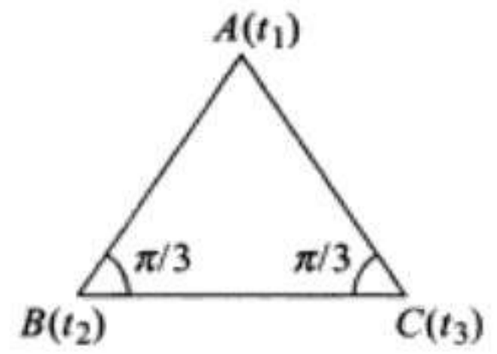

Fig. 1.23

8. Equation of a Straight Line

(i) Non-parametric form

An equation of a straight line joining the two points $A(z_1)$ and $B(z_2)$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or $$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

or $$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

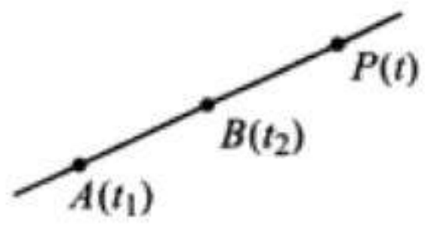

Fig. 1.24

(ii) Parametric form

An equation of the straight line joining the points $A(z_1)$ and $B(z_2)$ is

$$z = tz_1 + (1 - t)z_2$$

where t is a real parameter.

(iii) General Equation of a Straight Line

The general equation of a straight line is

$$\bar{a}z + a\bar{z} + b = 0$$

where a is a non-zero complex number and b is a real number.

9. Complex Slope of a Line

If $A(z_1)$ and $B(z_2)$ are two points in the complex plane, then complex slope of AB is defined to be

$$\mu = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

Two lines with complex slopes μ_1 and μ_2 are

(i) parallel, if $\mu_1 = \mu_2$

(ii) perpendicular, if $\mu_1 + \mu_2 = 0$

The complex slope of the line $\bar{a}z + a\bar{z} + b = 0$ is given by $-(a/\bar{a})$

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10. Length of Perpendicular from a Point to a Line

Length of perpendicular of point $A(\omega)$ from the line $\bar{a}z + a\bar{z} + b = 0$ ($a \in \mathbf{C} - \{0\}$, $b \in \mathbf{R}$) is given by

$$p = \frac{|\bar{a}\omega + a\bar{\omega} + b|}{2|a|}$$

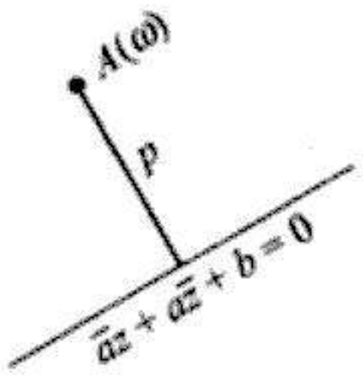

Fig. 1.25

11. Equation of Circle

(i) An equation of the circle with centre at z_0 and radius r is

$$|z - z_0| = r$$

or $z = z_0 + re^{i\theta}$, $0 \le \theta < 2\pi$ (parametric form)

or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z}_0 - r^2 = 0$

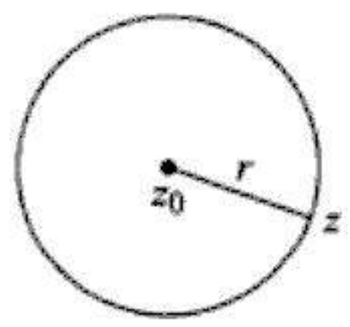

Fig. 1.26

(ii) General equation of a circle is

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0 \qquad (1)$$

where a is a complex number and b is a real number such that $a\bar{a} - b \ge 0$

Centre of (1) is $-a$ and its radius is $\sqrt{a\bar{a} - b}$.

(iii) Diameter Form of a Circle

An equation of the circle one of whose diameter is the segment joining $A(z_1)$ and $B(z_2)$ is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) = 0$$

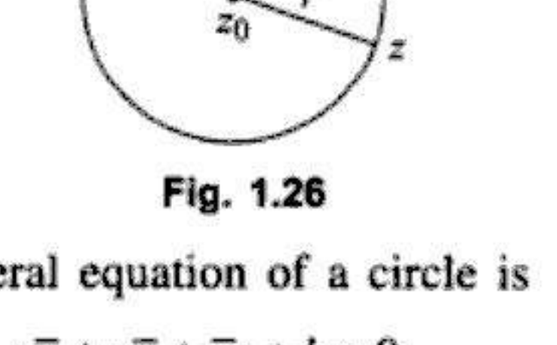

Fig. 1.27

(iv) An equation of the circle passing through two points $A(z_1)$ and $B(z_2)$ is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2)$$
$$+ k\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

where k is a parameter.

(v) Equation of a circle passing through three non-collinear points.

Let three non-collinear points be $A(z_1)$, $B(z_2)$ and $C(z_3)$. Let $P(z)$ be any point on the circle. Then either

$\angle ACB = \angle APB$

[when angles are in the same segment]

$\angle ACB + \angle APB = \pi$

[when angles are in the opposite segment]

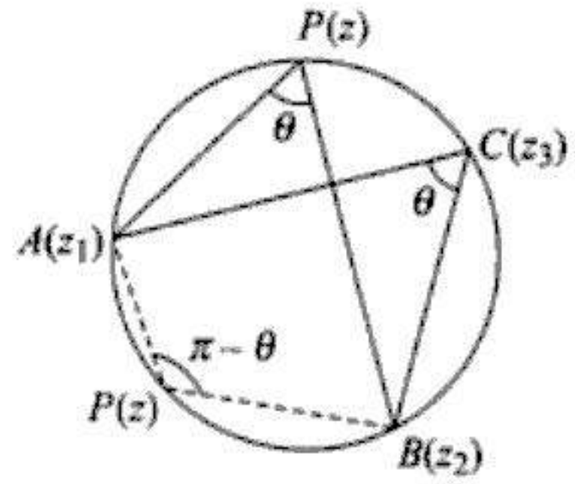

Fig. 1.28

$$\Rightarrow \quad \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) - \arg\left(\frac{z - z_2}{z - z_1}\right) = 0$$

or

$$\arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) + \arg\left(\frac{z - z_1}{z - z_2}\right) = \pi$$

$$\Rightarrow \quad \arg\left[\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\left(\frac{z - z_1}{z - z_2}\right)\right] = 0$$

$$\text{or} \quad \arg\left[\left(\frac{z - z_1}{z - z_2}\right)\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\right] = \pi$$

[using $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ and

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$]

In any case, we get $\frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$ is purely real.

$$\Leftrightarrow \quad \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}$$

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(vi) Condition for four points to be concyclic.
Four points z_1, z_2, z_3 and z_4 will lie on the same circle if and only if $\dfrac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}$ is purely real.

$\Leftrightarrow$ $\dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$ is purely real.

1.11 RECOGNIZING SOME LOCI BY INSPECTION

(i) $\arg(z) = \alpha$
If α is a real number, then $\arg(z) = \alpha$ represents a ray starting at the origin (excluding the origin) and making an angle α with the real axis. See Fig. 1.29.

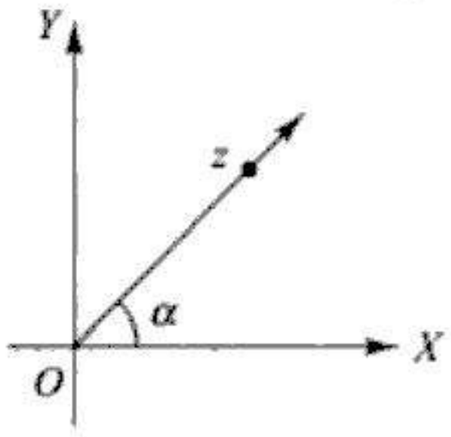

Fig. 1.29

(ii) $\arg(z - z_0) = \alpha$
If α is a real number and z_0 is a fixed point, then $\arg(z - z_0) = \alpha$ represents a ray starting at z_0 (excluding the point z_0) and making an angle α with the real axis.

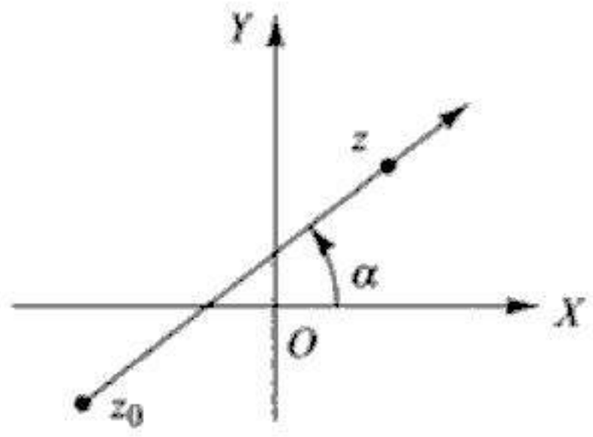

Fig. 1.30

(iii) If z_1 and z_2 are two fixed points, then
$|z - z_1| = |z - z_2|$
represents perdendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.

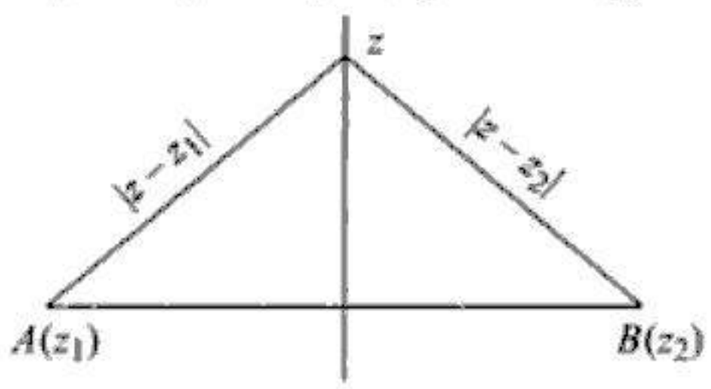

Fig. 1.31

(iv) If z_1 and z_2 are two fixed points, and $k > 0$, $k \neq 1$ is a real number, then
$$\frac{|z - z_1|}{|z - z_2|} = k$$
represents a circle.
For $k = 1$, it represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.

(v) $|z - z_1| + |z - z_2| = k$
Let z_1 and z_2 be two fixed points and k be a positive real number.
(a) If $k > |z_1 - z_2|$, then
$|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at $A(z_1)$ and $B(z_2)$ and length of major axis $= k$.

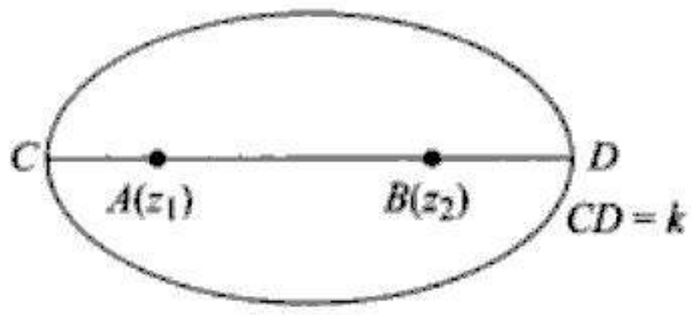

Fig. 1.32

(b) If $k = |z_1 - z_2|$, then
$|z - z_1| + |z - z_2| = k$
represents the segment joining z_1 and z_2.
(c) If $k < |z_1 - z_2|$, then
$|z - z_1| + |z - z_2| = k$
does not represent any curve in the Argand plane.

(vi) $|z - z_1| - |z - z_2| = k$
Let z_1 and z_2 be two fixed points, k be a positive real number.
(a) If $k \neq |z_1 - z_2|$, then
$|z - z_1| - |z - z_2| = k$
represents a hyperbola with foci at $A(z_1)$ and $B(z_2)$.

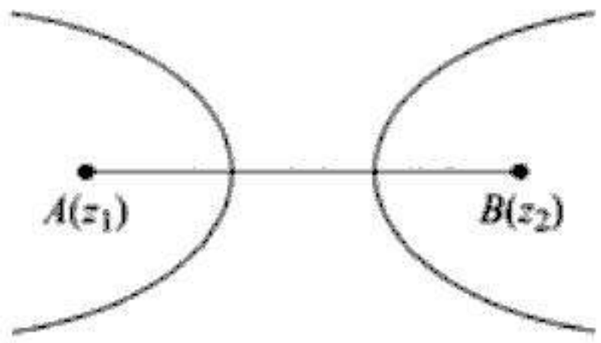

Fig. 1.33

(b) If $k = |z_1 - z_2|$, then $||z - z_1| - |z - z_2|| = k$ represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB. See Fig. 1.34.

Fig. 1.34

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(vii) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

If z_1 and z_2 are two fixed points, then

$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

represent a circle with z_1 and z_2 as the extremities of a diameter. See Fig. 1.35.

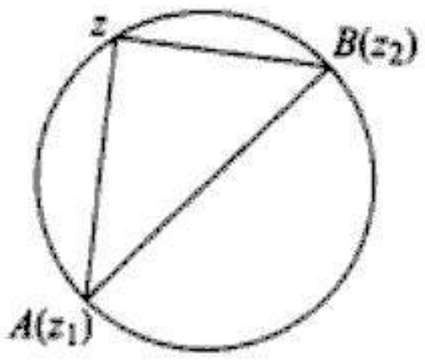

Fig. 1.35

(viii) $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \alpha$

Let z_1 and z_2 be two fixed points, and α be a real number such $0 \le \alpha \le \pi$.

(a) If $0 < \alpha < \pi$ and $\alpha \neq \pi/2$, then

$$\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha$$

represents a segment of the circle passing through $A(z_1)$ and $B(z_2)$. See Fig. 1.36.

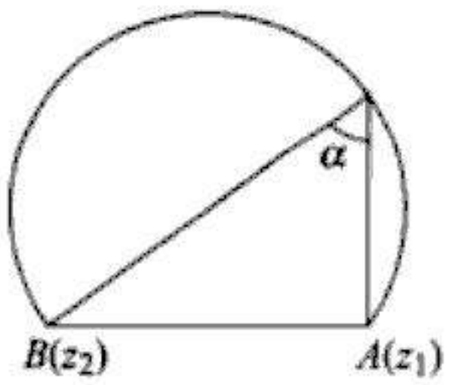

Fig. 1.36

(b) If $\alpha = \pi/2$, then

$$\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha\left(=\frac{\pi}{2}\right)$$

represents a circle with diameter as the segment joining $A(z_1)$ and $B(z_2)$. See Fig. 1.37.

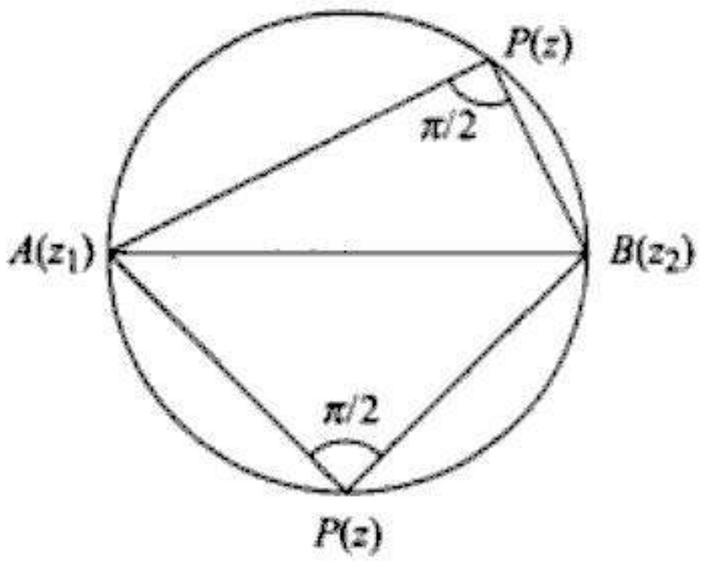

Fig. 1.37

(c) If $\alpha = \pi$, then

$$\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha\,(=\pi)$$

represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB. See Fig. 1.38

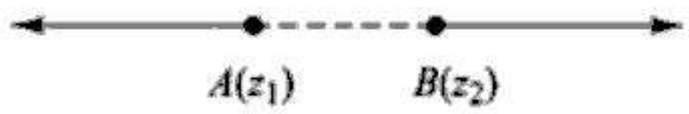

Fig. 1.38

(d) If $\alpha = 0$, then

$$\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha\,(=0)$$

represents the segment joining $A(z_1)$ and $B(z_2)$ see Fig. 1.39.

Fig. 1.39

1.12 GEOMETRIC INTERPRETATION OF MULTIPLYING A COMPLEX NUMBER BY $e^{i\alpha}$.

Let z be a non-zero complex number. We can write z in the polar form as follows:

$$z = r(\cos\,\theta + i\,\sin\,\theta) = re^{i\theta}$$

where $r = |z|$ and $\arg(z) = \theta$

We have

$$ze^{i\alpha} = re^{i\theta}\,e^{i\alpha} = re^{i(\theta+\alpha)}$$

Thus, $ze^{i\alpha}$ represents the complex number whose modulus is r and argument is $\theta + \alpha$. Geometrically, $ze^{i\alpha}$ can be obtained by rotating the segment joining O and $P(z)$ through an angle α in the anticlockwise direction. See Fig. 1.40

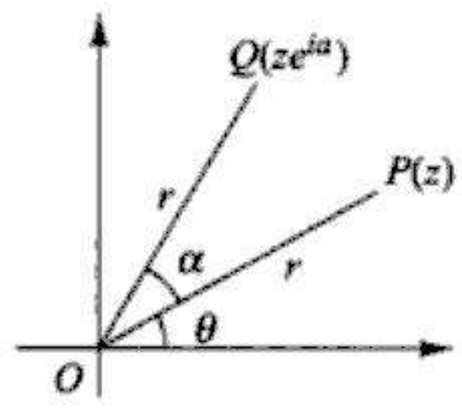

Fig. 1.40

Corollary If $A(z_1)$ and $B(z_2)$ are two complex number such that angle $\angle AOB = \theta$, then we can write

$$z_2 = \frac{|z_2|}{|z_1|}\,z_1 e^{i\theta}$$

Suppose $z_1 = r_1\,e^{i\alpha}$ and $z_2 = r_2\,e^{i\beta}$,

where $|z_1| = r_1$, $|z_2| = r_2$. Then

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$$\frac{z_2}{z_1} = \frac{r_2 e^{i\beta}}{r_1 e^{i\alpha}} = \frac{r_2}{r_1} e^{i(\beta-\alpha)}$$

Note that

$$\beta - \alpha = \theta$$

Thus,

$$\frac{z_2}{z_1} = \frac{r_2}{r_1} e^{i\theta}$$

$$\Rightarrow \qquad z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta}$$

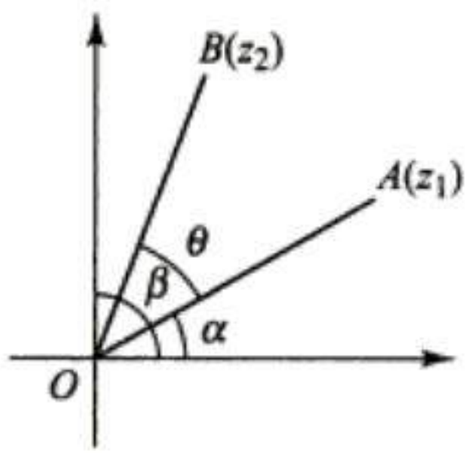

Fig. 1.41

1.13 DE MOIVRE'S THEOREM AND ITS APPLICATIONS

(a) *De Moivre's Theorem for integral index.*
If n is an integer, then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

(b) *De Moivre's Theorem for rational index.*
If n is a rational number, then value of or one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos(n\theta) + i\sin(n\theta)$. In fact, if $n = p/q$ where $p, q \in \mathbf{I}, q > 0$ and p,q have no factors in common, then $(\cos\theta + i\sin\theta)^n$ has q distinct values, one of which is $\cos(n\theta) + i\sin(n\theta)$.

Note
The values of $(\cos\theta + i\sin\theta)^{p/q}$ where $p, q \in \mathbf{I}$, $q \neq 0$, hcf $(p,q) = 1$ are given by

$$\cos\left[\frac{p}{q}(2k\pi+\theta)\right] + i\sin\left[\frac{p}{q}(2k\pi+\theta)\right]$$

where $k = 0, 1, 2, \ldots, q-1$.

The *n*th Roots of Unity

By an nth root of unity we mean any complex number z which satisfies the equation

$$z^n = 1 \qquad (1)$$

Since, an equation of degree n has n roots, there are n values of z which satisfy the equation (1). To obtain these n values of z, we write

$$1 = \cos(2k\pi) + i\sin(2k\pi)$$

where $k \in \mathbf{I}$ and

$$\Rightarrow \qquad z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

[using the De Moivre's Theorem]

where $k = 0, 1, 2, \ldots, n-1$.

Note
We may give any n consecutive integral values to k. For instance, in case of 3, we may take $-1, 0$ and 1 and in case of 4, we may take $-1, 0, 1$ and 2 or $-2, -1, 0$ and 1.

Notation $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$

By using the De Moivre's theorem, we can write the nth roots of unity as

$$1, \omega, \omega^2, \ldots, \omega^{n-1}.$$

Sum of the Roots of Unity is Zero

We have

$$1 + \omega + \ldots + \omega^{n-1} = \frac{1-\omega^n}{1-\omega}$$

But $\omega^n = 1$ as ω is a nth root of unity.

$$\therefore \qquad 1 + \omega + \ldots + \omega^{n-1} = 0$$

Also, note that

$$\frac{1}{x-1} + \frac{1}{x-\omega} \ldots + \frac{1}{x-\omega^{n-1}} = \frac{nx^{n-1}}{x^n - 1}$$

Writing *n*th Roots of Unity When *n* is Odd

If $n = 2m + 1$, then nth roots of unity are also given by

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

where $k = -m, -(m-1), \ldots, -1, \quad 0, 1, 2, \ldots m$.

Since $\cos\left(-\frac{2k\pi}{n}\right) = \cos\left(\frac{2k\pi}{n}\right)$

and $\sin\left(-\frac{2k\pi}{n}\right) = -\sin\left(\frac{2k\pi}{n}\right)$

we may take the roots as

$$1, \cos\left(\frac{2k\pi}{n}\right) \pm i\sin\left(\frac{2k\pi}{n}\right)$$

where $k = 1, 2, \ldots, m$.
In terms, ω we may take nth roots of unity to be $1, \omega^{\pm 1}, \omega^{\pm 2}, \ldots \omega^{\pm m}$.

Writing *n*th Roots of Unity When *n* is Even

If $n = 2m$, then nth roots of unity are given

$$z = \pm 1, \pm\omega, \pm\omega^2, \ldots \pm\omega^{m-1}$$

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where $\omega = \cos\left(\frac{2\pi}{2m}\right) + i\sin\left(\frac{2\pi}{2m}\right) = \cos\left(\frac{\pi}{m}\right) + i\sin\left(\frac{\pi}{m}\right)$

Cube Roots of Unity

Cube roots of unity are given by 1, ω, ω^2, where

$$\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2} \text{ and } \omega^2 = \frac{-1-\sqrt{3}i}{2}$$

Some Results Involving Complex Cube Root of Unity (ω)

(i) $\omega^3 = 1$

(ii) $1 + \omega + \omega^2 = 0$

(iii) $x^3 - 1 = (x - 1)(x - \omega)(x - \omega^2)$

(iv) ω and ω^2 are roots of $x^2 + x + 1 = 0$

(v) $a^3 - b^3 = (a - b)(a - b\omega)(a - b\omega^2)$

(vi) $a^2 + b^2 + c^2 - bc - ca - ab$
$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(vii) $a^3 + b^3 + c^3 - 3abc$
$= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(viii) $x^3+1 = (x + 1)(x + \omega)(x + \omega^2)$

(ix) $a^3 + b^3 = (a + b)(a + b\omega)(a + b\omega^2)$

(x) Cube roots of real number a are $a^{1/3}, a^{1/3}\omega, a^{1/3}\omega^2$.

To obtain cube roots of a, we write $x^3 = a$ as $y^3 = 1$ where $y = x/a^{1/3}$.

Solution of $y^3 = 1$ are $1, \omega, \omega^2$.

$\therefore$ $x = a^{1/3}, a^{1/3}\omega, a^{1/3}\omega^2$.

nth Roots of a Complex Number

Let $z \neq 0$ be a complex number. We can write z in the polar form as follows:

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arg(z)$. Recall $-\pi < \theta \leq \pi$.

The nth root of z has n values one of which is equal to

$z_0 = \sqrt[n]{|z|}\left[\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right]$ and is called as the principal value of $\sqrt[n]{|z|}$. To obtain other value of $\sqrt[n]{|z|}$, we write z as

$$z = r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$$

$$\Rightarrow z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$$

$$= z_0\,\omega^k \text{ where } k = 0, 1, 2, \ldots, n-1.$$

and $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is a complex nth root of unity. Thus, all the n nth roots of z can be obtained by multiplying the principal value of $\sqrt[n]{|z|}$ by different roots of unity.

Rational Power of a Complex Number

If z is a complex number and m/n is a rational number such that m and n are relatively prime integers and $n > 0$. We define

$$z^{m/n} = \left(\sqrt[n]{z}\right)^m$$

Thus, $z^{m/n}$ has n distinct values which are given by

$$z^{m/n} = \left(\sqrt[n]{|z|}\right)^m\left[\cos\left(\frac{m}{n}(\theta + 2k\pi)\right) + i\sin\left(\frac{m}{n}(\theta + 2k\pi)\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

SOLVED EXAMPLES

SECTION I

SINGLE CORRECT CHOICE TYPE

Example 1 The number of solutions of $z^3 + \bar{z} = 0$ is

(a) 2 (b) 3 (c) 4 (d) 5

Ans. (d)

Solution $z^3 + \bar{z} = 0 \Rightarrow z^3 = -\bar{z}$

$\Rightarrow$ $|z|^3 = |-\bar{z}| \Rightarrow |z|^3 = |z|$

$|z|(|z| - 1)(|z| + 1) = 0$

$\Rightarrow$ $|z| = 0$ or $|z| = 1$ $[\because |z| + 1 > 0]$

If $|z| = 0$, then $z = 0$

If $|z| = 1$ we get $|z|^2 = 1 \Rightarrow z\bar{z} = 1$

Thus, $z^3 + \bar{z} = 0 \Rightarrow z^3 + 1/z = 0$

$\Rightarrow$ $z^4 + 1 = 0$

This equation has four non-zero and distinct roots. Therefore, the given equation has five roots.

TIP It is unnecessary to find roots of $z^4 + 1 = 0$

Example 2 Let p and q be two complex numbers such that $q \neq 0$. If the roots of the equation $x^2 + px + q^2 = 0$ have the same absolute value, then p/q is

(a) an integer
(b) a rational number
(c) a real number
(d) a complex number.

Ans. (c)

Solution Let x_1, x_2 be the roots of $x^2 + px + q^2 = 0$ and $|x_1| = |x_2| = r$. Note that $x_1x_2 = q^2 \neq 0$. Now,

$$\frac{p^2}{q^2} = \frac{(x_1 + x_2)^2}{x_1x_2} = \frac{x_1}{x_2} + \frac{x_2}{x_1} + 2$$

$$= \frac{x_1\bar{x}_2}{|x_2|^2} + \frac{\bar{x}_1x_2}{|x_1|^2} + 2 = \frac{1}{r^2}(x_1\bar{x}_2 + \bar{x}_1x_2) + 2$$

$$= \frac{2}{r^2}\mathrm{Re}(x_1\bar{x}_2) + 2$$

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$\Rightarrow \frac{p^2}{q^2}$ is a real number.

Next, as $-\text{Re}(x_1, \bar{x}_2) \le |x_1 \bar{x}_2| = |x_1 x_2| = |x_1|\,|x_2| = r^2$, we get

$$\Rightarrow \frac{p^2}{q^2} \ge -\frac{2r^2}{r^2} + 2 = -2 + 2 = 0$$

Therefore, p/q is a real number.

Example 3 If $k > 0$, $|z| = k$ and

$w = \dfrac{z-k}{z+k}$, then Re(w) equals

(a) 0 (b) k

(c) $\dfrac{1}{k}$ (d) $-\dfrac{1}{k}$

Ans. (a)

Solution $|z| = k \Rightarrow |z|^2 = k^2$ or $z\bar{z} = k^2$ Next

$$2Re(w) = w + \bar{w} = \frac{z-k}{z+k} + \frac{\bar{z}-k}{\bar{z}+k}$$

Using $\bar{z} = k^2/z$, we get 2 Re $(w) = 0$.

Example 4 If $\left|z^3 + \dfrac{1}{z^3}\right| \le 2$, then $\left|z + \dfrac{1}{z}\right|$ cannot exceed

(a) 2 (b) 1

(c) $\sqrt{2}$ (d) $\sqrt{2} - 1$

Ans. (a)

Solution Let $\left|z + \dfrac{1}{z}\right| = a$.

Now, the identity,

$$z^3 + \frac{1}{z^3} = \left(z + \frac{1}{z}\right)^3 - 3\left(z + \frac{1}{z}\right)$$

gives us

$$\left|z + \frac{1}{z}\right|^3 \le \left|z^3 + \frac{1}{z^3}\right| + 3\left|z + \frac{1}{z}\right|$$

$$\Rightarrow \quad a^3 \le 2 + 3a \Rightarrow a^3 - 3a - 2 \le 0$$

$$\Rightarrow \quad (a-2)(a+1)^2 \le 0 \Rightarrow a - 2 \le 0$$

$$\Rightarrow \quad a \le 2$$

Example 5 If $iz^3 + z^2 - z + i = 0$, then $|z|$ equals

(a) 4 (b) 3 (c) 2 (d) 1

Ans. (d)

Solution We can write the given equation as

$$z^3 + \frac{1}{i}z^2 - \frac{1}{i}z + 1 = 0$$

or $\quad z^3 - iz^2 + iz - i^2 = 0$

$$\Rightarrow \quad z^2(z-i) + i(z-i) = 0$$

$$\Rightarrow \quad (z^2 + i)(z-i) = 0 \quad \Rightarrow \quad z^2 = -i,\ z = i$$

$$\Rightarrow \quad |z|^2 = |-i| \quad \text{and} \quad |z| = |i|$$

$$\Rightarrow \quad |z|^2 = 1 \quad \text{and} \quad |z| = 1$$

$$\Rightarrow \quad |z| = 1$$

Example 6 If $\omega = \cos\dfrac{\pi}{n} + i \sin\dfrac{\pi}{n}$, then value of $1 + \omega + \omega^2 + \ldots + \omega^{n-1}$ is

(a) $1 + i$ (b) $1 + i \tan(\pi/n)$

(c) $1 + i \cot(\pi/2n)$ (d) none of these.

Ans. (c)

Solution We have

$$S = 1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \frac{1-\omega^n}{1-\omega}$$

But $\quad \omega^n = \cos\left(\dfrac{n\pi}{n}\right) + i \sin\left(\dfrac{n\pi}{n}\right) = -1$

Thus, $\quad S = \dfrac{2}{1-\omega}$

But $\quad 1 - \omega = 1 - \cos\left(\dfrac{\pi}{n}\right) - i \sin\left(\dfrac{\pi}{n}\right)$

$$= 2\sin^2\left(\frac{\pi}{2n}\right) - 2i \sin\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right)$$

$$= -2i \sin\left(\frac{\pi}{2n}\right)\left[\cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)\right]$$

Thus, $\quad S = \dfrac{-2}{2i\sin(\pi/2n)}\left[\cos\left(\dfrac{\pi}{2n}\right) + i\sin\left(\dfrac{\pi}{2n}\right)\right]^{-1}$

$$= 1 + i \cot\left(\frac{\pi}{2n}\right)$$

Alternatively, we write

$$S = \frac{2e^{-\pi i/2n}}{e^{-\pi i/2n} - e^{\pi i/2n}}$$

$$= \frac{-1}{i\sin\left(\dfrac{\pi}{2n}\right)}\left[\cos\left(\frac{\pi}{2n}\right) - i\sin\left(\frac{\pi}{2n}\right)\right]$$

$$= 1 + i \cot\left(\frac{\pi}{2n}\right)$$

Example 7 For any complex number z, the minimum value of $|z| + |z - 2i|$ is

(a) 0 (b) 1

(c) 2 (d) none of these

Ans. (c)

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Solution We have, for $z \in \mathbf{C}$

$$|2i| = |z + (2i - z)| \le |z| + |2i - z|$$

$$\Rightarrow \quad 2 \le |z| + |z - 2i|$$

Thus, minimum value of $|z| + |z - 2i|$ is 2 and it is attained for any z lying on the segment joining $z = 0$ and $z = 2i$.

Example 8 For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

(a) 0 (b) 2 (c) 7 (d) 17

Ans. (b)

Solution $|z_1| = 12$ implies that z_1 lies on the circle with centre C_1 at the origin and radius 12 whereas $|z_2 - 3 - 4i| = 5$ implies z_2 lies on the circle with centre at C_2 $(3 + 4i)$ and radius 5. See Fig. 1.42. The quantity $|z_1 - z_2|$ will attain the least value 2 when z_1 and z_2 lie on the line joining C_1 and C_2 i.e. on the line $z = (3 + 4i)t$

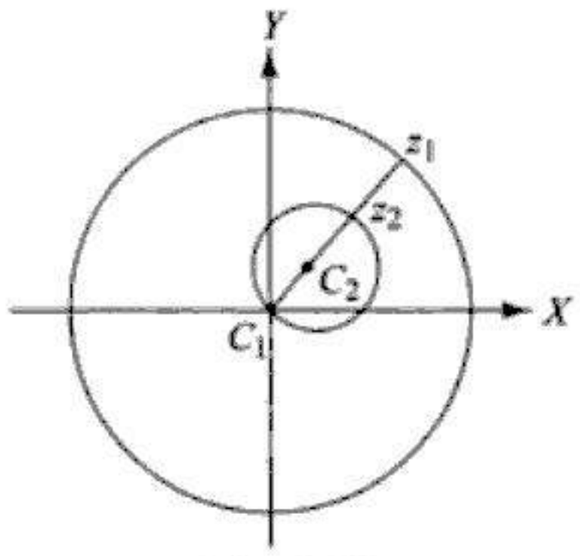

Fig. 1.42

In fact, when we take $z_2 = 6 + 8i$ and $z_1 = \frac{12}{5}(3 + 4i)$, we obtain $|z_1 - z_2| = \left|\frac{12}{5}(3+4i) - (6+8i)\right| = \frac{2}{5}|3+4i| = 2$.

Example 9 If z lies on the circle $|z - 1| = 1$, then $\frac{z-2}{z}$ equals

(a) 0 (b) 2
(c) -1 (d) none of these

Ans. (d)

Solution Note that $|z - 1| = 1$ represents a circle with the segment joining $z = 0$ and $z = 2 + 0i$ as a diameter. See Fig. 1.43. If z lies on the circle, then $\frac{z-2}{z-0}$ is purely imaginary.

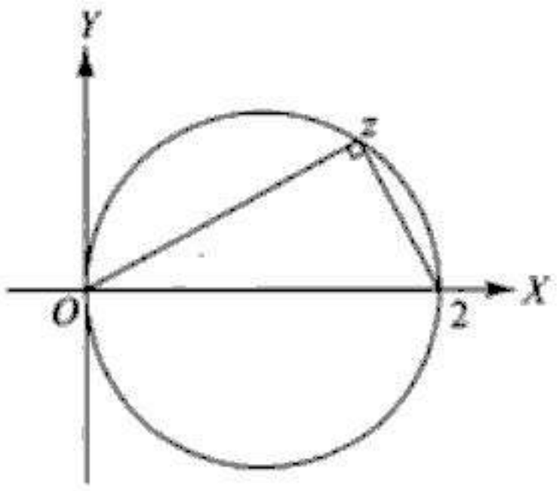

Fig. 1.43

Example 10 If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is

(a) equal to 1 (b) less than 1
(c) greater than 3 (d) equal to 3

Ans. (a)

Solution Since $|z_1| = |z_2| = |z_3| = 1$, we get $z_1 \bar{z}_1 = z_2 \bar{z}_2 = z_3 \bar{z}_3 = 1$

Now,

$$1 = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = |\overline{z_1 + z_2 + z_3}|$$

$$\Rightarrow \quad 1 = |z_1 + z_2 + z_3|$$

Example 11 If ω is an imaginary cube root of unity, then value of the expression

$$2(1 + \omega)(1 + \omega^2) + 3(2 + \omega)(2 + \omega^2) + \ldots + (n + 1)(n + \omega)(n + \omega^2)$$

is

(a) $\frac{1}{4}n^2(n+1)^2 + n$ (b) $\frac{1}{4}n^2(n+1)^2 - n$

(c) $\frac{1}{4}n(n+1)^2 - n$ (d) none of these

Ans. (a)

Solution rth term of the given expression is

$$(r + 1)(r + \omega)(r + \omega^2) = r^3 + 1$$

$\therefore$ value of the given expression is

$$\sum_{r=1}^{n}(r^3 + 1) = \frac{1}{4}n^2(n+1)^2 + n$$

Example 12 The greatest and the least value of $|z_1 + z_2|$ if $z_1 = 24 + 7i$ and $|z_2| = 6$ are respectively

(a) 31, 19 (b) 25, 19
(c) 31, 25 (d) none of these

Ans. (a)

Solution Note that $|z| = 6$ represents a circle. As $|z_2| = 6$,

$$|z_1 + z_2| = |z_2 - (-24 - 7i)|$$

represent distance between a point on the circle $|z| = 6$ and the point C $(-24 - 7i)$.

$|z_1 + z_2|$ will be greatest and least at points B and A which are the end points of the diameter of the circle through C. As $OC = 25$, $CA = OC - OA = 25 - 6 = 19$ and $CB = OC + OB = 25 + 6 = 31$.

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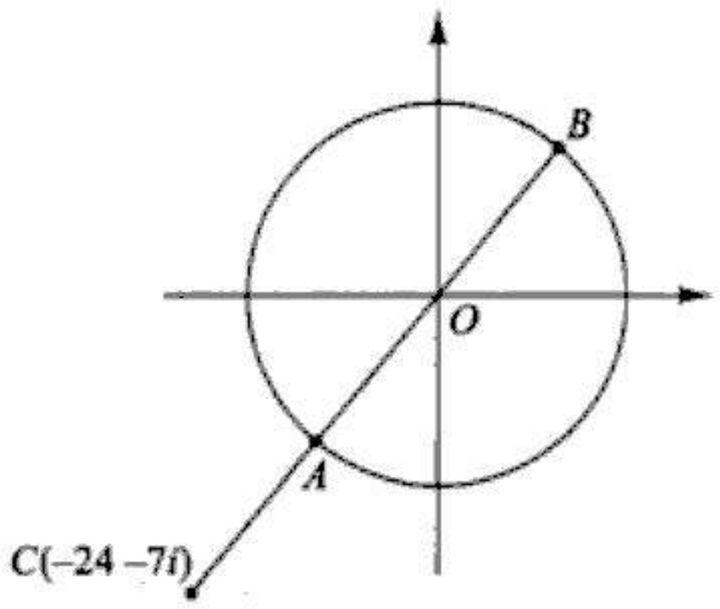

Fig. 1.44

Alternative Solution

$$|z_2| = 6 \Rightarrow z_2 = 6e^{i\theta} \text{ where } \theta \varepsilon \mathbf{R}.$$

$$\therefore \quad |z_1 + z_2|^2 = |24 + 7i + 6(\cos\theta + i\sin\theta)|^2$$
$$= (24 + 6\cos\theta)^2 + (7 + 6\sin\theta)^2$$
$$= 576 + 36\cos^2\theta + 288\cos\theta + 49 + 36\sin^2\theta + 84\sin\theta$$
$$= 625 + 36 + 12(24\cos\theta + 7\sin\theta)$$
$$= 661 + 12(25)\sin(\theta + \alpha)$$
$$[\text{put } 7 = r\cos\alpha \text{ and } 24 = r\sin\alpha]$$
$$= 661 + 300\sin(\theta + \alpha)$$

Thus, greatest possible value of $|z_1 + z_2|^2$ is $661 + 300 = 961$ and the least possible value of $|z_1 + z_2|^2$ is 361.

$\therefore$ greatest and least possible values of $|z_1 + z_2|$ are 31 and 19.

Example 13 If z_1 and z_2 are two complex numbers such that $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$, then

(a) $z_2 = kz_1,\ k \in \mathbf{R}$ (b) $z_2 = ikz_1,\ k \in \mathbf{R}$

(c) $z_1 = z_2$ (d) none of these

Ans. (b)

Solution Clearly both z_1 and z_2 cannot be 0. Suppose $z_2 \neq 0$. We can write $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$, as $\left|\frac{z_1/z_2 - 1}{z_1/z_2 + 1}\right| = 1$ or as $\left|\frac{z_1}{z_2} - 1\right| = \left|\frac{z_1}{z_2} + 1\right|$.

This shows that z_1/z_2 lies on the perpendicular bisector of the segment joining $A(-1 + i0)$ and $B(1 + i0)$ [See Theory].

Thus, z_1/z_2 lies on the imaginary axis.

$\therefore \quad z_1/z_2 = ia$ for some $a \in \mathbf{R}$.

$\Rightarrow \quad z_2/z_1 = 1/ia = -i/a$

$\Rightarrow \quad z_2 = ikz_1$ for some $k \in \mathbf{R}$

Alternative Solution

O(0), $A(z_1)$, $B(z_1 + z_2)$ and $c(z_2)$ are the vertices of a parallelogram.

Its diagonals $OB(z_1 + z_2)$ and $CA(z_1 - z_2)$ will be equal if and only if OABC is a rectangle, that is, z_2/z_1 is purely imaginary.

Example 14 For complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

(a) of area $\sqrt{3}$

(b) right-angled and isosceles

(c) equilateral

(d) obtuse-angled and isosceles

Ans. (c)

Solution We have

$$\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|\frac{1 - i\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4}(1+3)} = 1$$

$$\Rightarrow \quad |z_1 - z_3| = |z_2 - z_3|$$

Also, $\frac{z_1 - z_3}{z_2 - z_3} - 1 = \frac{1 - i\sqrt{3}}{2} - 1 \Rightarrow \frac{z_1 - z_2}{z_2 - z_3} = \frac{-1 - i\sqrt{3}}{2}$

$$\Rightarrow \quad \frac{|z_1 - z_2|}{|z_2 - z_3|} = \sqrt{\frac{1}{4}(1+3)} = 1$$

$$\Rightarrow \quad |z_1 - z_2| = |z_2 - z_3|$$

Thus, $|z_1 - z_3| = |z_2 - z_3| = |z_2 - z_1|$

Hence, z_1, z_2 and z_3 are the vertices of an equilateral triangle.

Example 15 If $a > 0$, and the equation $|z - a^2| + |z - 2a| = 3$ represents an ellipse, then a lies in

(a) (1, 3) (b) $(\sqrt{2}, \sqrt{3})$

(c) (0, 3) (d) $(1, \sqrt{3})$

Ans. (c)

Solution The equation $|z - a^2| + |z - 2a| = 3$ will represent an ellipse if

$$|a^2 - 2a| < 3 \qquad \text{[See Theory]}$$
$$\Leftrightarrow \quad -3 < a^2 - 2a < 3$$
$$\Leftrightarrow \quad -2 < (a-1)^2 < 4$$
$$\Leftrightarrow \quad (a-1)^2 < 4 \Leftrightarrow -2 < a - 1 < 2$$
$$\Leftrightarrow \quad -1 < a < 3$$
$$\Rightarrow \quad a \in (0, 3) \qquad [\because a > 0]$$

Example 16 If θ is real and z_1, z_2 are connected by $z_1^2 + z_2^2 + 2z_1 z_2 \cos\theta = 0$, then triangle with vertices 0, z_1 and z_2 is

(a) equilateral (b) right angled

(c) isosceles (d) none of these.

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Ans. (c)

Solution $z_1^2 + z_2^2 + 2z_1 z_2 \cos\theta = 0$

$$\Rightarrow \left(\frac{z_1}{z_2}\right)^2 + 2\left(\frac{z_1}{z_2}\right)\cos\theta + 1 = 0$$

$$\Rightarrow \left(\frac{z_1}{z_2} + \cos\theta\right)^2 = -(1 - \cos^2\theta) = -\sin^2\theta$$

$$\Rightarrow \frac{z_1}{z_2} = -\cos\theta \pm i\sin\theta$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \sqrt{(-\cos\theta)^2 + \sin^2\theta} = 1$$

$$\Rightarrow |z_1| = |z_2| \Rightarrow |z_1 - 0| = |z_2 - 0|$$

Thus, triangle with vertices 0, z_1 and z_2 are vertices of an isosceles triangle.

Example 17 If $\left|z - \frac{4}{z}\right| = 2$, then the greatest value of $|z|$ is

(a) $1 + \sqrt{2}$ (b) $2 + \sqrt{2}$
(c) $\sqrt{3} + 1$ (d) $\sqrt{5} + 1$

Ans. (d)

Solution We have $|z| = \left|z - \frac{4}{z} + \frac{4}{z}\right| \le \left|z - \frac{4}{z}\right| + \frac{4}{|z|}$

$$\Rightarrow |z| \le 2 + \frac{4}{|z|}$$

$$\Rightarrow |z|^2 \le 2|z| + 4 \Rightarrow (|z| - 1)^2 \le 5$$

$$\Rightarrow |z| - 1 \le \sqrt{5} \Rightarrow |z| \le \sqrt{5} + 1$$

Also, for $z = \sqrt{5} + 1$,

$$\left|z - \frac{4}{z}\right| = 2$$

Therefore, the greatest value of $|z|$ is $\sqrt{5} + 1$ which is attained when $z = \sqrt{5} + 1$

Example 18 Reflection of the line $\bar{a}z + a\bar{z} = 0$ in the real axis is

(a) $\bar{a}\bar{z} + az = 0$ (b) $\frac{\bar{a}}{a} = \frac{\bar{z}}{z}$
(c) $(a + \bar{a})(z + \bar{z}) = 0$ (d) none of these

Ans. (a)

Solution Let $a = \alpha + i\beta$ and $z = x + iy$, then $\bar{a}z + a\bar{z} = 0$ becomes $\alpha x + \beta y = 0$ or $y = \left(\frac{-\alpha}{\beta}\right)x$.

Its reflection in the x-axis is

$$y = \frac{\alpha}{\beta}x \quad \text{or} \quad \alpha x - \beta y = 0$$

or $$\left(\frac{a+\bar{a}}{2}\right)\left(\frac{z+\bar{z}}{2}\right) - \left(\frac{a-\bar{a}}{2i}\right)\left(\frac{z-\bar{z}}{2i}\right) = 0$$

or $$az + \bar{a}\bar{z} = 0$$

Example 19. If $\alpha, \beta, \gamma, \delta$ are four complex numbers such that $\frac{\gamma}{\delta}$ is real and $\alpha\delta - \beta\gamma \neq 0$, then $z = \frac{\alpha + \beta t}{\gamma + \delta t}$, $t \in \mathbf{R}$ represents a

(a) circle (b) parabola
(c) ellipse (d) straight line

Ans. (d)

Solution

Note that $\delta \neq 0$. Let $\frac{\gamma}{\delta} = k$, where $k \in \mathbf{R}$.

Two cases arise:

Case 1. $k = 0$.

In this case $\gamma = 0$ and

$$z = \frac{\alpha + \beta t}{\delta t} = \frac{\alpha}{\delta}t' + \frac{\beta}{\delta}, \text{ where } \frac{1}{t} = t'.$$

This shows that z lies on the line passing through β/δ and parallel to the complex number α/δ.

Case 2. $k \neq 0$.

In this case $\gamma \neq 0$ and

$$z = \frac{\alpha + \beta t}{k\delta + \delta t} = \frac{\alpha}{k\delta}\left(\frac{k}{k+t}\right) + \frac{\beta}{\delta}\left(\frac{t}{k+t}\right)$$

$$= \frac{\alpha}{\gamma}\left(\frac{k}{k+t}\right) + \frac{\beta}{\delta}\left(\frac{t}{k+t}\right)$$

This is a straight line passing through α/γ and β/δ.

Note that the condition $\alpha\delta - \beta\gamma \neq 0$ is necessary otherwise.

$$z = \frac{\alpha + \beta t}{\gamma + \delta t} = \frac{\beta}{\delta}\frac{\alpha/\beta + t}{\gamma/\delta + t} = \frac{\beta}{\delta},$$

and we merely get a single point in the Argand plane.

Example 20 If $n > 1$, the roots of $(z + 1)^n = z^n$ lie on a

(a) circle (b) ellipse
(c) hyperbola (d) straight line.

Ans. (d)

Solution Let z be complex number satisfying the equation $(z + 1)^n = z^n$, then

$$|(z + 1)^n| = |z^n| \Rightarrow |z + 1|^n = |z|^n \Rightarrow |z + 1| = |z|$$

This shows that z lies on the perpendicular bisector of the segment joining $A(-1 + 0i)$ and $B(0 + i0)$. That is, z lies on the line $z = -1/2$.

TIP It is unnecessary to find roots of $(z + 1)^n = z^n$

Example 21 If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to

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(a) $-\pi$ (b) $-\pi/2$ (c) $\pi/2$ (d) 0

Ans. (d)

Solution Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ where $r_1 = |z_1|$, $r_2 = |z_2|$, $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.

We have

$$|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$$

$$= (r_1 + r_2)^2 + 2r_1r_2\{\cos(\theta_1 - \theta_2) - 1\}$$

Now, $|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow \cos(\theta_1 - \theta_2) = 1 \Leftrightarrow \theta_1 - \theta_2 = 0 \Leftrightarrow \theta_1 = \theta_2$.

Alternatively, $|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow$ z_1/z_2 is a non-negative real number

$\Leftrightarrow$ $\arg(z_1) = \arg(z_2)$

Example 22 If $A(z_1)$, $B(z_2)$ and $C(z_3)$ are the vertices of a triangle such that $|z_1| = |z_2| = |z_3| > 0$, then orthocentre of ΔABC is

(a) $z_1 + z_2 + z_3$ (b) $z_1\, z_2\, z_3$
(c) 0 (d) none of these

Ans. (a)

Solution As $|z_1| = |z_2| = |z_3|$, circumcentre of ΔABC is $z_S = 0$. Also, centroid of ΔABC is $z_G = \frac{1}{3}(z_1 + z_2 + z_3)$

If z_H denotes the orthocentre of ΔABC, then

$$z_G = \frac{1}{3}(z_H + 2z_S)$$

$$\Rightarrow \frac{1}{3}(z_1 + z_2 + z_3) = \frac{1}{3}(z_H + 2(0))$$

$$\Rightarrow z_H = z_1 + z_2 + z_3$$

Example 23 If $|z - 25i| \le 15$, then
|maximum arg (z) – minimum arg (z)| equals

(a) $2\cos^{-1}(3/5)$ (b) $2\cos^{-1}(4/5)$
(c) $\pi/2 + \cos^{-1}(3/5)$ (d) $\sin^{-1}(3/5) - \cos^{-1}(3/5)$

Ans. (b)

Solution If $|z - 25i| \le 15$, then z lies either in the interior and or on the boundary of the circle with centre at $C(25i)$ and radius equal to 15.

From the Fig. 1.45, it is clear that argument is least for point A and argument is greatest for point B.

From right ΔOAC, $\cos(\frac{\pi}{2} - \theta) = \frac{OA}{OC} = \frac{20}{25}$

$$\Rightarrow \quad \pi/2 - \theta = \cos^{-1}(4/5)$$

Now, for $|z - 25i| \le 15$

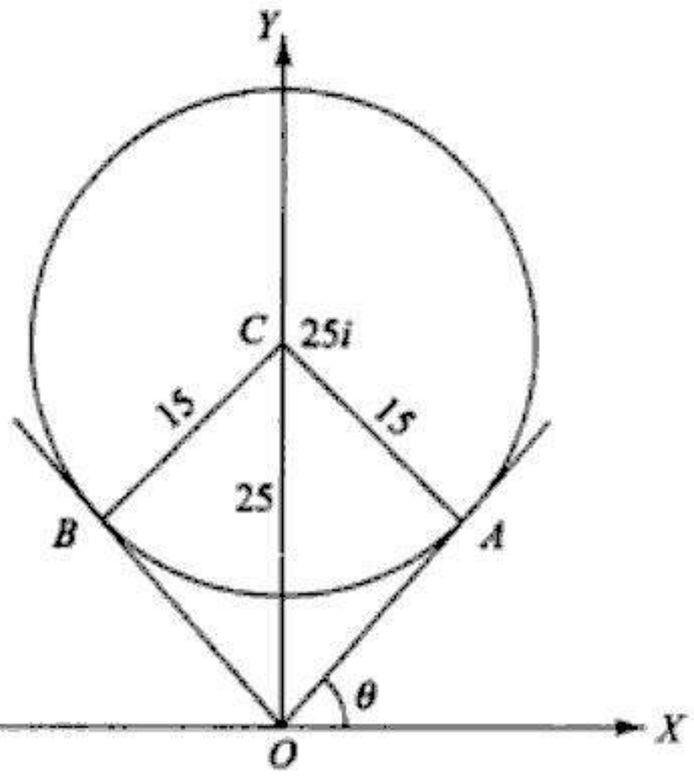

Fig. 1.45

|maximum (arg z) – minimum (arg z)|

$$= |\arg(B) - \arg(A)| = \angle BOA = 2\left(\frac{\pi}{2} - \theta\right)$$

$$= 2\cos^{-1}(4/5)$$

Example 24 If $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then $z_1^2 + z_2^2 + z_3^2$ equals

(a) -3 (b) $-1/3$
(c) 0 (d) 3

Ans. (c)

Solution $0 = (z_1 + z_2 + z_3)^2$

$$= z_1^2 + z_2^2 + z_3^2 + 2(z_2z_3 + z_3z_1 + z_1z_2)$$

But $z_2z_3 + z_3z_1 + z_1z_2 = z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$

$$= z_1z_2z_3\,(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) = 0$$

Thus, $z_1^2 + z_2^2 + z_3^2 = 0$

Example 25 If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, then area of the triangle whose vertices are z_1, z_2, z_3 is

(a) $3\sqrt{3}/4$ (b) $\sqrt{3}/4$ (c) 1 (d) 2

Ans. (a)

Solution $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

$\Rightarrow \quad |-z_3|^2 + |z_1 - z_2|^2 = 2(1 + 1)$

$$\Rightarrow \quad |z_1 - z_2|^2 = 3 \quad \Rightarrow \quad |z_1 - z_2| = \sqrt{3}$$

Similarly, $|z_2 - z_3| = |z_3 - z_1| = \sqrt{3}$

Thus, Δ whose vertices are z_1, z_2 and z_3 is equilateral and its side is of length $\sqrt{3}$.

Therefore, area of triangle

$$= \frac{\sqrt{3}}{4}(\sqrt{3})^3 = \frac{3\sqrt{3}}{4}$$

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Example 26 Shaded region is given by

(a) $|z + 2| \geq 6,\ 0 \leq \arg(z) \leq \frac{\pi}{6}$

(b) $|z + 2| \geq 6,\ 0 \leq \arg(z) \leq \frac{\pi}{3}$

(c) $|z + 2| \leq 6,\ 0 \leq \arg(z) \leq \frac{\pi}{2}$

(d) none of these.

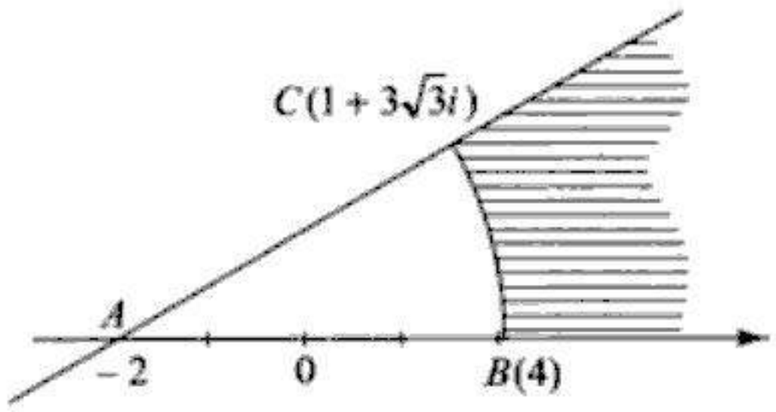

Fig. 1.46

Ans. (b)

Solution Note that $AB = 6$ and

$$1 + 3\sqrt{3}\mathrm{i} = -2 + 3 + 3\sqrt{3}\mathrm{i}$$

$$= -2 + 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= -2 + 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$\therefore \quad \angle BAC = \pi/3$

Thus, shaded region is given by

$$|z + 2| \geq 6 \quad \text{and} \quad 0 \leq \arg z \leq \pi/3$$

Example 27 If $-\pi < \arg(z) < -\frac{\pi}{2}$, then $\arg \bar{z} - \arg(-\bar{z})$ is

(a) π (b) $-\pi$ (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

Ans. (a)

Solution Let $\arg(z) = \theta$, then $\arg \bar{z} = -\theta$ and $\arg(-\bar{z}) = -\pi - \theta$. See Fig. 1.47

Thus, $\arg(\bar{z}) - \arg(-\bar{z})$

$$= -\theta - (-\pi - \theta) = \pi$$

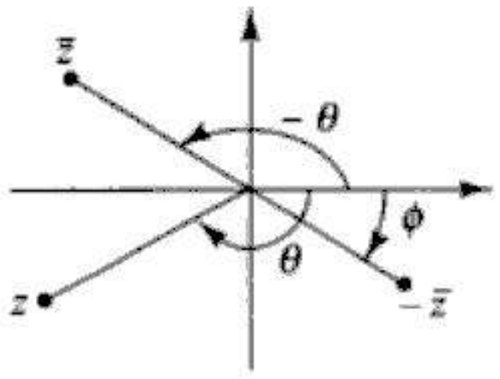

Fig. 1.47

Example 28 If $k > 1$, $|z_1| < k$ and $\left|\frac{k - z_1\bar{z}_2}{z_1 - k z_2}\right| = 1$, then

(a) $|z_2| < k$ (b) $|z_2| = k$
(c) $z_2 = 0$ (d) $|z_2| = 1$

Ans. (d)

$$\left|\frac{k - z_1\bar{z}_2}{z_1 - k z_2}\right| = 1$$

$$\Leftrightarrow \quad |k - z_1\ \bar{z}_2|^2 = |z_1 - k\ z_2|^2$$

$$\Leftrightarrow \quad k^2 + |z_1\bar{z}_2|^2 - kz_1\bar{z}_2 - k\bar{z}_1z_2$$

$$= |z_1|^2 + k^2|z_2|^2 - kz_1\bar{z}_2 - k\bar{z}_1z_2$$

$$\Rightarrow \quad |z_1|^2\,|z_2|^2 - |z_1|^2 - k^2|z_2|^2 + k^2 = 0$$

$$\Rightarrow \quad (|z_1|^2 - k^2)\,(|z_2|^2 - 1) = 0$$

As $|z_1| < k$, we get $|z_2|^2 - 1 = 0$ or $|z_2| = 1$

Example 29 If $k > 0$, $|z| = |w| = k$, and $\alpha = \frac{z - \bar{w}}{k^2 + z\bar{w}}$,

then Re (α) equals

(a) 0 (b) $k/2$
(c) k (d) none of these

Ans. (a)

Solution $\bar{\alpha} = \frac{\bar{z} - w}{k^2 + \bar{z}w}$

But $z\bar{z} = w\bar{w} = k^2$, thus,

$$\bar{\alpha} = \frac{\frac{k^2}{z} - \frac{k^2}{\bar{w}}}{k^2 + \frac{k^2}{z}\frac{k^2}{\bar{w}}} = \frac{\bar{w} - z}{z\bar{w} + k^2} = -\alpha$$

$$\Rightarrow \quad \alpha + \bar{\alpha} = 0 \qquad \Rightarrow \mathrm{Re}(\alpha) = 0$$

Example 30 Roots of the equations are

$$(z + 1)^5 = (z - 1)^5 \qquad (1)$$

are

(a) $\pm i \tan\left(\frac{\pi}{5}\right), \pm i \tan\left(\frac{2\pi}{5}\right)$

(b) $\pm i \cot\left(\frac{\pi}{5}\right), \pm i \cot\left(\frac{2\pi}{5}\right)$

(c) $\pm i \cot\left(\frac{\pi}{5}\right), \pm i \tan\left(\frac{2\pi}{5}\right)$

(d) none of these

Ans. (b)

Solution For $z \neq 1$, (1) can be written as

$$\left(\frac{z+1}{z-1}\right)^5 = 1 = e^{2k\pi i},\ k \in \mathbf{I}$$

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$$\Rightarrow \qquad \frac{z+1}{z-1} = e^{2k\pi i/5},$$

where $k = -2, -1, 1, 2$

[For $k = 0$, we get $1 = -1$.]

If we denote this value of z by z_k, then

$$z_k = \frac{e^{2k\pi i/5}+1}{e^{2k\pi i/5}-1} = \frac{e^{k\pi i/5}+e^{-k\pi i/5}}{e^{k\pi i/5}-e^{-k\pi i/5}}$$

$$= -i \cot\left(\frac{k\pi}{5}\right),\ k = -2, -1, 1, 2$$

$\therefore$ Roots of (1) are

$$\pm i \cot\left(\frac{\pi}{5}\right),\ \pm i \cot\left(\frac{2\pi}{5}\right)$$

Example 31 If $z \in \mathbf{C} - \mathbf{R}$ then $\frac{1+z+z^2}{1-z+z^2} \in \mathbf{R}$, is equivalent to say

(a) $|z| < 1/3$ (b) $|z| = 1$

(c) $|z| = 2$ (d) $|z| > 3$

Ans. (b)

Solution As

$$1 + \frac{2z}{1-z+z^2} = \frac{1+z+z^2}{1-z+z^2} \in \mathbf{R}$$

$$\Leftrightarrow \qquad \frac{z}{1-z+z^2} \in \mathbf{R} \Leftrightarrow \frac{1-z+z^2}{z} \in \mathbf{R}$$

$$\Leftrightarrow \qquad \frac{1}{z} + z - 1 \in \mathbf{R} \Leftrightarrow z + \frac{1}{z} \Leftrightarrow \mathbf{R}$$

$$\Leftrightarrow \qquad z + \frac{1}{z} = \bar{z} + \frac{1}{\bar{z}}$$

$$\Leftrightarrow \qquad z - \bar{z} - \left(\frac{1}{z} - \frac{1}{\bar{z}}\right) = 0$$

$$\Leftrightarrow \qquad (z - \bar{z})\left(1 - \frac{1}{\bar{z}\,z}\right) = 0$$

$$\Leftrightarrow \qquad z\bar{z} = 1 \text{ or } |z| = 1.$$

$[\because\ z \neq \bar{z}\,]$

Example 32 Let z_1, z_2, z_3 be three distinct complex numbers lying on a circle with centre at the origin such that $z_1 + z_2z_3$, $z_2 + z_3z_1$ and $z_3 + z_1z_2$ are real numbers, then $z_1z_2z_3$ equals

(a) -1 (b) 0

(c) 1 (d) none of these

Ans. (c)

Solution As z_1, z_2, z_3 lie on a circle with centre at the origin,

$$|z_1| = |z_2| = |z_3| = r(\text{say}).$$

As $z_1 + z_2z_3 \in \mathbf{R}$, $z_1 + z_2z_3 = \bar{z}_1 + \bar{z}_2\,\bar{z}_3$

$$= \frac{r^2}{z_1} + \frac{r^4}{z_2\,z_3}$$

$$= \frac{r^2(z_2\,z_3 + r^2 z_1)}{z_1 z_2 z_3}$$

$$\Rightarrow \qquad \frac{r^2}{z_1z_2z_3} = \frac{z_1 + z_2\,z_3}{z_2z_3 + r^2z_1} \qquad (1)$$

Similarly, $$\frac{r^2}{z_1z_2z_3} = \frac{z_2 + z_3\,z_1}{z_3z_1 + r^2z_2} \qquad (2)$$

and $$\frac{r^2}{z_1z_2z_3} = \frac{z_3 + z_1\,z_2}{z_1z_2 + r^2z_3} \qquad (3)$$

From (1), (2) and (3), we get

$$\frac{r^2}{z_1z_2z_3} = \frac{z_1 + z_2\,z_3}{z_2z_3 + r^2z_1} = \frac{z_2 + z_3\,z_1}{z_3z_1 + r^2z_2} = \frac{z_3 + z_1\,z_2}{z_1z_2 + r^2z_3}$$

$$= \frac{z_1 + z_3\,z_3 - (z_2 + z_3\,z_1)}{z_2z_3 + r^2z_1 - (z_3 + z_1r^2z_2)}$$

$$= \frac{(z_1 - z_2)(z_3 - 1)}{(z_1 - z_2)(z_3 - r^2)} = \frac{z_3 - 1}{z_3 - r^2}$$

$\therefore$ $$\frac{r^2}{z_1z_2z_3} = \frac{z_3 - 1}{z_3 - r^2} = \frac{z_2 - 1}{z_2 - r^2} = \frac{z_1 - 1}{z_1 - r^2}$$

$$= \frac{(z_2 - 1) - (z_1 - 1)}{(z_2 - r^2) - (z_1 - r^2)}$$

$$= \frac{z_2 - z_1}{z_2 - z_1} = 1$$

$$\Rightarrow \qquad z_3 - 1 = z_3 - r^2 \Rightarrow r^2 = 1.$$

Hence, $z_1\,z_2\,z_3 = 1$

Example 33 If the points $z, -iz$ and 1 are collinear then z lies on

(a) a straight line (b) a circle

(c) an ellipse (d) a pair of straight lines.

Ans. (b)

Solution As $z, -iz$ and 1 are collinear

$$0 = \begin{vmatrix} z & \bar{z} & 1 \\ -iz & i\bar{z} & 1 \\ 1 & 1 & 1 \end{vmatrix} = (-i)\begin{vmatrix} z & \bar{z} & 1 \\ z & -\bar{z} & i \\ 1 & 1 & 1 \end{vmatrix}$$

[taking $-i$ common from R_2]

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$$\Rightarrow \quad \begin{vmatrix} z-1 & \bar{z}-1 & 1 \\ z-i & -\bar{z}-i & i \\ 0 & 0 & 1 \end{vmatrix} = 0$$

[using $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$]

$$\Rightarrow \quad (z-1)(\bar{z}+i) + (z-i)(\bar{z}-1) = 0$$

$$\Rightarrow \quad z\bar{z} - \bar{z} + iz - i + z\bar{z} - i\bar{z} - z + i = 0$$

$$\Rightarrow \quad 2z\bar{z} - (1-i)z - (1+i)\bar{z} = 0$$

$$\Rightarrow \quad z\bar{z} - \frac{1-i}{2}z - \frac{1+i}{2}\bar{z} = 0$$

$$\Rightarrow \quad \left|z - \frac{1+i}{2}\right| = \frac{1}{\sqrt{2}}$$

Thus, z lies on a circle.

Example 34 If $|z - i| = 1$ and $\arg(z) = \theta$ where $\theta \in (0, \pi/2)$, then

$$\cot\theta - \frac{2}{z}$$

equals

(a) $2i$ (b) $-i$ (c) i (d) $1+i$

Ans. (c)

Solution $|z-i| = 1$ represents a circle with centre at i and radius 1. As $0 < \theta < \pi/2$, θ lies on the semi-circle in the first quadrant. See Fig. 1.48.

As $\angle OPA = \pi/2$ and $\angle POA = \pi/2 - \theta$, we get

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{OP}{OA}$$

$$\Rightarrow \quad \sin\theta = \frac{|z|}{2}$$

Also, since $\angle POA = \pi/2 - \theta$

$$\frac{2i}{z} = \frac{OP}{OA}\left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]$$

$$= \frac{1}{\sin\theta}[\sin\theta + i\cos\theta] = 1 + i\cot\theta$$

$$\Rightarrow \quad -\frac{2}{z} = i - \cot\theta \Rightarrow \cot\theta - \frac{2}{z} = i$$

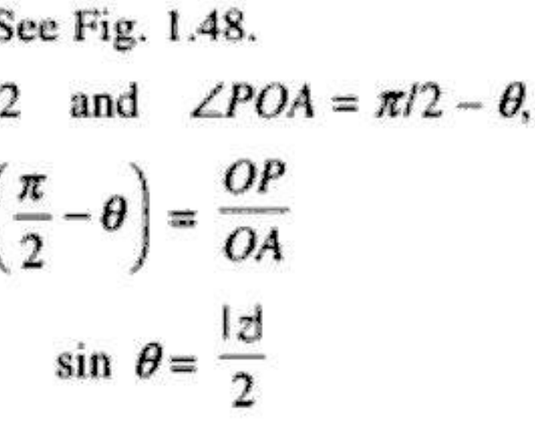

Fig. 1.48

Example 35 If $|z_1| = 2, |z_2| = 3, |z_3| = 4$ and $|2z_1 + 3z_2 + 4z_3| = 4$, then absolute value of $8z_2z_3 + 27z_3z_1 + 64z_1z_2$ equals

(a) 24 (b) 48 (c) 72 (d) 96

Ans. (d)

Solution $|8z_2z_3 + 27z_3z_1 + 64z_1z_2|$

$$= |z_1|\,|z_2|\,|z_3|\left|\frac{8}{z_1} + \frac{27}{z_2} + \frac{64}{z_3}\right|$$

$$= (2)(3)(4)\left|\frac{8\bar{z}_1}{|z_1|^2} + \frac{27\bar{z}_2}{|z_2|^2} + \frac{64\bar{z}_3}{|z_3|^2}\right|$$

$$= 24\left|2\bar{z}_1 + 3\bar{z}_2 + 4\bar{z}_3\right|$$

$$= 24\left|\overline{2z_1 + 3z_2 + 4z_3}\right|$$

$$= 24\,|2z_1 + 3z_2 + 4z_3|$$

$$= (24)(4) = 96$$

Example 36 If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = \sqrt{2+i}$, then the complex number $z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2$ is

(a) purely imaginary
(b) purely real
(c) positive real number
(d) none of these

Ans. (a)

Solution $\left|\sqrt{2} + i\right|^2 = |z_1 + z_2 + z_3|^2$

$$\Rightarrow \quad 3 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\,Re(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2)$$

$$\Rightarrow \quad Re(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2) = 0$$

Example 37 If a, b, c are distinct integers and $\omega \neq 1$ is a cube root of unity then minimum value of

$$x = |a + b\omega + c\omega^2| + |a + b\omega^2 + c\omega|$$

(a) $6\sqrt{2}$ (b) 3
(c) $4\sqrt{2}$ (d) 2

Ans. (a)

Solution Let $z = a + b\omega + c\omega^2$

Then

$$|z|^2 = z\bar{z} = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

$$= a^2 + b^2 + c^2 - bc - ca - ab$$

$$= \frac{1}{2}[(b-c)^2 + (c-a)^2 + (a-b)^2]$$

Thus,

$$x = |z| + |\bar{z}| = 2|z|$$

$$= \sqrt{2}\,[(b-c)^2 + (c-a)^2 + (a-b)^2]$$

As a, b, c are distinct integers, minimum value of $(b-c)^2 + (c-a)^2 + (a-b)^2$ is $1^2 + 1^2 + 2^2 = 6$.

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Therefore minimum value of x is $6\sqrt{2}$.

Example 38 Let $r_k > 0$ and $z_k = r_k(\cos\alpha_k + i\sin\alpha_k)$ for $k = 1, 2, 3$ be such that

$$\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3} = 0$$

Let A_k be the point in the complex plane given by $w_k = \dfrac{\cos 2\alpha_k + i\sin 2\alpha_k}{z_k}$ for $k = 1, 2, 3$. The origin, O is the

(a) incentre of $\Delta A_1A_2A_3$
(b) orthocentre of $\Delta A_1A_2A_3$
(c) circumcentre of $\Delta A_1A_2A_3$
(d) centroid of $\Delta A_1A_2A_3$

Ans. (d)

Solution $$w_k = \frac{(\cos\alpha_k + i\sin\alpha_k)^2}{z_k} = \frac{z_k^2}{r_k^2 z_k}$$

$$= \frac{z_k^2}{z_k\bar{z}_k z_k} = \frac{1}{\bar{z}_k}$$

We have

$$w_1 + w_2 + w_3 = \frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}$$

$$= \overline{\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)} = 0$$

$\therefore$ The origin O is the centroid of $\Delta A_1A_2A_3$.

Example 39 If a and b are two real numbers and z_1 and z_2 are two non zero complex numbers such that $a|z_1| = b|z_2|$, then $z = \dfrac{az_1}{bz_2}+\dfrac{bz_2}{az_1}$ is

(a) purely real
(b) purely imaginary
(c) $|z| = a/b$
(d) none of these

Ans. (a)

Solution Let $w = \dfrac{az_1}{bz_2}$, then $|w| = \dfrac{a|z_1|}{b|z_2|} = 1$

$\Rightarrow$ $w = e^{i\theta}$ for some θ, $0 \le \theta < 2\pi$

Now, $$z = w + \frac{1}{w} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$\Rightarrow$ z is purely real.

Example 40 z_1 and z_2 lie on a circle with centre at the origin. The point of intersection z_3 of the tangents at z_1 and z_2 is given by

(a) $\dfrac{1}{2}(\bar{z}_1 + \bar{z}_2)$
(b) $\dfrac{2z_1z_2}{z_1+z_2}$
(c) $\dfrac{1}{2}\left(\dfrac{1}{z_1}+\dfrac{1}{z_2}\right)$
(d) $\dfrac{z_1+z_2}{\bar{z}_1\bar{z}_2}$

Ans. (b)

Solution Refer Fig. 1.49. As ΔOAC is a right triangle with right angle at A, $|z_1|^2 + |z_3 - z_1|^2 = |z_3|^2$

$\Rightarrow$ $$2|z_1|^2 - \bar{z}_3z_1 - \bar{z}_1z_3 = 0$$

$\Rightarrow$ $$2\bar{z}_1 - \bar{z}_3 - \frac{\bar{z}_1}{z_1}z_3 = 0 \qquad (1)$$

Similarly, $$2\bar{z}_2 - \bar{z}_3 - \frac{\bar{z}_2}{z_2}z_3 = 0 \qquad (2)$$

Subtracting (2) from (1) we get

$$2(\bar{z}_2 - \bar{z}_1) = z_3\left(\frac{\bar{z}_1}{z_1}-\frac{\bar{z}_2}{z_2}\right)$$

$\Rightarrow$ $$\frac{2r^2(z_1 - z_2)}{z_1z_2} = z_3r^2\left(\frac{z_2^2 - z_1^2}{z_1^2z_2^2}\right)$$

$[\because |z_1|^2 = |z_2|^2 = r^2]$

$\Rightarrow$ $$z_3 = \frac{2z_1z_2}{z_2+z_1}$$

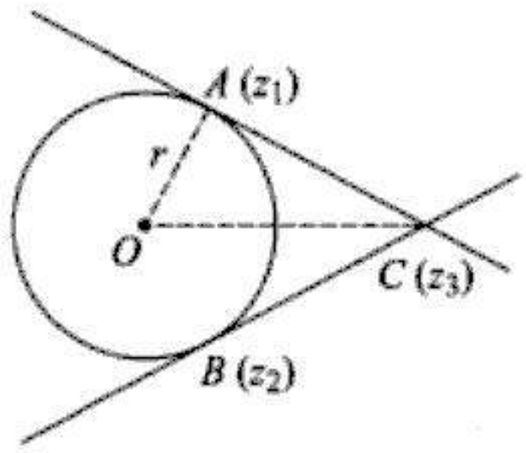

Fig. 1.49

Example 41 Let $z = \cos\theta + i\sin\theta$. Then value of

$$\sum_{m=1}^{15}\text{Im}\,(z^{2m-1})$$

at $\theta = 3°$ is

(a) -1
(b) 0
(c) $\dfrac{1}{2\sin 3°}$
(d) $\dfrac{-1}{2\sin 3°}$

Ans. (c)

Solution $$\sum_{m=1}^{15}\text{Im}\,(z^{2m-1}) = \text{Im}\left(\sum_{m=1}^{15}z^{2m-1}\right)$$

$$= \text{Im}\left[\frac{z(1-(z^2)^{15})}{1-z^2}\right]$$

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$$= \text{Im}\left[\frac{1-z^{30}}{\bar{z}-z}\right]$$

$$= \text{Im}\left[\frac{-1}{2i\sin\theta}\{1-\cos 30\theta - i\sin 30\theta\}\right]$$

$$= \frac{1}{2\sin\theta} = \frac{1}{2\sin 3^\circ}$$

Example 42 Let $z = x + iy$ be a complex numbers where x and y are integers. Then area of the rectangle whose vertices are roots of

$$z\bar{z}^3 + \bar{z}z^3 = 738$$

is

(a) 80 (b) 48
(c) 56 (d) 96

Ans. (a)

Solution $z\bar{z}^3 + \bar{z}z^3 = 738$

$\Rightarrow$ $z\bar{z}(\bar{z}^2 + z^2) = 738$

$\Rightarrow$ $(x^2 + y^2)[(x - iy)^2 + (x + iy)^2] = 738$

$\Rightarrow$ $(x^2 + y^2)(2x^2 - 2y^2) = 738$

$\Rightarrow$ $x^4 - y^4 = 369 \Rightarrow x^4 \geq 369 \Rightarrow |x| \geq 4$

Let us try $x = \pm 5$

$\Rightarrow$ $625 - 369 = y^4 \Rightarrow y = \pm 4$

Thus, roots of $z\bar{z}^3 + \bar{z}z^3 = 738$ are $\pm 5 \pm 4i$.

Area of the rectangle $ABCD$ whose vertices are $\pm 5 + 4i$ is $(10)(8) = 80$

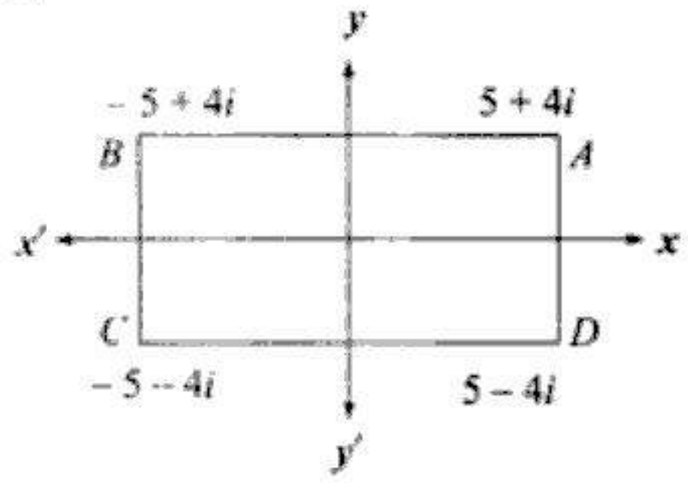

Fig. 1.50

Example 43 A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then position of P in the argand plane is

(a) $3e^{i\pi/4} + 4i$ (b) $(3 - 4i)e^{i\pi/4}$
(c) $(4 + 3i)e^{i\pi/4}$ (d) $(3 + 4i)e^{i\pi/4}$

Ans. (d)

Solution Suppose the man reaches A after walking 3 units in N 45° E and B after walking a distance of 4 units in N 45° W. Position of A in the Argand diagram is $3e^{i\pi/4}$. Let position of B be z. Since $\angle OAB = \pi/2$, we have

$$\arg\left(\frac{0 - 3e^{i\pi/4}}{z - 3e^{i\pi/4}}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{-3e^{i\pi/4}}{z - 3e^{i\pi/4}} = \frac{OA}{OB}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$\Rightarrow \quad \frac{-3e^{i\pi/4}}{z - 3e^{i\pi/4}} = \frac{3i}{4}$$

$$\Rightarrow \quad z - 3e^{i\pi/4} = 4ie^{i\pi/4}$$

$$\Rightarrow \quad z = (3 + 4i)e^{i\pi/4}$$

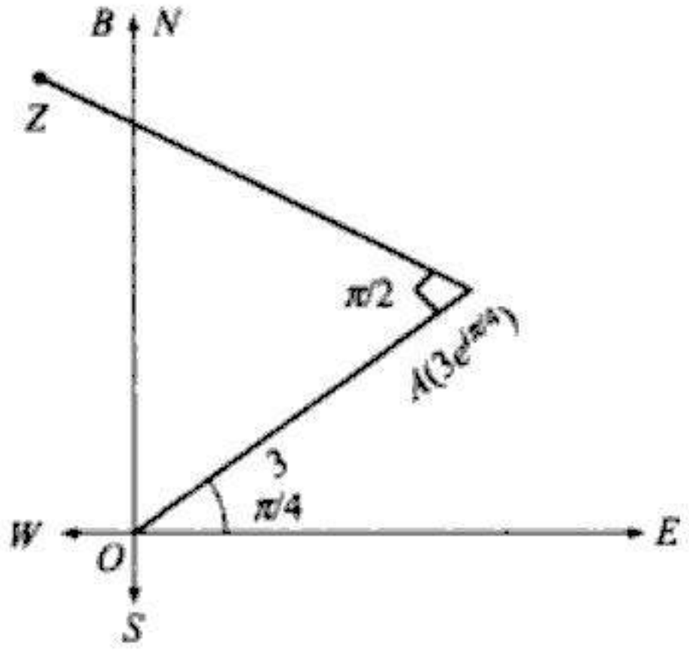

Fig. 1.51

Example 44 A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from the origin by 5 units and then vertically away from the origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of $\hat{i} + \hat{j}$ and then it moves through an angle $\pi/2$ in the anticlockwise direction on a circle with centre at origin, to reach point z_2. The point z_2 is given by

(a) $6 + 7i$ (b) $-7 + 6i$
(c) $7 + 6i$ (d) $-6 + 7i$

Ans. (d)

Solution

Affix of B is $(1 + 2i) + 5 = 6 + 2i$

Affix of C is $(6 + 2i) + 3i = 6 + 5i$

Affix of D is $(6 + 5i) + \sqrt{2}\left(\frac{1+i}{\sqrt{2}}\right) = 7 + 6i$

Affix of E is $(7 + 6i)e^{\pi i/2} = (7 + 6i)i = -6 + 7i$

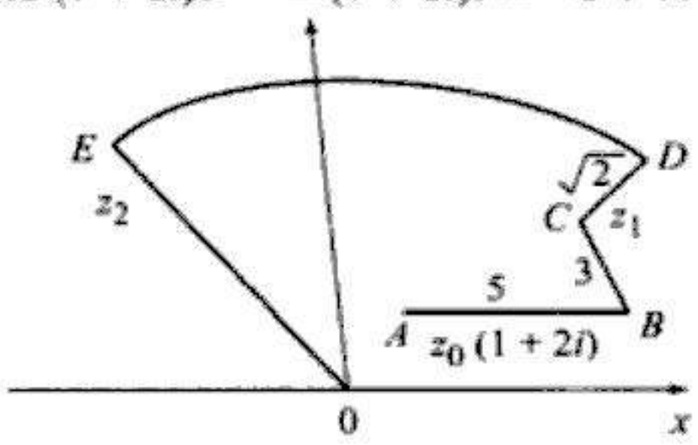

Fig. 1.52

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Example 45 Let a, b, c be three points lying on the circle $|z| = 1$ and suppose $\alpha \in (0, \pi/2)$ be such that $a + b\cos\alpha + c\sin\alpha = 0$, then

(a) $b = \pm ic$ (b) $2a^2 + b^2 + c^2 = 0$

(c) $a^2 + b^2 + c^2 = 0$ (d) none of these

Ans. (a)

Solution $1 = |-a|^2 = |b\cos\alpha + c\sin\alpha|^2$

$$= |b|^2\cos^2\alpha + |c|^2\sin^2\alpha + \left(b\bar{c} + \bar{b}c\right)\cos\alpha\sin\alpha$$

$$= 1 + \left(\frac{b}{c} + \frac{c}{b}\right)\cos\alpha\sin\alpha$$

$$1 = 1 + \frac{b^2 + c^2}{2bc}(\sin 2\alpha)$$

$$\Rightarrow \frac{b^2 + c^2}{2bc}(\sin 2\alpha) = 0 \Rightarrow b^2 + c^2 = 0$$

$$\Rightarrow \qquad b = \pm ic$$

SECTION II

MULTIPLE CORRECT CHOICE TYPE

Example 46 If $S = \sum_{k=1}^{10}\left(\sin\frac{2\pi k}{11} - i\cos\frac{2\pi k}{11}\right)$ then

(a) $S + \bar{S} = 0$ (b) $S\bar{S} = 1$

(c) $\sqrt{S} = \frac{1}{\sqrt{2}}(1 \pm i)$ (d) $S - \bar{S} = 0$

Ans. (a), (b), (c)

Solution Put $\omega = \cos\frac{2\pi}{11} + i\sin\frac{2\pi}{11}$,

so that for $1 \le k \le 10$

$$\sin\frac{2\pi k}{11} - i\cos\frac{2\pi k}{11}$$

$$= -i\left(\cos\frac{2\pi k}{11} + i\sin\frac{2\pi k}{11}\right)$$

$$= -i\omega^k \qquad \text{[De Moivre's theorem]}$$

thus,

$$S = -i\sum_{k=1}^{10}\omega^k = \frac{i\omega(1-\omega^{10})}{1-\omega} = \frac{i(\omega-\omega^{11})}{1-\omega}$$

But $\omega^{11} = \cos 2\pi + i\sin 2\pi = 1 + i0 = 1$

$\therefore \qquad S = i$

$\Rightarrow \qquad S + \bar{S} = 0,\ S\bar{S} = 1$

and $\sqrt{S} = \frac{1}{\sqrt{2}}(1 \pm i)$

Example 47 If $|z_1| = |z_2| = |z_3| = 1$ and

$$S = |z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$$

Let α = minimum value of S

and β = maximum value of S

as z_1, z_2, z_3 vary on $|z| = 1$, then

(a) $\alpha = 0$ (b) $\beta = 9$

(c) $\beta = 3$ (d) $\alpha = 2$

Ans. (a), (b)

Solution Note that $S \ge 0$ and $S = 0$ for $z_1 = z_2 = z_3$

Thus, $\alpha = 0$

Since

$|a - b|^2 = |a|^2 + |b|^2 - \bar{a}b - a\bar{b}$, we get

$|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$

$= 6 - (z_2\bar{z}_3 + \bar{z}_2z_3 + z_3\bar{z}_1 + \bar{z}_3z_1 + z_1\bar{z}_2 + \bar{z}_1z_2)$

$= 6 - \bar{z}_1(z_2 + z_3) - \bar{z}_2(z_3 + z_1) - \bar{z}_3(z_1 + z_2)$ (1)

Also $0 \le |z_1 + z_2 + z_3|^2$

$= |z_1|^2 + |z_2|^2 + |z_3|^2 + \bar{z}_1(z_2 + z_3)$

$+ \bar{z}_2(z_3 + z_1) + \bar{z}_3(z_1 + z_2)$

$\Rightarrow \bar{z}_1(z_2 + z_3) + \bar{z}_2(z_3 + z_1) + \bar{z}_3(z_1 + z_2) \ge -3$ (2)

Thus, from (1) and (2), we get

$$|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2 \le 9$$

The value 9 is attained when

$$z_1 = 1 \qquad z_2 = \omega \qquad z_3 = \omega^2$$

where $\omega \ne 1$ is a cube root of unity.

Example 48 Suppose $z_1 + z_2 + z_3 + z_4 = 0$ and $|z_1| = |z_2| = |z_3| = |z_4| = 1$. If $z_1, z_2, z_3\ z_4$ are the vertices of a quadrilateral, then the quadrilateral must be a

(a) parallelogram (b) rhombus

(c) rectangle (d) square

Ans. (a), (c)

Solution $z_1 + z_2 + z_3 + z_4 = 0$ (1)

$\Rightarrow \quad \bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \bar{z}_4 = 0$

$\Rightarrow \quad \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} = 0$

$\Rightarrow \quad z_2z_3z_4 + z_1z_3z_4 + z_1z_2z_4 + z_1z_2z_3 = 0$ (2)

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Also, $|z_1 z_2 z_3 z_4| = 1$

The equation whose roots are z_1, z_2, z_3 and z_4 is

$$(z - z_1)(z - z_2)(z - z_3)(z - z_4) = 0 \qquad (3)$$

or $z^4 + bz^2 + d = 0$ [use (1) and (2)]

where $b = (z_1 + z_2)(z_3 + z_4) + z_1 z_2 + z_3 z_4$

and $d = z_1 z_2 z_3 z_4$

Clearly, if α is a root of (4), then $-\alpha$ is a root of (4).

Thus, root of (3) are of the form $\pm \alpha, \pm \beta$.

If $z_1 + z_3 = 0$ then $z_2 + z_4 = 0$

$\Rightarrow z_1, z_2, z_3, z_4$ are vertices of a parallelogram.

(Similarly for other pairs)

Also, $|z_1 - z_3|^2 = |z_1|^2 + |z_3|^2 - z_1 \bar{z}_3 - \bar{z}_1 z_3$

$= 2 - z_1(-\bar{z}_1) - \bar{z}_1(-z_2) = 4$

$\Rightarrow |z_1 - z_3| = 2$

Similarly $|z_2 - z_4| = 2$

Thus, $z_1, z_2, z_3\, z_4$ are vertices of a rectangle.

Example 49 Let $P(x)$ and $Q(x)$ be two polynomials. Suppose that $f(x) = P(x^3) + x\,Q(x^3)$ is divisible by $x^2 + x + 1$, then

(a) $P(x)$ is divisible by $(x - 1)$ but $Q(x)$ is not divisible by $x - 1$

(b) $Q(x)$ is divisible by $(x - 1)$ but $P(x)$ is not divisible by $x - 1$

(c) Both $P(x)$ and $Q(x)$ are divisible by $x - 1$

(d) $f(x)$ is divisible by $x - 1$.

Ans. (c), (d).

Solution We have $x^2 + x + 1 = (x - \omega)(x - \omega^2)$.

Since $f(x)$ is divisible by $x^2 + x + 1$, $f(\omega) = 0$, $f(\omega^2) = 0$.

$\therefore$ $P(\omega^3) + \omega Q(\omega^3) = 0 \Rightarrow P(1) + \omega Q(1) = 0$ (1)

and $P(\omega^6) + \omega^2 Q(\omega^6) = 0 \Rightarrow P(1) + \omega^2 Q(1) = 0$ (2)

Solving (1) and (2) we obtain

$P(1) = 0$ and $Q(1) = 0$.

$\therefore$ Both $P(x)$ and $Q(x)$ are divisible by $x - 1$.

$\Rightarrow$ $P(x^3)$ and $Q(x^3)$ are divisible by $x^3 - 1$

and hence by $x - 1$

Since $f(x) = P(x^3) + x\,Q(x^3)$, we get $f(x)$ is divisible by $x - 1$.

Example 50 If z_1, z_2, z_3, z_4 are the vertices of a square in that order, then

(a) $z_1 + z_3 = z_2 + z_4$

(b) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_4| = |z_4 - z_1|$

(c) $|z_1 - z_3| = |z_2 - z_4|$

(d) $(z_1 - z_3)/(z_2 - z_4)$ is purely imaginary.

Ans. (a), (b), (c), (d)

Solution Let the four points represented by z_1, z_2, z_3 and z_4 be A, B, C and D respectively. Since $ABCD$ is a square, the mid point of AC = the mid point of BD

$\Rightarrow \quad \frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4)$ or $z_1 + z_3 = z_2 + z_4$.

Also, $AB = BC = CD = DA$

$\Rightarrow |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_4| = |z_4 - z_1|$

Since diagonals of the square $ABCD$ are equal

$AC = BD$ or $|z_1 - z_3| = |z_2 - z_4|$

Also, since $AC \perp BD$, $(z_1 - z_3)/(z_2 - z_4)$ is purely imaginary.

Example 51 If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$, then

(a) $z_1 \bar{z}_2$ is purely imaginary

(b) z_1/z_2 is purely imaginary

(c) $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 0$

(d) O, z_1, z_2 are the vertices of a right triangle.

Ans. (a), (b), (c), (d)

Solution $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + |z_2|^2$

$\Rightarrow |z_1|^2 + z_2 \bar{z}_1 + z_1 \bar{z}_2 + |z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow z_2 \bar{z}_1 + z_1 \bar{z}_2 = 0 \Rightarrow \overline{z_1 \bar{z}_2} = \bar{z}_1 z_2 = -z_1 \bar{z}_2$

$\Rightarrow z_1 \bar{z}_2$ is purely imaginary.

$\Rightarrow \frac{z_1}{z_2}|z_2|^2$ is purely imaginary.

Since $|z_2|^2$ is real, we get z_1/z_2 is purely imaginary

As z_1/z_2 is purely imaginary, $\arg(z_1/z_2) = \pi/2$

Thus, O, z_1, z_2 forms a right triangle.

Example 52 If $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the nth roots of unity, then

$$(2 - \omega)(2 - \omega^2) \ldots (2 - \omega^{n-1})$$

equals

(a) $2^n - 1$

(b) ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1}$

(c) $[{}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n]^{1/2} - 1$

(d) none of these.

Ans. (a), (b), (c)

Solution We have

$$z^n - 1 = (z - 1)(z - \omega)(z - \omega^2) \ldots (z - \omega^{n-1})$$

$$\Rightarrow \quad (z - \omega)(z - \omega^2) \ldots (z - \omega^{n-1}) = \frac{z^n - 1}{z - 1}$$

Putting $z = 2$, we get

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$$(2-\omega)(2-\omega^2)\ldots(2-\omega^{n-1}) = \frac{2^n-1}{2-1} = 2^n - 1$$

Example 53 If a, b, c are real numbers and z is a complex number such that $a^2+b^2+c^2 = 1$ and $b+ic = (1+a)z$, then $\frac{1+iz}{1+iz}$ equals

(a) $\frac{b-ic}{1-ia}$ (b) $\frac{a+ib}{1+c}$

(c) $\frac{1-c}{a+ib}$ (d) $\frac{1+a}{b+ic}$

Ans. (b), (c)

Solution We have $z = \frac{b+ic}{1+a} \Rightarrow \bar{z} = \frac{b-ic}{1+a}$

$\Rightarrow z+\bar{z} = \frac{2b}{1+a}, z-\bar{z} = \frac{2ic}{1+a},$

and $z\bar{z} = \frac{b^2+c^2}{(1+a)^2} = \frac{1-a^2}{(1+a)^2} = \frac{1-a}{1+a}$

We have

$$\frac{1+iz}{1-iz} = \frac{1+iz}{1-iz}\times\frac{1+i\bar{z}}{1+i\bar{z}} = \frac{1+i(z+\bar{z})-z\bar{z}}{1-i(z-\bar{z})+z\bar{z}}$$

$$= \frac{1+\frac{2bi}{1+a}-\frac{1-a}{1+a}}{1+\frac{2c}{1+a}+\frac{1-a}{1+a}} = \frac{1+a+2bi-1+a}{1+a+2c+1-a}$$

$$= \frac{2(a+ib)}{2(1+c)} = \frac{a+ib}{1+c} \quad (1)$$

$$= \frac{a+ib}{1+c}\times\frac{a-ib}{a-ib} = \frac{a^2+b^2}{(1+c)(a-ib)}$$

$$= \frac{1-c^2}{(1+c)(a-ib)} = \frac{1-c}{a-ib} \quad (2)$$

From (1) and (2) we get (b) and (c) are true.

Example 54 Let a be a positive real number and let

$$S_a = \{z \in \mathbf{C} : z \neq 0 \text{ and } \left|z+\frac{1}{z}\right| = a\},$$

then

(a) $\min_{z\in S_a} |z| = \frac{1}{2}\left(\sqrt{a^2+4}-a\right)$

(b) $\max_{z\in S_a} |z| = \frac{1}{2}\left(\sqrt{a^2+4}+a\right)$

(c) $\min_{z\in S_a} \frac{1}{|z|} = \frac{1}{2}\left(\sqrt{a^2+4}-a\right)$

(d) $\max_{z\in S_a} \frac{1}{|z|} = \frac{1}{2}\left(\sqrt{a^2+4}+a\right)$

Ans. (a), (b), (c), (d)

Solution $a^2 = \left|z+\frac{1}{z}\right|^2 = |z|^2 + \frac{1}{|z|^2} + \frac{z}{\bar{z}} + \frac{\bar{z}}{z}$

$$= |z|^2 + \frac{1}{|z|^2} + \frac{z^2+\bar{z}^2}{|z|^2}$$

$$\Rightarrow |z|^2\left(a^2 - |z|^2 - \frac{1}{|z|^2}\right) = (z+\bar{z})^2 - 2|z|^2 \geq -2|z|^2$$

$$\Rightarrow |z|^4 - (2+a^2)|z|^2 + 1 \leq 0 \quad (1)$$

Roots of $x^2 - (2+a^2)x + 1 = 0$ are

$$x = \frac{(2+a^2)\pm\sqrt{(2+a^2)^2-4}}{2}$$

$$= \frac{1}{2}\left[(2+a^2)\pm\sqrt{a^4+4a^2}\right]$$

$$= \frac{1}{4}\left[a^2+(a^2+4)\pm 2a\sqrt{a^2+4}\right]$$

$$= \frac{1}{4}\left(\sqrt{a^2+4}\pm a\right)^2$$

From (1) we get

$$|z|^2 \in \left[\frac{1}{4}\left(\sqrt{a^2+4}-a\right)^2, \frac{1}{4}\left(\sqrt{a^2+4}+a\right)^2\right]$$

$$\text{or } |z| \in \left[\frac{1}{2}\left(\sqrt{a^2+4}-a\right), \frac{1}{2}\left(\sqrt{a^2+4}+a\right)\right]$$

Note that the minimum value of $|z|$ is obtained when $z = \frac{1}{2}\left(\sqrt{a^2+4}-a\right) = z_1$ (say) and the maximum value of $|z|$ is obtained when $z = \frac{1}{2}\left(\sqrt{a^2+4}+a\right) = z_2$ (say).

Next, note that $z \in S_a \Leftrightarrow \frac{1}{z} \in S_a$.

Since, $\frac{1}{2}\left(\sqrt{a^2+4}-a\right) \leq |z| \leq \frac{1}{2}, \left(\sqrt{a^2+4}+a\right) \forall\ z \in S_a$

we get

$$\frac{2}{\sqrt{a^2+4}+a} \leq \frac{1}{|z|} \leq \frac{2}{\sqrt{a^2+4}-a} \quad \forall\ z \in S_a$$

$$\text{or } \frac{1}{2}\left(\sqrt{a^2+4}-a\right) \leq \frac{1}{|z|} \leq \frac{1}{2}\left(\sqrt{a^2+4}+a\right) \quad \forall\ z \in S_a$$

These extreme values are obtained when $z = z_2$ and $z = z_1$.

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Example 55 If the roots of $(z-1)^n = 2\omega(z+1)^n$ (where $n \geq 3$ and ω is a complex cube root of unity) are plotted in the argand plane, then these roots lie on

(a) straight line (b) circle
(c) ellipse (d) rectangular hyperbola.

Ans. (b)

Solution If z is a root of

$$(z-1)^n = 2\omega\,(z+1)^n, \text{ then}$$

$$\left(\frac{z-1}{z+1}\right)^n = 2\omega \quad \Rightarrow \left|\frac{z-1}{z+1}\right|^n = |2\omega| = 2$$

$$\Rightarrow \qquad \left|\frac{z-1}{z+1}\right| = 2^{1/n}$$

As $2^{1/n} > 1$, we get z lies on a circle.

[See section 1.11 in Theory]

Example 56 If θ_1, θ_2, θ_3 and θ_4 are four real numbers, then any root of the equation

$$\sin\theta_1\, z^3 + \sin\theta_2\, z^2 + \sin\theta_3\, z + \sin\theta_4 = 3 \qquad (1)$$

lying inside the unit circle $|z| = 1$, satisfies the inequality

(a) $|z| < 2/3$ (b) $|z| > 2/3$
(c) $|z| < 1/2$ (d) $|z| > 1/2$

Ans. (b), (d)

Solution If z satisfies (1), then $z \neq 0$ for otherwise, $\sin\theta_4 = 3$. Also

$$3 = |3| = |\sin\theta_1\, z^3 + \sin\theta_2\, z^2 + \sin\theta_3\, z + \sin\theta_4|$$
$$\leq |\sin\theta_1|\,|z|^3 + |\sin\theta_2|\,|z|^2 + |\sin\theta_3|\,|z| + |\sin\theta_4|$$
$$\leq |z|^3 + |z|^2 + |z| + 1 \qquad [\because |\sin\theta| \leq 1]$$
$$\leq 1 + |z| + |z|^2 + |z|^3 + |z|^4 + \ldots \leq \frac{1}{1-|z|}$$
$$\Rightarrow 1 - |z| < 1/3 \Rightarrow |z| > 2/3$$

Thus, $|z| > 2/3$ and hence $|z| > 1/2$

Example 57 If α is a root of $z^5 + z^3 + z + 3 = 0$, then

(a) $|\alpha| \geq 1$
(b) $|\alpha| < 1$
(c) α lies on or outside the unit circle $|z| = 1$
(d) α lies inside the unit circle $|z| = 1$.

Ans. (a), (c).

Solution Suppose $|\alpha| < 1$. We have

$$|-3| = |\alpha^5 + \alpha^3 + \alpha| \leq |\alpha|^5 + |\alpha|^3 + |\alpha| < 1 + 1 + 1$$
$$\Rightarrow 3 < 3.$$

A contradiction

Thus, $|\alpha| \geq 1$.

$\Rightarrow$ α lies either on the or outside the unit circle $|z| = 1$.

Example 58 If $z \neq 0$ is a complex number, then z, iz, $-z$ and $-iz$ are the vertices of a

(a) square (b) rectangle
(c) rhombus (d) none of these

Ans. (a), (b), (c)

Solution Suppose z, iz, $-z$ and $-iz$ are represented by A, B, C and D in the complex plane. We have

mid-point of AC is $\frac{1}{2}(z + (-z)) = 0$, and

mid-point of BD is $\frac{1}{2}(iz + (-iz)) = 0$

Thus, the diagonals bisect each other.

Also, $AC = |-z - z| = 2|z|$

and $BD = |-iz - iz| = 2|-i|\,|z| = 2|z|$

$\therefore$ $AC = BD$

Finally, $AB = |iz - z| = |i - 1|\,|z| = \sqrt{2}\,|z|$

and $BC = |-z - iz| = |-1 - i|\,|z| = \sqrt{2}\,|z|$.

Hence, $ABCD$ is a square.

Example 59 The equation $|z - z_1|^2 + |z - z_2|^2 = k$ (where k is a real number) will represent a circle if.

(a) $k = |z_1 - z_2|$ (b) $k \leq |z_1 - z_2|$
(c) $k \geq \frac{1}{2}|z_1 - z_2|^2$ (d) none of these

Ans. (c)

Solution $|z - z_1|^2 + |z - z_2|^2 = k$ implies

$$|z|^2 + |z_1|^2 - 2\text{Re}(z\bar{z}_1) + |z|^2 + |z_2|^2 - 2\text{Re}(z\bar{z}_2) = k$$
$$\Rightarrow 2|z|^2 - 2\text{Re}|z(\bar{z}_1 + \bar{z}_2)] = k - (|z_1|^2 + |z_2|^2)$$
$$\Rightarrow \quad |z|^2 - \text{Re}\left[z\left(\overline{\frac{z_1 + z_2}{2}}\right)\right] + \frac{1}{4}|z_1 + z_2|^2$$
$$= \frac{k}{2} + \frac{1}{4}\,[|z_1 + z_2|^2 - 2\,|z_1|^2 - 2|z_2|^2]$$
$$\Rightarrow \quad \left|z - \frac{z_1 + z_2}{2}\right|^2 = \frac{1}{2}\left[k - \frac{1}{2}\left\{|z_1|^2 + |z_2|^2 - 2\,\text{Re}\,(z_1\bar{z}_2)\right\}\right]$$
$$\Rightarrow \quad \left|z - \frac{z_1 + z_2}{2}\right|^2 = \frac{1}{2}\left[k - \frac{1}{2}|z_1 - z_2|^2\right]$$

This will represent a circle with centre at $\frac{1}{2}(z_1 + z_2)$ and radius $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$ provided $k \geq \frac{1}{2}|z_1 - z_2|^2$.

Example 60 The locus of z satisfying the inequality

$$\log_{(0.8)}|z + 1| > \log_{(0.8)}|z - 1|$$

(a) $Re(z) < 0$ (b) $Re(z) > 0$
(c) $Im(z) < 0$ (d) $Im(z) > 0$

Ans. (a)

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Solution We have $\log_{0.8}|z+1| > \log_{(0.8)}|z-1|$

$\Rightarrow \quad |z+1| < |z-1| \quad [\because\ 0 < 0.8 < 1]$

$\Rightarrow \quad (x+1)^2 + y^2 < (x-1)^2 + y^2$ [where $z = x + iy$]

$\Rightarrow \quad 4x < 0 \quad$ or $\quad x < 0 \quad$ or $\quad R(z) < 0$

Example 61 For any two complex numbers z_1, z_2 and a, $b \in \mathbf{R}$,

$$|az_1 + bz_2|^2 + |bz_1 - az_2|^2 =$$

(a) $(a^2 + b^2)\ (|z_1|^2 + |z_2|^2)$
(b) $(a^2 - b^2)\ (|z_1|^2 - |z_2|^2)$
(c) $a^2 + b^2$
(d) none of these.

Ans. (a)

Solution We have

$$|az_1 + bz_2|^2 + |bz_1 - az_2|^2$$

$$= |az_1|^2 + |bz_2|^2 + 2R\left(az_1\,\overline{bz_2}\right)$$

$$+ |bz_1|^2 + |az_2|^2 - 2R\left(bz_1\,\overline{az_2}\right)$$

$$= a^2|z_1|^2 + b^2|z_2|^2 + 2abR(z_1\bar{z}_2)$$

$$+ b^2|z_1|^2 + a^2|z_2|^2 - 2abR(z_1\bar{z}_2)$$

[$\because$ a and b are real numbers]

$$= (a^2 + b^2)\ (|z_1|^2 + |z_2|^2)$$

Example 62 If $|z^2 - 1| = |z|^2 + 1$, then z lies on

(a) the real axis
(b) the imaginary axis
(c) $y = x$
(d) circle

Ans. (b)

Solution

$$|z^2 + (-1)| = |z^2| + |-1|$$

$\Leftrightarrow$ $z^2/(-1)$ is non negative real number

$\Leftrightarrow$ z is purely imaginary

Example 63 If $\omega \neq 1$ is a cube root of unity and ω and ω^2 satisfy the equation

$$\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} = \frac{2}{x} \qquad (1)$$

then value of $\dfrac{1}{a+1} + \dfrac{1}{b+1} + \dfrac{1}{c+1} + \dfrac{1}{d+1}$ is

(a) 1 (b) 2
(c) 3 (d) 0

Ans. (b)

Solution We can write (1) as

$\Sigma x(b+x)(c+x)(d+x) = 2(a+x)(b+x)(c+x)(d+x)$

$\Rightarrow \quad \Sigma x\{x^3 + (b+c+d)x^2 + (bc+cd+bd)x + bcd\}$

$$= 2\{x^4 + (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 + (abc+abd+acd+bcd)x + abcd\}$$

$\Rightarrow \quad 2x^4 + (a+b+c+d)x^3 + 0x^2 - (abc+abd+acd+bcd)x - 2abcd = 0$

This is a fourth degree equation whose two roots are w and w^2. Let α, β be the two other roots, then

$$(\alpha+\beta)(w+w^2) + \alpha\beta + \omega \cdot \omega^2 = 0$$

[sum of the products taken two at a time]

$\Rightarrow \quad (\alpha+\beta)(-1) + \alpha\beta + 1 = 0$

$\Rightarrow \quad (1-\alpha)(1-\beta) = 0$

$\Rightarrow \quad \alpha = 1 \quad$ or $\quad \beta = 1$

$\therefore$ 1 satisfies the equation (1).

Thus, $\dfrac{1}{a+1} + \dfrac{1}{b+1} + \dfrac{1}{c+1} + \dfrac{1}{d+1} = 1.$

Example 64 If $\sin^6 x + \cos^6 x = A + B\cos 4x$, then

(a) $A = \dfrac{5}{3}$ (b) $B = \dfrac{1}{8}$

(c) $A = \dfrac{5}{8}$ (d) $B = \dfrac{3}{8}$

Ans. (b), (d).

Solution We have

$$A + B\cos 4x = \frac{1}{64}\left[(e^{ix} + e^{-ix})^6 - (e^{ix} - e^{-ix})^6\right]$$

$$= \frac{1}{32}\left[{}^6C_1\,(e^{ix})^5\,(e^{-ix}) + {}^6C_3(e^{ix})^3\,(e^{-ix})^3 + {}^6C_5\,(e^{ix})\,(e^{-ix})^5\right]$$

$$= \frac{1}{32}\left[6(e^{4ix} + e^{-4ix}) + 20\right]$$

$$= \frac{1}{32}\left[(6)\,(2\cos 4x) + 20\right]$$

$$= \frac{3}{8}\cos 4x + \frac{5}{8}$$

Thus $\quad A = 5/8 \quad$ and $\quad B = 3/8$

Example 65 Let z_1, z_2, z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and $\dfrac{z_1^2}{z_2 z_3} + \dfrac{z_2^2}{z_3 z_1} + \dfrac{z_3^2}{z_1 z_2} = -1$, then value of $|z_1 + z_2 + z_3|$ can be

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (a), (b)

Solution We have ${z_1}^3 + {z_2}^3 + {z_3}^3 = -z_1 z_2 z_3$.

$\Rightarrow \quad -4z_1z_2z_3 = {z_1}^3 + {z_2}^3 + {z_3}^3 - 3z_1z_2z_3$

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$= (z_1 + z_2 + z_3)(z_1^2 + z_2^2 + z_3^2 - z_2z_3 - z_3z_1 - z_1z_2)$

$= (z_1 + z_2 + z_3)[(z_1 + z_2 + z_3)^2 - 3(z_2z_3 + z_3z_1 + z_1z_2)]$

$\Rightarrow \quad z^3 - 3z(z_2z_3 + z_3z_1 + z_1z_2) + 4z_1z_2z_3 = 0$

where $z = z_1 + z_2 + z_3$

$$\Rightarrow \quad z^3 = z_1z_2z_3\left[3z\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) - 4\right]$$

$= z_1z_2z_3\,[3z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) - 4\,]$

$[\because \quad |z_1| = |z_2| = |z_3| = 1]$

$= z_1z_2z_3\,[3z\,\bar{z} - 4]$

$\Rightarrow \quad |z|^3 = |z_1|\,|z_2|\,|z_3|\,|3|z|^2 - 4|$

$\Rightarrow \quad |z|^3 - |3|z|^2 - 4| = 0$

If $|z| \geq 2/\sqrt{3}$, we get

$$|z|^3 - 3|z|^2 + 4 = 0$$

$\Rightarrow \quad (|z| - 2)(|z|^2 - |z| - 2) = 0$

$\Rightarrow \quad (|z| - 2)^2(|z| + 1) = 0$

$\Rightarrow \quad |z| - 2 = 0$ or $|z| = 2$

If $|z| < 2/3$, we get

$$|z|^3 + 3|z|^2 - 4 = 0$$

$\Rightarrow \quad (|z| - 1)(|z|^2 + 4|z| + 4) = 0$

$\Rightarrow \quad |z| - 1 = 0 \Rightarrow |z| = 1.$

Example 66 If $\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0$, then

(a) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$
(b) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$
(c) $\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0$
(d) $\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0$

Ans. (a), (b), (c), (d)

Solution Let $a = \cos\alpha + i\sin\alpha$,

$\beta = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$,

then $$a + b + c = 0 \qquad (1)$$

and $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

$$\Rightarrow \quad bc + ca + ab = 0 \qquad (2)$$

From (1) and (2)

$$a^2 + b^2 + c^2 = 0 \qquad (3)$$

From (2)

$\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) +$

$i\,[\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta)] = 0$

From this (c) and (d) follow. Similarly, (a), (b) follow from (3).

Example 67 If $\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0$ then

(a) $\cos 4\alpha + \cos 4\beta + \cos 4\gamma = 0$
(b) $\sin 4\alpha + \sin 4\beta + \sin 4\gamma = 0$
(c) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$
(d) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Ans. (a), (b), (c), (d)

Solution (a), (b) follows from (a) and (b) of Example 61. Using notation of Example 61,

$$a^3 + b^3 + c^3 - 3abc$$
$$= (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$$
$$= 0$$

From this identity both (c) and (d) follow.

Example 68 If $\omega \neq 1$ is a cube root of unity, then sum of the series

$$S = 1 + 2\omega + 3\omega^2 + \ldots + 3n\,\omega^{3n-1} \quad (n \in \mathbf{N}) \text{ is}$$

(a) $\dfrac{3n}{\omega - 1}$ (b) $3n(\omega^2 - 1)$

(c) 0 (d) none of these.

Ans. (a), (b)

Solution $1 + 2x + 3x^2 + \ldots + 3nx^{3n-1}$

$$= \frac{d}{dx}\left[x + x^2 + \ldots + x^{3n}\right] = \frac{d}{dx}\left[\frac{x - x^{3n+1}}{1 - x}\right]$$

$$= \frac{(1-x)(1-(3n+1)x^{3n}) + (x - x^{3n+1})}{(1-x)^2}$$

Put $x = \omega$, we get

$$1 + 2\omega + 3\omega^2 + \ldots + 3n\,\omega^{3n-1}$$

$$= \frac{(1-\omega)(1-3n-1) + (\omega - \omega)}{(1-\omega)^2} = \frac{-3n}{(1-\omega)}$$

$$= \frac{3n(\omega^2 - 1)}{(\omega - 1)(\omega^2 - 1)} = n(\omega^2 - 1)$$

Example 69 Suppose $A(z_1)$, $B(z_2)$ and $C(z_3)$ are vertices of a triangle lying on the unit circle $|z| = 1$. AD is altitude of the ΔABC meeting the unit circle in E.

(a) orthocentre of ΔABC is $z_1 + z_2 + z_3$
(b) affix of E is $-z_2z_3/z_1$
(c) if $z_1^2 = z_2z_3$ and $z_2^2 = z_3z_1$, then ΔABC is equilateral.
(d) if $z_2 + z_3 = 0$, then ΔABC is a right angled.

Ans. (a), (b), (c), (d)

Solution Centroid of ΔABC is $z_G = \frac{1}{3}(z_1 + z_2 + z_3)$, circumcentre of ΔABC is $z_S = 0$. As z_G divides the join of

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z_S and z_H (orthocentre) in the ratio 1 : 2, we get

$$z_G = \frac{2z_s + 1z_H}{3} = \frac{1}{3} z_H$$

$$\Rightarrow \frac{1}{3}(z_1 + z_2 + z_3) = \frac{1}{3} z_H \Rightarrow z_H = z_1 + z_2 + z_3$$

As AD passes through H, its equation is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_1 + z_2 + z_3 & \bar{z}_1 + \bar{z}_2 + \bar{z}_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 & 0 \\ z_1 & \bar{z}_1 & 1 \\ z_2 + z_3 & \bar{z}_2 + \bar{z}_3 & 0 \end{vmatrix} = 0$$

[use $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$]

$$\Rightarrow (z - z_1)(\bar{z}_2 + \bar{z}_3) - (\bar{z} - \bar{z}_1)(z_2 + z_3) = 0$$

$$\Rightarrow (z - z_1)\left(\frac{1}{z_2} + \frac{1}{z_3}\right) - \left(\bar{z} - \frac{1}{z_1}\right)(z_2 + z_3) = 0$$

$$\Rightarrow (z - z_1)\frac{(z_2 + z_3)}{z_2 z_3} - \left(\bar{z} - \frac{1}{z_1}\right)(z_2 + z_3) = 0$$

$$\Rightarrow \frac{z - z_1}{z_2 z_3} = \bar{z} - \frac{1}{z_1}$$

As E lies on the circle $|z| = 1$, we have $\bar{z} = 1/z$ for the affix of E. Thus,

$$\frac{z - z_1}{z_2 z_3} = \frac{1}{z} - \frac{1}{z_1} = -\frac{z - z_1}{z z_1}$$

$$\Rightarrow z = -\frac{z_2 z_3}{z_1}$$

Next, $z_1^2 = z_2 z_3,\ z_2^2 = z_3 z_1$

$$\Rightarrow z_1^2 z_2^2 = z_1 z_2 z_3^2 \Rightarrow z_3^2 = z_1 z_2$$

Thus, $z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$

$\Rightarrow \Delta ABC$ is equilateral.

Lastly, if $z_2 + z_3 = 0$, $z_H = z_1$, that is, orthocentre of ΔABC is at $A \Rightarrow \Delta ABC$ is right angled at A.

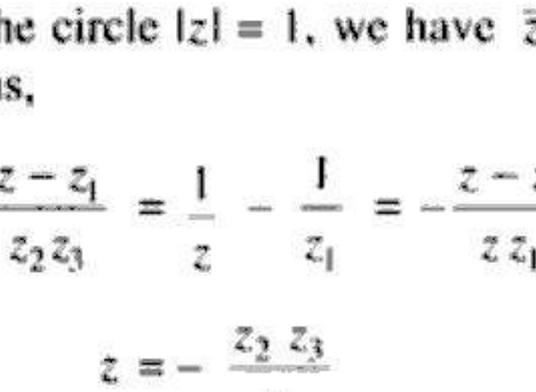

Fig. 1.53

Example 70 Let z and w be two complex numbers such that $|z| \le 1$, $|w| \le 1$, $|z + iw| = |z - i\bar{w}| = 2$, then z can be

(a) 1 (b) – 1
(c) i (d) – i

Ans. (a), (b)

Solution Note that z, w, $-iw$ and $i\bar{w}$ lie inside or on the boundary of the disc $|z| \le 1$. Since $|z + iw| = |z - i\bar{w}| = 2$, we get $-iw$, $i\bar{w}$ are the other end points of the diameter having one end point as z. Thus, $-iw = i\bar{w} = -z$.
Therefore, $|z + z| = 2 \Rightarrow |z| = 1$.
Now, $-z = -iw = i\bar{w}$

$$\Rightarrow w = -iz \text{ and } \bar{w} = iz$$

$$\therefore \bar{w} = i\bar{z} \text{ and } \bar{w} = iz$$

$$\therefore i\bar{z} = iz \Rightarrow z = \bar{z}$$

$\Rightarrow$ z is real.
As $|z| = 1$, we get $z = 1, -1$.

SECTION III

COMPREHENSION TYPE

Paragraph for Example Nos. 71 to 75.

If $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are root nth roots of unity, then

$$\alpha_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

where $0 \le k \le n - 1$. Also

$$x^n - 1 = (x - \alpha_0)(x - \alpha_1) \ldots (x - \alpha_{n-1}) \quad (1)$$

Example 71 Value of $(1 + \alpha_0)(1 + \alpha_1) \ldots (1 + \alpha_{n-1})$ is

(a) 3 (b) $(-1)^n$
(c) 0 (d) $1 + (-1)^{n-1}$

Example 72 Value of $(1 - \alpha_1) \ldots (1 - \alpha_{n-1})$ is

(a) n (b) $n - 1$
(c) $(-1)^n$ (d) 0

Example 73 Value of

$$2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \ldots \sin\left(\frac{n-1}{n}\pi\right) \text{ is}$$

(a) n (b) –1
(c) $n - 1$ (d) $n/3$

Example 74 If $n = 5$ in (1), then value of

$$(z + 1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 + 2z\cos\frac{\pi}{5} + 1\right) \text{ is}$$

(a) $z^5 - 1$ (b) z^5
(c) $z^5 + 1$ (d) 0

Example 75 Value of $\sin\frac{\pi}{10}\cos\frac{\pi}{5}$ is

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(a) 1/4 (b) 1/2 (c) 1/3 (d) $1/\sqrt{3}$

Ans. 71. (d), 72. (a), 73. (a), 74. (c), 75. (a)

Solution 71. Put $x = -1$ in (1) so that

$(-1)^n - 1 = (-1 - \alpha_0)(-1 - \alpha_1) \ldots (-1 - \alpha_{n-1})$

$\Rightarrow (-1)^n [1 + (-1)^{n-1}] = (-1)^n (1 + \alpha_0) \ldots (1 + \alpha_{n-1})$

$\Rightarrow (1 + \alpha_0)(1 + \alpha_1) \ldots (1 + \alpha_{n-1}) = 1 + (-1)^{n-1}$

72. Note that $\alpha_0 = 1$ and (1) can be written as

$$(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_{n-1}) = \frac{x^n - 1}{x - 1}$$

$$= 1 + x + \ldots + x^{n-1}$$

Put $x = 1$, so that

$$(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{n-1}) = n \qquad (2)$$

73. We have for $1 \le k \le n - 1$

$$|1 - \alpha_k|^2 = \left(1 - \cos\frac{2k\pi}{n}\right)^2 + \sin^2\frac{2k\pi}{n}$$

$$= 2\left(1 - \cos\frac{2k\pi}{n}\right) = 4\sin^2\frac{k\pi}{n}$$

From (2)

$|(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{n-1})| = n$

$$\Rightarrow \quad 2^{n-1} \sin\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) \ldots \sin\left(\frac{n-1}{n}\pi\right) = n$$

74. For $n = 5$, note that $\alpha_3 = e^{6\pi i/5} = \bar{\alpha}_2$ and $\alpha_4 = \bar{\alpha}_1$

Thus, (1) can be written as

$$x^5 - 1 = (x - 1)(x - \alpha_1)(x - \bar{\alpha}_1)(x - \alpha_2)(x - \bar{\alpha}_2)$$

$$= (x - 1)[x^2 - (\alpha_1 + \bar{\alpha}_1)x + 1]$$

$$[x^2 - (\alpha_2 + \bar{\alpha}_2)x + 1]$$

Putting $x = -z$, we get

$$z^5 + 1 = (z + 1)\left(z^2 + 2\cos\frac{2\pi}{5}z + 1\right)$$

$$\left(z^2 + 2\cos\frac{4\pi}{5}z + 1\right) \qquad (3)$$

Now use $\cos(4\pi/5) = -\cos(\pi/5)$

75. In (3) put $z = 1$, so that

$$2 = 2^3\left(1 + \cos\frac{2\pi}{5}\right)\left(1 + \cos\frac{4\pi}{5}\right)$$

$$\Rightarrow \quad 2 = 2^5 \cos^2\frac{\pi}{5}\cos^2\frac{2\pi}{5}$$

$$\Rightarrow \quad \frac{1}{4} = \cos\frac{\pi}{5}\cos\left(\frac{\pi}{2} - \frac{\pi}{10}\right)$$

$$\Rightarrow \quad \frac{1}{4} = \cos\frac{\pi}{5}\sin\frac{\pi}{10}$$

Paragraph for Example Nos. 76 to 80.

If n is a natural number define polynomial $P_n(x)$ of degree n as follows:

$\cos n\theta = P_n(\cos\theta)$

For example, $P_2(x) = 2x^2 - 1$ and $P_3(x) = 4x^3 - 3x$.

Example 76 $\frac{1}{2x}[P_{n+1}(x) + P_{n-1}(x)]$ equals

(a) $P_{n+2}(x)$ (b) $P_{n-1}(x) + P_n(x)$

(c) $P_n(x)$ (d) $P_{n+1}(x) - P_n(x)$

Example 77 $(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n$ equals

(a) $P_n(x)$ (b) $P_{n+1} + P_{n-1}(x)$

(c) $2P_n(x)$ (d) none of these

Example 78 $P_6(x)$ equals

(a) $36x^6 - 45x^4 + 18x^2 - 8$

(b) $32x^6 - 48x^4 + 18x^2 - 1$

(c) $36x^6 - 48x^4 + 18x^2 - 5$

(d) none of these

Ans. 76. (c), 77. (c), 78. (b)

Solution 76. We have $P_1(x) = x$ and

$\cos(n + 1)\theta + \cos(n - 1)\theta = 2\cos(n\theta)\cos\theta$

$\Rightarrow P_{n+1}(x) + P_{n-1}(x) = 2x P_n(x)$

$$\Rightarrow \quad P_n(x) = \frac{1}{2x}[P_{n+1}(x) + P_{n-1}(x)]$$

77. Put $x = \cos\theta$ so that $\sqrt{x^2 - 1} = i\sin\theta$

Now, $(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n$

$= 2\cos(n\theta)$ [using DeMoivre's theorem]

$= 2P_n(\cos\theta)$

Thus, $(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n = 2P_n(x)$.

78. $2P_6(x) = (x + \sqrt{x^2 - 1})^6 + (x - \sqrt{x^2 - 1})^6$

$= 2[x^6 + {}^6C_2 x^4(x^2 - 1) + {}^6C_4 x^2(x^2 - 1)^2$

$+ {}^6C_6 (x^2 - 1)^3]$

$P_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

Paragraph for Example Nos. 79 to 83.

For non-zero complex numbers z_1 and z_2 define the dot product and the cross product of z_1 and z_2 as follows:

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Dot product:

$$z_1 \cdot z_2 = |z_1|\,|z_2| \cos\theta$$
$$= x_1 x_2 + y_1 y_2 = Re(\bar{z}_1 z_2)$$

Cross product:

$$z_1 \times z_2 = |z_1|\,|z_2| \sin\theta$$
$$= x_1 y_2 - x_2 y_1 = Im(\bar{z}_1 z_2)$$

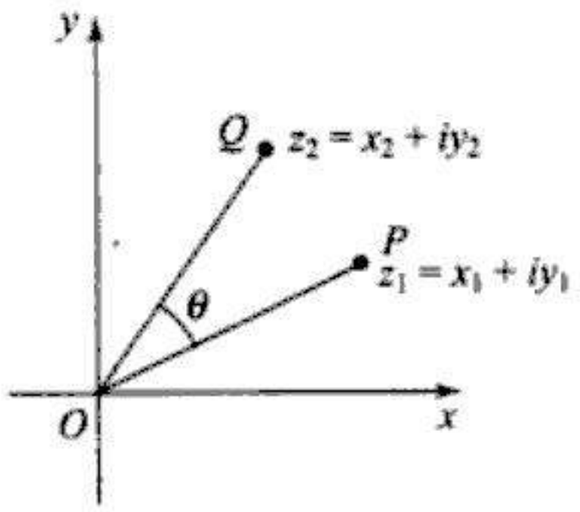

Fig. 1.54

Example 79 $z_1 \cdot z_2 + i(z_1 \times z_2)$ equals

(a) $z_1 z_2$ (b) $z_1 \bar{z}_2$

(c) $\bar{z}_1 z_2$ (d) $\bar{z}_1 \bar{z}_2$

Example 80 $z_1 \cdot z_2 = 0$ is equivalent to say

(a) z_2/z_1 is purely imaginary
(b) z_1/z_2 is purely real
(c) $z_1 = 0$ or $z_2 = 0$
(d) none of these

Example 81 OP coincides with OQ if and only if

(a) $z_1 \cdot z_2 = 0$
(b) $z_1 \times z_2 = 0$
(c) $z_1 \cdot z_2 + z_1 \times z_2 = 0$
(d) none of these

Example 82 The area of parallelogram with sides OP and OQ is

(a) $|z_1 \cdot z_2|$
(b) $|z_1 \times z_2|$
(c) $|z_1 \cdot z_2| + |z_1 \times z_2|$
(c) $|z_1 \cdot z_2|\,|z_1 \times z_2|$

Example 83 The length of projection of OP on OQ is

(a) $|z_1 \cdot z_2|/|z_1|$ (b) $|z_1 \cdot z_2|/|z_2|$

(c) $|z_1 \cdot z_2|$ (d) $|z_1 \times z_2|$

Ans. 79. (c), 80. (a), 81. (b), 82. (b), 83. (b)

Solution 79. $z_1 \cdot z_2 + i(z_1 \times z_2)$

$$= (x_1 x_2 + y_1 y_2) + i(x_1 y_2 - x_2 y_1)$$
$$= (x_1 - iy_1)(x_2 + iy_2) = \bar{z}_1 z_2$$

80. $z_1 \cdot z_2 = 0 \Leftrightarrow Re(\bar{z}_1 z_2) = 0$

$$\Leftrightarrow Re\left(|z_1|^2 \frac{z_2}{z_1}\right) = 0 \Leftrightarrow \frac{z_2}{z_1} \text{ is purely imaginary}$$

81. OP coincides with OQ

$$\Leftrightarrow \theta = 0 \Leftrightarrow z_1 \times z_2 = 0$$

82. Area of parallelogram with sides OP and OQ

$$= 2\,[\text{area}\,(\Delta OPQ)] = 2\,|\tfrac{1}{2}|z_1|\,|z_2| \sin\theta| = 2|z_1 \times z_2|$$

83. The length of projection of OP and OQ

$$= |z_1| \cos\theta| = \frac{|\,|z_1|\,|z_2| \cos\theta|}{|z_2|} = \frac{|z_1 \cdot z_2|}{|z_2|}$$

Paragraph for Example Nos. 84 to 86

Let α, β, γ be three real numbers such that $\alpha^2 + \beta^2 + \gamma^2 - \gamma = 0$ and $z = \dfrac{\alpha + i\beta}{1 - \gamma}$.

Example 84 $|z|^2$ equals

(a) γ (b) $1 - \gamma$

(c) $\dfrac{\gamma}{1-\gamma}$ (d) $\dfrac{1-\gamma}{\gamma}$

Example 85 α equals

(a) $\dfrac{z + \bar{z}}{2(1+|z|^2)}$ (b) $\dfrac{(\bar{z} - z)i}{2(1+|z|^2)}$

(c) $\dfrac{(z - \bar{z})i}{2(1+|z|^2)}$ (d) none of these

Example 86 β equals

(a) $\dfrac{z - \bar{z}}{2(1+|z|^2)}$ (b) $\dfrac{(\bar{z} - z)i}{2(1+|z|^2)}$

(c) $\dfrac{z}{2(1+|z|^2)}$ (d) $\dfrac{2z}{1+|z|^2}$

Ans. 84 (c), 85 (a), 86 (b)

Solution 84. $|z|^2 = \dfrac{\alpha^2 + \beta^2}{(1-\gamma)^2} = \dfrac{\gamma - \gamma^2}{(1-\gamma)^2} = \dfrac{\gamma}{1-\gamma}$

85. From Example 84,

$$|z|^2 = \frac{\gamma}{1-\gamma} = -1 + \frac{1}{1-\gamma}$$

$$\Rightarrow \quad 1 + |z|^2 = \frac{1}{1-\gamma} \Rightarrow \gamma = \frac{|z|^2}{1+|z|^2}$$

Now, $$\frac{2\alpha}{1-\gamma} = z + \bar{z}$$

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$$\Rightarrow \qquad 2\alpha = (z + \bar{z})(1 - \gamma) = \frac{z + \bar{z}}{1 + |z|^2}$$

$$\Rightarrow \qquad \alpha = \frac{z + \bar{z}}{2(1 + |z|^2)}$$

86. $$\frac{2i\beta}{1 - \gamma} = z - \bar{z}$$

$$\Rightarrow \qquad 2i\beta = (z - \bar{z})(1 - \gamma) = \frac{z - \bar{z}}{1 + |z|^2}$$

$$\Rightarrow \qquad \beta = \frac{(\bar{z} - z)i}{2(1 + |z|^2)}$$

Paragraph for Example Nos. 87 to 91

Let $\lambda \in \mathbf{C}$ and $z_1, z_2, z_3 \in \mathbf{C}$ are such that

$$\frac{z_3 - z_1}{z_2 - z_1} = \lambda$$

Example 87 If λ is a given purely imaginary number then all the triangles with vertices z_1, z_2, z_3 are

(a) similar (b) acute angled
(c) equilateral (d) obtuse angled

Example 88 Value of λ for which triangle with vertices z_1, z_2, z_3 is equilateral, is

(a) $\frac{1}{2}(1 - \sqrt{3}i)$ (b) $\frac{1}{2}(1 + \sqrt{3}i)$
(c) $\sqrt{3}i$ (d) $-\sqrt{3}i$

Example 89 If $\lambda \in \mathbf{R}$, then z_1, z_2, z_3

(a) lie on a circle
(b) are vertices of a right triangle
(c) lie on a straight line
(d) lie on a parabola

Example 90 If $\lambda = it$ ($t \in \mathbf{R}$) and z_2, z_3 are fixed, then locus of z_1 is

(a) a straight line
(b) an ellipse
(c) a circle
(d) a pair of straight lines

Example 91 If $\lambda = e^{it}$ ($t \in \mathbf{R}$) and z_2, z_3 are fixed, then locus of z_1 is

(a) a straight line
(b) an ellipse
(c) a circle
(d) a pair of straight lines

Ans. 87. (a), 88. (b), 89. (c), 90. (c), 91. (a)

Solution 87. Let $\lambda = ia$ where $a \in \mathbf{R}$.

Let z_1, z_2, z_3 and w_1, w_2, w_3 be vertices of two triangles ABC and PQR respectively, such that

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{w_3 - w_1}{w_2 - w_1} = ia \qquad (1)$$

$$\Rightarrow \qquad \frac{|z_3 - z_1|}{|z_2 - z_1|} = \frac{|w_3 - w_1|}{|w_2 - w_1|} = |a|$$

$$\Rightarrow \qquad \frac{AC}{AB} = \frac{PR}{PQ} = |a| \qquad (2)$$

Also, $$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg\left(\frac{w_3 - w_1}{w_2 - w_1}\right) = \frac{\pi}{2}$$

$$\Rightarrow \qquad \angle BAC = \angle QPR = \pi/2 \qquad (3)$$

From (2) and (3) we get ΔABC and ΔPQR are similar.

88. $$\lambda = \frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= \frac{1}{2}(1 + \sqrt{3}i)$$

89. $z_3 - z_1 = \lambda(z_2 - z_1)$

$$\Rightarrow \qquad z_3 = (1 - \lambda)z_1 + \lambda z_2$$

$\Rightarrow$ z_3 lies on the line joining z_1 and z_2.

90. As $\frac{z_3 - z_1}{z_2 - z_1}$ is purely imaginary, angle between the segment joining z_3z_1 and z_2z_1 is a right angle.

$\Rightarrow$ z_1 lies on a circle with the segment joining z_2z_3 as diameter.

91. $$\left|\frac{z_3 - z_1}{z_2 - z_1}\right| = |e^{it}| = 1$$

$\Rightarrow$ z_1 is equidistant from z_2 and z_3

$\Rightarrow$ z_1 lies the perpendicular bisector of the segment joining z_3 and z_3.

Paragraph for Example Nos. 92 to 95

Let A, B, C be three sets of complex numbers defined as follows:

$A = \{z : \text{Im}(z) \geq 1\}$

$B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}$.

Example 92 The number of elements in the set $A \cap B \cap C$ is

(a) 0 (b) 1
(c) 2 (d) ∞

Example 93 Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

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(a) 25 and 29 (b) 30 and 34
(c) 35 and 39 (d) 40 and 44

Example 94 Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

(a) -6 and 3 (b) -3 and 9
(c) -6 and 6 (d) -3 and 6

Example 95 If z is any point $A \cap B \cap C$, then

$$\arg\left(\frac{5+i-z}{i-1-z}\right) \text{ equals}$$

(a) $\pi/2$ (b) $\pi/3$
(c) $2\pi/3$ (d) $\pi/4$

Ans. 92. (b), 93. (c), 94. (b), 95. (a)

Solution 92. A consists all point in the half plane $\text{Im} z \geq 1$; B all points on the circle $|z - (2 + i)| = 3$ and C all the points on the line $x + y = \sqrt{2}$. These three curve intersect in just one point as shown in Fig. 1.51

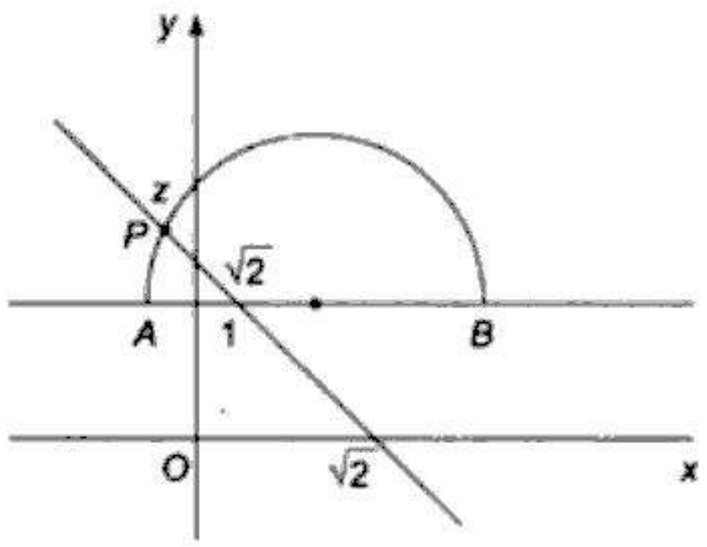

Fig. 1.55

93. Points A and B are end points of a diameter of $|z - (2 + i)| = 3$. Affix of A is $-1 + i$ and that of B is $5 + i$.

$|z + 1 - i|^2 + |z - 5 - i|^2 = PA^2 + PB^2 = AB^2 = 36$

where lies between 35 and 39.

94. $||z| - |w|| \leq |z - w|$

Here z is fixed. Thus distance between z and w would be maximum for diametrically opposite points.

$\therefore \quad ||z| - |w|| \leq |z - w| < 6$

$\Rightarrow \quad -6 < |z| - |w| < 6 \Rightarrow -3 < |z| - |w| + 3 < 9$

95. As $\angle APB = \pi/2$, $\arg\left(\frac{5+i-z}{i-1-z}\right) = \frac{\pi}{2}$.

SECTION IV

INTEGER ANSWER TYPE

Example 96 If $a_1, a_2, \ldots, a_n$ are real numbers and $\cos\alpha + i\sin\alpha$ is a root of

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0 \qquad (1)$$

find the value of $|a_1 \cos\alpha + a_2 \cos 2\alpha + \ldots + a_n \cos(n\alpha)|$

Ans. 1.

Solution As $a_1, a_2, \ldots a_n$ are real numbers and $z = \cos\alpha + i\sin\alpha$ is a root of (1),

$\bar{z} = \cos\alpha - i\sin\alpha$ is also a root of (1).

Thus, $\cos\alpha - i\sin\alpha = \dfrac{1}{\cos\alpha + i\sin\alpha} = \dfrac{1}{z}$ is a root of (1).

Therefore,

$$\frac{1}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \ldots + a_n = 0$$

$$\Rightarrow \quad 1 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$$

$$\Rightarrow \quad 1 + a_1(\cos\alpha + i\sin\alpha) + a_2(\cos 2\alpha + i\sin 2\alpha) + \ldots + a_n(\cos n\alpha + i\sin n\alpha) = 0$$

[using DeMoivre's Theorem]

Equating real parts, we get

$$1 + a_1\cos\alpha + a_2\cos 2\alpha + \ldots + a_n\cos n\alpha = 0$$

$$\Rightarrow \quad |a_1\cos\alpha + a_2\cos 2\alpha + \ldots + a_n\cos n\alpha| = 1.$$

Example 97 If $z \neq 0$ and $2 + \cos\theta + i\sin\theta = 3/z$, then find the value of $2(z + \bar{z}) - |z|^2$.

Ans. 3

Solution We have

$$\frac{9}{|z|^2} = (2 + \cos\theta)^2 + \sin^2\theta = 5 + 4\cos\theta \qquad (1)$$

and

$$\frac{3}{z} + \frac{3}{\bar{z}} = 4 + 2\cos\theta \qquad (2)$$

Eliminating θ from (1) and (2), we get

$$\frac{9}{|z|^2} - 6\left(\frac{1}{z} + \frac{1}{\bar{z}}\right) = -3$$

$$\Rightarrow \quad 3 = 2(\bar{z} + z) - |z|^2$$

Example 98 If $z = \frac{1}{2}(\sqrt{3} - i)$, find the smallest value of positive integer n for which $(z^{89} + i^{97})^{94} = z^n$

Ans. 10.

Solution We have $z = \frac{1}{2}(\sqrt{3} - i) = -\frac{1}{2}i(1 + i\sqrt{3}) = i\omega^2$

where $\omega \neq 1$ is a cube of unity.

$\therefore \quad z^{89} = (i\omega^2)^{89} = i^{89}\,\omega^{178} = i\omega$

Also, $\quad i^{97} = i^{96}\,i = i$

Thus, $\quad (z^{89} + i^{97})^{94} = (i\omega + i)^{94} = [i(-\omega^2)]^{94} = -\omega^2$

Also, $\quad z^n = i^n\,\omega^{2n}$

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$\therefore$ The given equation becomes

$$-\omega^2 = i^n\,\omega^{2n} \quad \Rightarrow \quad i^n\,\omega^{2n-2} = -1$$

This is possible if n is of the type $4k + 2$ and $2n - 2$ is a multiple of 3.
That is, $2(4k + 2) - 2 = 8k + 2$ is multiple of 3. The least value of k for which this is possible is 2. Therefore, $n = 10$.

Example 99 If $1, x_1, x_2, x_3$ are the roots of $x^4 - 1 = 0$ and ω is a complex cube root of unity, find the value of

$$\frac{(\omega^2 - x_1)(\omega^2 - x_2)(\omega^2 - x_3)}{(\omega - x_1)(\omega - x_2)(\omega - x_3)}.$$

Ans. 1

Solution We have

$$x^4 - 1 = (x - 1)(x - x_1)(x - x_2)(x - x_3).$$

Thus,
$$\frac{(\omega^2 - x_1)(\omega^2 - x_2)(\omega^2 - x_3)}{(\omega - x_1)(\omega - x_2)(\omega - x_3)}$$

$$= \frac{1}{\omega + 1}\,\frac{(\omega^2 - 1)(\omega^2 - x_1)(\omega^2 - x_2)(\omega^2 - x_3)}{(\omega - 1)(\omega - x_1)(\omega - x_2)(\omega - x_3)}$$

$$= \frac{1}{\omega + 1}\,\frac{(\omega^2)^4 - 1}{\omega^4 - 1}$$

$$= \frac{1}{\omega + 1}\,\frac{\omega^6 \cdot \omega^2 - 1}{\omega^3 \cdot \omega - 1} = \frac{\omega^2 - 1}{(\omega + 1)(\omega - 1)} = 1$$

Example 100 If $z_n = \left(1 + i\sqrt{3}\right)^n$, find the value of $\sqrt{3}\ Im\,(z_5\ \bar{z}_4)$.

Ans. 1536

Solution
$$z_5\ \bar{z}_4 = \left(1 + i\sqrt{3}\right)^5 \left(1 - i\sqrt{3}\right)^4$$

$$= \left(1 + i\sqrt{3}\right)(1 + 3)^4$$

$$\Rightarrow \quad \sqrt{3}\,Im\,(z_5\,\bar{z}_4) = 3(2^8) = 1536$$

Example 101 Let $a, b, c, d \in \mathbf{R}$ and $a^2 + b^2 = 4, c^2 + d^2 = 2$ and $a + ib = (c + id)\sqrt{x + iy}$, find the value of $x^2 + y^2$.

Ans. 4

Solution
$$\sqrt{x+iy} = \frac{a+ib}{c+id}$$

$$\Rightarrow \quad x + iy = \left(\frac{a+ib}{c+id}\right)^2$$

$$\Rightarrow \quad |x + iy| = \left|\frac{(a+ib)^2}{(c+id)^2}\right| = \left|\frac{a+ib}{c+id}\right|^2$$

$$\Rightarrow \quad \sqrt{x^2+y^2} = \frac{a^2+b^2}{c^2+d^2} = 2$$

$$\Rightarrow \quad x^2 + y^2 = 4$$

Example 102 If $a, b \in \mathbf{R}$, find the number of complex numbers $a + ib$ for which $(a + ib)^2 = (a - ib)^3$.

Solution Let $z = a + ib$, the given equation can be written as

$$z^2 = \bar{z}^3$$

$$\Rightarrow \quad |z|^2 = |\bar{z}|^3$$

$$\Rightarrow \quad |z|^2 = |z|^3$$

$$\Rightarrow \quad |z| = 0 \text{ or } |z|^4 = 1$$

$$\Rightarrow \quad |z| = 0 \text{ or } |z| = 1$$

If $|z| = 0$, then $z = 0$

If $|z| = 1$, we get $\bar{z} = 1/z$ and the equation becomes

$$z^2 = \frac{1}{z^3}$$

$$\Rightarrow \quad z^5 = 1$$

This equation has 5 roots.
Thus, there are 6 complex numbers satisfying the given equation.

Example 103 If the complex number z is such that $|z - 1| \le 1$ and $|z - 2| = 1$ find the maximum possible value $|z|^2$.

Ans. 3

Solution $|z - 1| \le 1$ represent the interior and boundary of the circle with centre at $1 + 0i$ and radius 1 and $|z - 2| = 1$ represent circle with centre at $2 + 0i$ and radius 1.
Clearly the points z satisfying $|z - 1| \le 1$ and $|z - 2| = 1$ lie on the arc DAC.

$\therefore$ $\quad OA \le |z| \le OC\ (= OD)$

As $\angle OCB = \pi/2$, $OC^2 = OB^2 - BC^2 = 4 - 1 = 3$

$$\Rightarrow \quad OC = \sqrt{3}$$

Thus, $\quad |z|^2 \le 3$

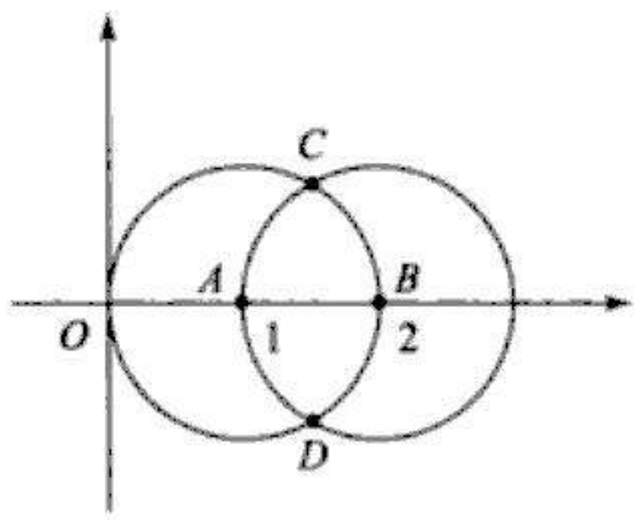

Fig. 1.56

Example 104 If $\left|\frac{z-25}{z-1}\right| = 5$, find the value of $|z|$.

Ans. 5

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Solution $\left|\dfrac{z-25}{z-1}\right| = 5$

$\Rightarrow \quad |z-25|^2 = 25\,|z-1|^2$

$\Rightarrow \quad |z|^2 - 25z - 25\,\bar{z} + 625$

$= 25\,\{|z|^2 - z - \bar{z} + 1\}$

$\Rightarrow \quad 24|z|^2 = 600 \quad \Rightarrow |z|^2 = 25$

$\Rightarrow \quad |z| = 5$

Example 105 If $0 \le \arg z \le \pi/4$, find the least value of $\sqrt{2}\,|z - i|$.

Ans. 1

Solution $0 \le \arg(z) \le \pi/4$ represent the region of the complex plane lying in the first quadrant and bounded by the x-axis and the line $y = x$.

The least value of $|z - i|$ is the length of perpendicular from (0, 1) to $y = x$, which is $\dfrac{1}{\sqrt{2}}$.

Thus, least value of $\sqrt{2}\ |z - i| = 1$

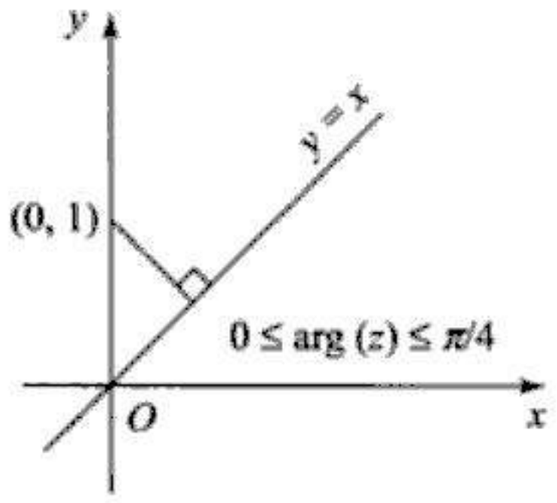

Fig. 1.57

SECTION V

MATRIX MATCH TYPE

Example 106 Number solutions of

(a)	$z^2 + \|z\| = 0$	(p)	1
(b)	$z^2 + \bar{z}^2 = 0$	(q)	3
(c)	$z^2 + 8\bar{z} = 0$	(r)	4
(d)	$\|z - 2\| = 1$ and $\|z - 1\| = 2$	(s)	infinite

Ans.

	p	q	r	s
a		●		
b				●
c		●		
d	●			

Solution

(a) $z^2 = -|z|$

$\Rightarrow \quad |z|^2 = |z| \Rightarrow |z| = 0 \quad$ or $\quad |z| = 1$

If $\quad |z| = 0, \quad$ then $\quad z = 0$

If $\quad |z| = 1, \quad$ then $\quad z^2 = -1 \Rightarrow z = \pm i$

$\therefore \quad z^2 + |z| = 0$ has 3 solutions.

(b) For $z \neq 0$, $z^2 = -\bar{z}^2$

$\Rightarrow \quad \left(\dfrac{z}{\bar{z}}\right)^2 = -1 \quad \Rightarrow \quad \dfrac{z}{\bar{z}} = \pm i$

$\Rightarrow \quad z = \pm i\bar{z}$

Each complex number of the form $a(1 \pm i)$ where $a \in \mathbf{R}$ satisfies the given equation.

(c) $|z^2| = |-8\bar{z}|$

$\Rightarrow \quad |z|^2 = 8|z| \Rightarrow |z| = 0 \quad$ or $\quad |z| = 8$

If $\quad |z| = 8 \Rightarrow z\bar{z} = 64$

$\therefore \quad z^2 = -8\bar{z} = -8^3/z$

$\Rightarrow \quad z^3 = -8^3$

$\Rightarrow \quad z = -8, -8w, -8w^2$

Thus, $z^2 + 8\bar{z}^2 = 0$ has 4 solutions.

(d) Circles $|z - 2| = 1$ and $|z - 1| = 2$ touch each other at $z = 3$.

Thus, $|z - 2| = 1$ and $|z - 1| = 2$ has just one solution.

Example 107 If a, b, c are distinct integers, and $\omega \neq 1$ is a cube root of unity, then value of

(a)	$\|a + b\omega + c\omega^2\|^2$	(p)	1
(b)	$\|a + b\omega + c\omega^2\|^2 + \|a + b\omega^2 + c\omega\|^2$	(q)	≥ 1
(c)	$\left\|\dfrac{a + b\omega + c\omega^2}{a + b\omega^2 + c\omega}\right\|$	(r)	≥ 2
(d)	$\|a + b\omega^2 + c\omega\|$	(s)	2

Ans.

	p	q	r	s
a		●	●	
b			●	
c	●			
d		●	●	

Solution $|a + bw + cw^2|^2$

$= (a + bw + cw^2)\,(a + b\bar{w} + c\bar{w}^2)$

$= (a + bw + cw^2)\,(a + bw^2 + cw)$

$= a^2 + b^2 + c^2 - bc - ca - ab$

$= \dfrac{1}{2}\,[(b - c)^2 + (c - a)^2 + (a - b)^2]$

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As a, b, c are distinct integers the least value of $(b-c)^2 + (c-a)^2 + (a-b)^2 \geq 2$

Also, $|a + b\omega^2 + c\omega|^2 = |a + b\bar{\omega}^2 + c\bar{\omega}|^2$

$= |a + b\omega + c\omega^2|^2$

Tip By just noting that $|a + b\omega + c\omega^2| = |a + b\omega^2 + c\omega|$ we can guess all the answers.

Example 108 Centre of circle

(a) $|z - 2|^2 + |z - 4i|^2 = 20$ (p) $1 - i$

(b) $\left|\frac{z-1}{z+1}\right| = \frac{1}{2}$ (q) $5/3 + 0i$

(c) $z\bar{z} - (1 + i)z - (1 - i)\bar{z} + 7 = 0$ (r) $-4 - i$

(d) $\arg\left(\frac{z+3+4i}{z+5-2i}\right) = \frac{\pi}{2}$ (s) $1 + 2i$

Ans.

	p	q	r	s
a			●	
b	●			
c				●
d		●		

Solution

(a) The given equation represents a circle with the segment joining 2 and $4i$ as diameter. Therefore, its centre is $1 + 2i$

(b) $4|z - 1|^2 = |z + 1|^2$

$\Rightarrow \quad 4(z\bar{z} - z - \bar{z} + 1) = z\bar{z} + z + \bar{z} + 1$

$\Rightarrow \quad 3z\bar{z} - 5z - 5\bar{z} + 3 = 0$

Its circle with centre at $5/3 + 0i$

(c) It represents a circle with centre at $1 - i$

(d) Diameter of the circle is the segment joining $-3 - 4i$ and $-5 + 2i$. Thus, its centre to $-4 - i$.

Example 109 Modulus of complex number whose reciprocal is

(a) $\frac{1}{a} + \frac{1}{b+ic}$ (p) $\frac{\sqrt{a^2+b^2}\sqrt{a^2+c^2}}{|b-c|}$

(b) $\frac{1}{a-ib} - \frac{1}{a-ic}$ (q) $\sqrt{a^2 + (b+c)^2}$

(c) $\frac{b}{a+ib} + \frac{c}{a-ic}$ (r) $\frac{|a|\sqrt{b^2+c^2}}{\sqrt{(a+b)^2+c^2}}$

(d) $\frac{1}{a+ib+ic}$ (s) $\frac{\sqrt{a^2+b^2}\sqrt{a^2+c^2}}{|a||b+c|}$

Ans.

	p	q	r	s
a			●	
b	●			
c		●		
d		●		

Solution

(a) $\frac{1}{z} = \frac{a+b+ic}{ab+iac}$

$\Rightarrow \quad \frac{1}{|z|} = \frac{\sqrt{(a+b)^2+c^2}}{|a|\sqrt{b^2+c^2}}$

$\Rightarrow \quad |z| = \frac{|a|\sqrt{b^2+c^2}}{\sqrt{(a+b)^2+c^2}}$

(b) $\frac{1}{z} = \frac{i(b-c)}{(a-ib)(a-ic)}$

$\Rightarrow \quad \frac{1}{|z|} = \frac{|b-c|}{\sqrt{a^2+b^2}\sqrt{a^2+c^2}}$

$\Rightarrow \quad |z| = \frac{\sqrt{a^2+b^2}\sqrt{a^2+c^2}}{|b-c|}$

(c) $\frac{1}{z} = \frac{a(b+c)}{(a+ib)(a-ic)}$

$\Rightarrow \quad |z| = \frac{\sqrt{a^2+b^2}\sqrt{a^2+c^2}}{|a||b+c|}$

(d) $\frac{1}{z} = \frac{1}{\sqrt{a^2+(b+c)^2}}$

$\Rightarrow \quad |z| = \sqrt{a^2+(b+c)^2}$

Example 110 Modulus of $z = \frac{1-\cos\theta-i\sin\theta}{1+\cos\theta-i\sin\theta}$, when

(a) $-\pi < \theta < 0$ (p) 0

(b) $\theta = 0$ (q) $\tan\left(\frac{1}{2}\theta\right)$

(c) $0 < \theta < \pi$ (r) $-\tan\left(\frac{1}{2}\theta\right)$

(d) $\theta = 2\pi/3$ (s) $\sqrt{3}$

Ans.

	p	q	r	s
a			●	
b	●	●		
c		●		
d		●		

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Solution
$$|z| = \frac{\sqrt{(1-\cos\theta)^2 + \sin^2\theta}}{\sqrt{(1+\cos\theta)^2 + \sin^2\theta}}$$
$$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \left|\tan\left(\frac{\theta}{2}\right)\right|$$
$\therefore$
$$|z| = -\tan\left(\frac{\theta}{2}\right) \quad \text{if } -\pi < \theta < 0$$
$$= 0 \quad \text{if } \theta = 0$$
$$= \tan\left(\frac{\theta}{2}\right) \quad \text{if } 0 < \theta < \pi$$

Example 111 If $z = \cos\alpha + i\sin\alpha$, $0 < \alpha < \pi/6$, then the argument of

(a)	$1 + z^3$	(p)	$2\alpha - \frac{\pi}{2}$
(b)	$1 - z^4$	(q)	$\frac{\pi}{2} + \frac{\alpha}{2}$
(c)	$\frac{1+z^3}{1-z^4}$	(r)	$\frac{3\alpha}{2}$
(d)	$\frac{z^4-1}{z^3+1}$	(s)	$\frac{\pi}{2} - \alpha$

Ans.

	p	q	r	s
a			●	
b	●			
c				
d		●		

Solution

(a) $1 + z^3 = 1 + \cos 3\alpha + i\sin 3\alpha$
$$= 2\cos^2\frac{3\alpha}{2} + 2i\sin\frac{3\alpha}{2}\cos\frac{3\alpha}{2}$$
$$= 2\cos\left(\frac{3\alpha}{2}\right)\left[\cos\left(\frac{3\alpha}{2}\right) + i\sin\left(\frac{3\alpha}{2}\right)\right]$$
$\Rightarrow$ $\arg(1 + z^3) = 3\alpha/2$

(b) $1 - z^4 = 1 - \cos 4\alpha - i\sin 4\alpha$
$$= 2\sin^2 2\alpha - 2i\cos 2\alpha\sin 2\alpha$$
$$= 2\sin 2\alpha\,[\cos(2\alpha - \pi/2) + i\sin(2\alpha - \pi/2)]$$
$\Rightarrow$ $\arg(1 - z^4) = 2\alpha - \pi/2$

(c) $\frac{1+z^3}{1-z^4} = \frac{\cos(3\alpha/2)}{\sin(2\alpha)}\left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$

$\Rightarrow$ $\arg\left(\frac{1+z^3}{1-z^4}\right) = \frac{\pi}{2} - \frac{\alpha}{2}$

(d) $\frac{z^4-1}{z^3+1}$
$$= \frac{\sin(2\alpha)}{\cos(3\alpha/2)}\left[\cos\left(\frac{\alpha}{2} - \frac{\pi}{2}\right) + i\sin\left(\frac{\alpha}{2} - \frac{\pi}{2}\right)\right]$$
$$= \frac{\sin(2\alpha)}{\cos(3\alpha/2)}\left[\cos\left(\frac{\pi}{2} + \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} + \frac{\alpha}{2}\right)\right]$$

Thus, $\arg\left(\frac{z^4-1}{z^3+1}\right) = \frac{\pi}{2} + \frac{\alpha}{2}$

Example 112 Match the equation on the left with the curve they represent on the right

(a)	$\|z - 3\| + \|z - i\| = 10$	(p)	circle
(b)	$\left\|\frac{2z-3}{z-i}\right\| = 2$	(q)	hyperbola
(c)	$z^2 + \bar{z}^2 = 5$	(r)	straight line
(d)	$\left\|\frac{z-6}{z-2i}\right\| = 3$	(s)	ellipse

Ans.

	p	q	r	s
a	●	●	●	●
b	●	●	●	●
c	●	●	●	●
d	●	●	●	●

Solution For (a) and (d) see theory. (b) can be written as
$$\left|\frac{z - 3/2}{z - i}\right| = 1$$
$\Rightarrow$ z lies on the perpendicular bisector of the segment joining 3/2 and i.

(c) Let $z = x + iy$
$\Rightarrow$ $z^2 = x^2 - y^2 + 2ixy$
Thus, $z^2 + \bar{z}^2 = 5$
$\Rightarrow$ $2(x^2 - y^2) = 5$
$\Rightarrow$ $x^2 - y^2 = 5/2$
Thus, z lies on a hyperbola.

Example 113 Value of

(a)	$4\alpha(\beta^4 - \alpha^4)$ if $\alpha + i\beta$, $\beta \neq 0$ is a root of $z^5 = 1$	(p)	0
(b)	$\left\|2 + \frac{1}{z}\right\|^2 + \|2 - z\|^2$ if $z\bar{z} = 1$	(q)	1

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(c) $\left|\frac{1}{2}(z_1+z_2)+w\right|$ (r) 4

$+\left|\frac{1}{2}(z_1+z_2)-w\right|$

if $w=\sqrt{z_1z_2}$, $|z_1|=3$,

$|z_2|=1$

(d) $z_1-z_2+z_3-z_4$ if (s) 10

z_1, z_2, z_3, z_4 represent vertices of a parallelogram.

Ans.

	p	q	r	s
a	(p)	(q)	r	s
b	(p)	q	r	s
c	p	q	r	s
d	(p)	q	r	s

Solution

(a) Let $z=a+i\beta$. then $z^5=1$.

Also, $|z|^5=1 \Rightarrow |z|=1 \Rightarrow \bar{z}=1/z$

Now,

$$4\alpha(\beta^4-\alpha^4)=4\alpha(\beta^2-\alpha^2)|z|^2$$

$$=-\frac{1}{2}(z+\bar{z})\left[(z-\bar{z})^2+(z+\bar{z})^2\right]$$

$$=-(z+\bar{z})(z^2+\bar{z}^2)=-\left(z^3+z+\frac{1}{z}+\frac{1}{z^3}\right)$$

$$=-\frac{1}{z^3}\left(\frac{z^8-1}{z^2-1}\right)=-\left(\frac{z^8-1}{z^5-z^3}\right)=\left(\frac{1-z^3}{1-z^3}\right)$$

$[\because z^5=1]$

$=1$

(b) $|2+\bar{z}|^2+|2-z|^2=2\{2^2+|z|^2\}=10$

(c) $\left|\frac{1}{2}(z_1+z_2)+w\right|+\left|\frac{1}{2}(z_1+z_2)-w\right|$

$$=\frac{1}{2}\left|\left(\sqrt{z_1}+\sqrt{z_2}\right)^2\right|+\frac{1}{2}\left|\left(\sqrt{z_1}-\sqrt{z_2}\right)^2\right|$$

$=|z_1|+|z_2|=4$

(d) z_1, z_2, z_3, z_4 are vertices of a parallelogram if and only if diagonals bisect each other.

Example 114 If $(1+x)^{100}=\sum_{k=0}^{100}C_k x^k$, then value of

(a) $C_0-C_2+C_4-\ldots+C_{100}$ (p) 0

(b) $C_1-C_3+C_5-\ldots-C_{99}$ (q) $2^{49}(2^{49}-1)$

(c) $C_0+C_4+\ldots+C_{100}$ (r) 2^{98}

(d) $C_1+C_5+\ldots+C_{97}$ (s) -2^{50}

Ans.

	p	q	r	s
a	p	q	r	(s)
b	(p)	q	r	s
c	p	q	r	s
d	p	q	r	s

Solution Put $x=i$

$$\Rightarrow \sum_{k=0}^{100}C_k(i)^k=(1+i)^{100}$$

$$=2^{50}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{100}$$

$$=2^{50}(-1+0i)$$

$$\Rightarrow C_0-C_2+C_4-\ldots+C_{100}=-2^{50} \quad (1)$$

and $C_1-C_3+C_5-\ldots-C_{99}=0$ (2)

Also $C_0+C_2+\ldots+C_{100}=2^{99}$ (3)

and $C_1+C_3+\ldots+C_{99}=2^{99}$ (4)

From (1) and (3)

$$C_0+C_4+\ldots+C_{100}=2^{49}(2^{49}-1)$$

From (2) and (4)

$$C_1+C_5+\ldots+C_{97}=2^{98}$$

Example 115 z_1, z_2, z_3 are vertices of a triangle. Match the condition on the left with type of triangle on the right

(a) $z_1^2+z_2^2+z_3^2=z_2z_3+z_3z_1+z_1z_2$ (p) right angled

(b) $Re\left(\frac{z_3-z_1}{z_3-z_2}\right)=0$ (q) obtuse angled

(c) $Re\left(\frac{z_3-z_1}{z_3-z_2}\right)<0$ (r) isosceles and right angled

(d) $\frac{z_3-z_1}{z_3-z_2}=i$ (s) equilateral

Ans.

	p	q	r	s
a	p	q	r	(s)
b	(p)	q	r	s
c	p	q	r	s
d	p	q	(r)	s

Solution

(a) See Theory

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(b) As $\frac{z_3 - z_1}{z_3 - z_2}$ is purely imaginary the triangle is right angled with right angle at z_3.

(c) It is an obtuse angled triangle.

(d) As $|z_3 - z_1| = |z_3 - z_2|$ and $\frac{z_3 - z_1}{z_3 - z_2}$ is purely imaginary, triangle is right angled and isosceles.

SECTION VI

REASONING TYPE

Example 116 **Statement-1:** $\left|\frac{3z+i}{2z+3+4i}\right| = 1.5$ represents a straight line.

Statement-2: Perpendicular bisector of a segment is a straight line.

Ans. (a)

Solution $\left|\frac{3z+i}{2z+3+4i}\right| = 1.5$

$$\Rightarrow \quad \frac{3\left|z+\frac{i}{3}\right|}{2\left|z+\frac{3}{2}+2i\right|} = \frac{3}{2}$$

$$\Rightarrow \quad \left|z+\frac{i}{3}\right| = \left|z+\frac{3}{2}+2i\right|$$

$\Rightarrow z$ is equidistant from $A\left(\frac{-i}{3}\right)$ and $B\left(-\frac{3}{2}-2i\right)$.

Thus, z lies on the perpendicular bisector of AB and hence is a straight line.

Example 117 **Statement-1:** $|z + i|^2 + |z - 3|^2 = 16$ represents a circle.

Statement-2: The set of points in a plane such that sum of the squares of their distances from two fixed points in a plane is a constant is an ellipse.

Ans. (c)

Solution $|z + i|^2 + |z - 3|^2 = 16$

$$\Rightarrow \quad 2z\bar{z} - (3 + i)z - (3 - i)\bar{z} - 6 = 0$$

which represents a circle.

Example 118 **Statement -1:** a, b, c are three distinct real numbers and $\omega \neq 1$ is a cube root of unity, then

$$\left|\frac{a+bw+cw^2}{aw^2+b+cw}\right| = 1$$

Statement-2: If $z \neq 0$, then $\left|\frac{z}{\bar{z}}\right| = 1$.

Ans. (b)

Solution We have

$$|a + b\omega + c\omega^2| = |a\omega^3 + b\omega + c\omega^2| \qquad [\because \omega^3 = 1]$$

$$= |\omega(a\omega^2 + b + c\omega)|$$

$$= |\omega|\,|a\omega^2 + b + c\omega|$$

$$= |a\omega^2 + b + c\omega| \qquad [\because |\omega| = 1]$$

$$\Rightarrow \quad \left|\frac{a+b\omega+c\omega^2}{a\omega^2+b+c\omega}\right| = 1$$

Also, since $|\bar{z}| = |z|$, we get $\left|\frac{z}{\bar{z}}\right| = 1 \ \forall\ z \neq 0$.

Example 119 **Statement-1:** a, b, c are three non-zero real numbers such that $a + b + c = 0$ and z_1, z_2, z_3 are three complex numbers such that $az_1 + bz_2 + cz_3 = 0$, then z_1, z_2 and z_3 lie on a circle.

Statement 2: If z_1, z_2 and z_3 are collinear then

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

Ans. (d)

Solution $a + b + c = 0 \ \Rightarrow \ c = -(a + b)$

$\therefore \quad az_1 + bz_2 + cz_3 = 0$

$\Rightarrow \quad az_1 + bz_2 - (a + b)z_3 = 0$

$$\Rightarrow \quad z_3 = \frac{az_1 + bz_2}{a+b}$$

$\Rightarrow$ z_3 divides the segment joining z_1 and z_2 in the ratio $b : a$

$\Rightarrow$ z_1, z_2 and z_3 are collinear.

For truth of statement-2, see theory.

Example 120 **Statement-1:** If $z \in C$, least value of $y = |z| + |z - 1| + |z - 2|$ is

Statement-2: If $k > 0$, then $|z - z_1| = |z - z_2|$ represents the perpendicular bisector of the segment joining z_1 and z_2.

Ans. (b)

Solution We have

$$y = |z| + |2 - z| + |z - 1| \geq |z + (2 - z) + |z - 1|$$

$$= 2 + |z - 1|$$

$\therefore$ least value of y is 2 which is attained at $z = 1$

For statement-2, see Theory.

Example 121 **Statement-1:** Two nonzero complex numbers z_1 and z_2 lie on a straight line through origin if and only if $z_1\bar{z}_2$ is real.

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Statement-2: Two non-zero complex numbers z_1 and z_2 always lie on a straight line passing through origin if and only if $\bar{z}_1 z_2$ is real.

Ans. (a)

Solution z_1, z_2 will lie on a straight line through the origin if the origin O divides the join of z_1, z_2 is some ratio.

$$\Rightarrow \quad 0 = \frac{z_1 + kz_2}{1+k} \text{ for some } k \in \mathbf{R}.$$

$$\Rightarrow \frac{z_1}{z_2} = -k \in \mathbf{R} \quad \Rightarrow \quad z_1 \bar{z}_2 = -k|z_2|^2 \in \mathbf{R}.$$

Next $z_1 \bar{z}_2 \in \mathbf{R} \Leftrightarrow \bar{z}_1 z_2 \in \mathbf{R}$.

Example 122 **Statement-1:** If A, B, C, D are four points in the complex plane, then

$$AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2$$

Statement-2: If z_1, z_2, z_3, z_4 are four complex numbers then

$$|z_1 + z_2 - z_3 - z_4|^2 \geq 0$$

Ans. (a)

Solution Let z_1, z_2, z_3 and z_4 represent the points A, B, C and D in the complex plane, then

$$AC^2 + BD^2 + AD^2 + BC^2 - AB^2 - CD^2$$
$$= |z_1 - z_3|^2 + |z_2 - z_4|^2 + |z_1 - z_4|^2 + |z_2 - z_3|^2 - |z_1 - z_2|^2 - |z_3 - z_4|^2$$
$$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 - \text{Re}(z_1 \bar{z}_3) - \text{Re}(z_2 \bar{z}_4) - \text{Re}(z_1 \bar{z}_4) - \text{Re}(z_2 \bar{z}_3) - (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 - 2\text{Re}(z_1 \bar{z}_2) - 2\text{Re}(z_3 \bar{z}_4)$$
$$= |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + 2\text{Re}[z_1(z_3 \bar{z}_4)] - 2\text{Re}[z_2(\bar{z}_3 + \bar{z}_4) + 2\text{Re}(z_3 \bar{z}_4)$$
$$= |z_1 + z_2 - z_3 - z_4|^2 \geq 0.$$

EXERCISE

LEVEL 1

SECTION I

SINGLE CORRECT CHOICE TYPE

1. If $|z_1| \neq 1$, $\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = 1$, then
   (a) $|z_2| = 1$ (b) $|z_2| = 0$
   (c) $|z_2| > 1$ (d) $0 < |z_2| < 1$

2. Let $z_1, z_2, z_3 \in \mathbf{C}$ and
$$f(z_1, z_2, z_3) = \text{sgn} \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{vmatrix}$$
where $\text{sgn } x = \frac{x}{|x|}$ if $x \neq 0$ and $\text{sgn}(0) = 0$, then for $a \neq 0$,
   (a) $f(z_1, z_2, z_3) = f(az_1, az_2, az_3)$
   (b) $f(z_1, z_2, z_3) = f(az_2, az_1, az_3)$
   (c) $f(z_1, z_2, z_3) = f(az_3, az_2, az_1)$
   (d) none of these

3. If f is as in Question 2, then for $a \in \mathbf{C}$,
   (a) $f(z_1, z_2, z_3) = f(z_1 + a, z_2 + a, z_3 + a)$
   (b) $f(z_1, z_2, z_3) = |a| f(z_1, z_2, z_3)$
   (c) $f(z_1, z_2, z_3) = |a| f(a_1, a_2, a_3)$
   where a_1, a_2, a_3 is a permutation of z_1, z_2, z_3.
   (d) none of these

4. If z_1, z_2, z_3 are three complex numbers, then $|z_2 + z_3|^2 + |z_3 + z_1|^2 + |z_1 + z_2|^2$ equal
   (a) $2|z_1 + z_2 + z_3|^2$
   (b) $|z_1 + z_2 + z_3|^2$
   (c) $2|z_1 + z_2 + z_3|^2 - |z_1|^2 - |z_2|^2 - |z_3|^2$
   (d) none of these

5. If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then
$2i \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ equals
   (a) $z_1 \bar{z}_2 - z_2 \bar{z}_1$ (b) $\bar{z}_1 z_2 - z_1 \bar{z}_2$
   (c) $|z_1|^2 - |z_2|^2$ (d) $|z_1|^2 - |z_1 - z_2|^2$

6. If $\text{Re}(a), \text{Re}(b) > 0$, and $x = |a - b| - |\bar{a} + b|$, then
   (a) $x < 0$ (b) $x > 0$
   (c) $x \geq 1$ (d) $0 < x < 1$

7. If $|z| = 1$, $|a| \neq 1$, and $x = z/(z - a)(1 - z\bar{a})$, then
   (a) $x < 0$ (b) $x > 0$
   (c) $x \geq 1$ (d) $0 < x < 1$

8. Let $z_1, z_2 \in \mathbf{C}$ and $x = |z_1 z_2| - \text{Re}(z_1 z_2) - \frac{1}{2}|\bar{z}_1 - z_2|^2 + \frac{1}{2}(|z_2| - |z_1|)^2$ then
   (a) $x < 0$ (b) $x = 0$
   (c) $x \geq 1$ (d) $0 < x < 1$

9. If the ratio $\frac{1 - z}{1 + z}$ is purely imaginary, then
   (a) $|z| \leq 1/2$ (b) $|z| = 1$
   (c) $|z| > 1$ (d) $0 < |z| < 1$

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10. If $|a| \neq |b|$, then the equation $az + b\bar{z} + c = 0$ represents
    (a) a circle (b) an ellipse
    (c) a straight line (d) a point
11. If $|a| = |b|$ and $\bar{a}c \neq b\bar{c}$, then the equation $az + b\bar{z} + c = 0$ has
    (a) no solution
    (b) exactly one solution
    (c) finitely many solutions
    (d) infinitely many solutions
12. If $|a| = |b| \neq 0$ and $\bar{a}c = b\bar{c}$, then $az + b\bar{z} + c = 0$ represents
    (a) a circle (b) an ellipse
    (c) a straight line (d) a point
13. The equation $\bar{z} = \bar{z}_0 + A(z - z_0)$, where A is a constant, represents
    (a) straight line (b) a circle
    (c) a point (d) none of these
14. If m is slope of the straight line in Question 13, then
    (a) $A = \dfrac{1+im}{1-im}$ (b) $A = \dfrac{1-im}{1+im}$
    (c) $A = \dfrac{i+m}{1-mi}$ (d) $A = \dfrac{i-m}{1+mi}$
15. If θ is the angle which the line of Question 13 makes with the positive direction of real axis, then
    (a) $A = e^{2i\theta}$ (b) $A = e^{-2i\theta}$
    (c) $A = e^{i\theta}$ (d) $A = e^{-i\theta}$
16. The equation
    $$\bar{z} = \bar{a} + \frac{r^2}{z-a}, \quad r > 0$$
    represents
    (a) an ellipse
    (b) a parabola
    (c) a circle
    (d) a straight line through point $\bar{a}$
17. If $\bar{a} \neq b$, then the equation $z\bar{z} + az + b\bar{z} + c = 0$ represents
    (a) a circle
    (b) an ellipse
    (c) a straight line
    (d) finite number of points in **C**
18. If $\bar{a} = b$, $c \in \mathbf{R}$ and $|b|^2 > c$, then $z\bar{z} + az + b\bar{z} + c = 0$ represents
    (a) a circle
    (b) a parabola
    (c) a straight line
    (d) finite number of points in
19. If $\bar{a} = b$, $c \in \mathbf{R}$, and $|b|^2 < c$, then the equation $z\bar{z} + az + b\bar{z} + c = 0$
    (a) has no solution
    (b) exactly two solutions
    (c) infinite number of solution
    (d) none of these
20. If $z_1 + z_2 + z_3 = 0$, then $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$ equals
    (a) $\dfrac{1}{3}|z_1|^2 + 2|z_2|^2 + 2|z_3|^2$
    (b) $\dfrac{2}{3}(|z_1|^2 + |z_2|^2 + |z_3|^2)$
    (c) $2(|z_1|^2 + |z_2|^2 + |z_3|^2)$
    (d) $3(|z_1|^2 + |z_2|^2 + |z_3|^2)$
21. If $|z_1| = |z_2| = |z_3| = |z_4| = 1$, and $z_1 + z_2 + z_3 + z_4 = 0$, then least value of the expression $E = |z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2 + |z_4 - z_1|^2$ is
    (a) 6 (b) 8
    (c) 10 (d) 12
22. If $z_1 + z_2 + z_3 + z_4 = 0$, then the expression $|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2 + |z_4 - z_1|^2 - 2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)$ is equal to 0, if and only if,
    (a) $z_1 = -z_3$ and $z_4 = -z_2$
    (b) $z_1 = -z_4$ and $z_2 = -z_3$
    (c) $z_1 = z_2$ and $z_3 = z_4$
    (d) $z_1 = z_3$ and $z_2 = z_4$
23. If $|u| = |v| = 1$, $uv \neq -1$, and $z = \dfrac{u-v}{1+uv}$, then
    (a) $|z| = 1$ (b) $\mathrm{Re}(z) = 0$
    (c) $\mathrm{Im}(z) = 0$ (d) $\mathrm{Re}(z) = \mathrm{Im}(z)$
24. If $|u| < 1$, $|v| < 1$, and $z = \dfrac{u-v}{1+\bar{u}v}$, then least value of $|z|$ is
    (a) $\dfrac{|u|-|v|}{1+|u||v|}$ (b) $\dfrac{|u|+|v|}{1-|u||v|}$
    (c) $\dfrac{||u|-|v||}{1-|u||v|}$ (d) none of these
25. Let L be the straight line $\bar{a}z + a\bar{z} + c = 0$, $a \neq 0$, $c \in \mathbf{R}$, and OA be the straight line joining O and $A(a)$, then
    (a) $L \parallel OA$
    (b) $L \perp OA$
    (c) L makes an angle of $\pi/4$ with OA
    (d) none of these.

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26. Locus of point z so that z, i, and iz are collinear, is
(a) a straight line
(b) a circle
(c) an ellipse
(d) a rectangular hyperbola

27. The number of roots of the equation $|z^2| - 5|z| + 1 = 0$ is
(a) 0 (b) 2
(c) 4 (d) infinite

28. The set of points in $\mathbf{C}$ satisfying the inequality $|\arg(z) - \pi/2| < \pi/2$ is given by
(a) $\{z : \text{Re}(z) > 0\}$
(b) $\{z : \text{Im}(z) < 0\}$
(c) $\{z : \text{Im}(z) > 0\}$
(d) $\{z : \text{Re}(z) = \text{Im}(z)\}$

29. If $n \geq 3$ and $1, \alpha_1, \alpha_2, \ldots \alpha_{n-1}$ are nth roots of unity, then value of $\sum_{1 \leq i < j \leq n-1} \alpha_i \alpha_j$ is
(a) 0 (b) 1
(c) -1 (d) $(-1)^n$

30. The quadrilateral formed by the points with affix $0, z, z + iz$ and iz is
(a) parallelogram (b) rhombus
(c) rectangle (d) square

31. If $|z - 2| = 2|z - 1|$, then $3|z|^2 - 4\text{Re}(z)$ equals
(a) 0 (b) -1
(c) 1 (d) none of these

32. Sum of the common roots of $z^{2006} + z^{100} + 1 = 0$ and $z^3 + 2z^2 + 2z + 1 = 0$ is
(a) 0 (b) -1
(c) 1 (d) 2

33. If z_1 and z_2 lie on the same side the line $\bar{a}z + \bar{a}z + b = 0$, where $a \in \mathbf{C}$, $a \neq 0$, $b \in \mathbf{R}$, then the ratio $\dfrac{\bar{a}z_1 + a\bar{z}_1 + b}{\bar{a}z_2 + a\bar{z}_2 + b}$ is
(a) purely imaginary
(b) a positive real number
(c) a negative real number
(d) 0

34. If z_1 and z_2 are the roots of the equation $z^2 + az + b = 0$, then prove that the origin, z_1 and z_2 form an equilateral triangle if and only if
(a) $a^2 = 3b$ (b) $b^2 = 3a$
(c) $a^2 + 3b = 0$ (d) $b^2 + 3a = 0$

35. If $a > 0$ and $z|z| + az + 2a = 0$ then z must be
(a) purely imaginary
(b) a positive real number

36. If $\dfrac{3}{2 + \cos\theta + i\sin\theta} = a + ib$, then $(a-2)^2 + b^2$ equals
(a) 0 (b) 1
(c) -1 (d) 2

37. If $z = 4 + i\sqrt{7}$, then value of $z^3 - 4z^2 - 9z + 91$ equals
(a) 0 (b) 1
(c) -1 (d) 2

38. The number of complex numbers satisfying the equation $|z| = 2$ and $|z| = |z - 1|$ is
(a) 0 (b) 1
(c) 2 (d) infinite

39. If $z = \dfrac{mz_1 + z_2}{m+1}$, then distance of point z from the line joining $z_1 + 1$ and $z_2 + 1$ is
(a) 0 (b) 1
(c) $\dfrac{2m}{m+1}$ (d) $\dfrac{m}{m+1}$

40. If z_1, z_2, z_3 are three complex numbers such that $4z_1 - 7z_2 + 3z_3 = 0$, then z_1, z_2, z_3 are
(a) vertices of a scalane triangle
(b) vertices of a right triangle
(c) points on a circle
(d) collinear points

SECTION II

MULTIPLE CORRECT CHOICE TYPE

41. If a complex number z has modulus 1 and argument $\pi/3$, then $z^2 + z$
(a) is purely imaginary
(b) has modulus $\sqrt{3}$
(c) lies on the imaginary axis
(d) none of these

42. If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1 \bar{z}_2) = 0$, then the pair of complex numbers, $w_1 = a + ic$ and $w_2 = b + id$ satisfy
(a) $|w_1| = 1$ (b) $|w_2| = 1$
(c) $|w_1 \bar{w}_2| = 1$ (d) $Re(\bar{w}_1 w_2) = 0$

43. If $2\cos\theta = x + \dfrac{1}{x}$ and $2\cos\varphi = y + \dfrac{1}{y}$, then
(a) $x^n + \dfrac{1}{x^n} = 2\cos(n\theta)$
(b) $\dfrac{x}{y} + \dfrac{y}{x} = 2\cos(\theta - \varphi)$

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(c) $xy + \dfrac{1}{xy} = 2\cos(\theta + \varphi)$

(d) none of these

44. If z_1, z_2 are two complex numbers satisfying the relation

$$\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = 1$$

then

(a) $\dfrac{z_1}{z_2}$ is purely imaginary

(b) $\dfrac{z_1}{z_2} > 0$

(c) $z_1 = ikz_2$

(d) $\angle z_1 O z_2 = \pi/2$

45. Points A, B and with affixes z_1, z_2 and $(1 - i)z_1 + iz_2$ are the vertices of

(a) an isosceles triangle

(b) an equilateral triangle

(c) a right triangle

(d) an obtuse angled triangle

46. If $|z_1 - z_2| = |z_1| + |z_2|$, then

(a) $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{\pi}{2}$

(b) $\arg\left(\dfrac{z_1}{z_2}\right) = \pi$

(c) $z_1\bar{z}_2 + \bar{z}_1 z_2 \le 0$

(d) z_1, O, z_2 lie on a straight line.

47. If z_1 lies on $|z| = 1$ and z_2 lies on $|z| = 2$, then

(a) $3 \le |z_1 - 2z_2| \le 5$

(b) $1 \le |z_1 + z_2| \le 3$

(c) $|z_1 - 3z_2| \ge 5$

(d) $|z_1 - z_2| \ge 1$

48. If z satisfies the inequality $|z - 1 - 2i| \le 1$, then

(a) $\min(\arg(z)) = \tan^{-1}\left(\dfrac{3}{4}\right)$

(b) $\max(\arg(z)) = \dfrac{\pi}{2}$

(c) $\min(|z|) = \sqrt{5} - 1$

(d) $\max(|z|) = \sqrt{5} + 1$

49. If O, z_1 and z_2 are vertices of an equilateral triangle, then

(a) $|z_1| = |z_2| = |z_1 - z_2|$

(b) $(z_1 + z_2)^2 = 3z_1z_2$

(c) $|\arg(z_1) - \arg(z_2)| = \pi/3$

(d) $|z_1 + z_2| = 2|z_1| + |z_2|$

50. Which of the following value(s) of k, the equation

$$\left|\frac{3z - 1 - i}{z + 2 + i}\right| = k$$

represents a circle?

(a) 3 (b) 2

(c) 5 (d) π

51. Let $z \in \mathbf{C}$ and $x = |z - 1| + |z| + |z + 1|$, then

(a) minimum value of x is 2

(b) minimum value is attained on at least three points

(c) minimum value of x is attained at infinite number of points

(d) maximum value of x is attained on the circle $|z| = \sqrt{2}$

52. If z_1, z_2 and z_3 are vertices of an equilateral triangle whose orthocentre is, the origin, then

(a) $z_1 + z_2 + z_3 = 0$

(b) $|z_1| = |z_2| = |z_3|$

(c) $z_2z_3 + z_3z_1 + z_1z_2 = 0$

(d) Oz_1 is perpendicular to z_2z_3.

53. If the point z_1 is the reflection of a point z_2 through the line $\bar{b}z + b\bar{z} = c$, where $b \ne 0$, then $b\bar{z}_1 + b\bar{z}_2$ is

(a) proportional to $|z_1|^2 - |z_2|^2$

(b) equal to c

(c) inversely proportional to c

(d) none of these

54. Equation of line making equal intercepts on the axes and touching $\sqrt{2}\,|z| = 1$ is

(a) $x + y = 1$ (b) $x - y = -1$

(c) $x - y = 1$ (d) $x + y = -1$

55. If $z_1, z_2 \in \mathbf{C}$ are such that

$$\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = 1,$$

then

(a) $|z_1| = 1$ (b) $|z_2| = 1$

(c) $z_1 = e^{i\theta}$, $\theta \in \mathbf{R}$ (d) $z_2 = e^{i\phi}$, $\phi \in \mathbf{R}$

56. If z_1 and z_2 are two complex numbers such that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$, then

(a) $z_1\bar{z}_2$ is purely imaginary

(b) z_1/z_2 is purely imaginary

(c) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$

(d) O, z_1, z_2 are vertices of a right triangle

57. If $z^2/(z - 1)$ is always real, then z can lie on

(a) the real axis (b) the imaginary axis

(c) a circle (d) a parabola

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58. If z_1 and z_2 are two complex numbers and $c > 0$, then $|z_1 - z_2|^2 \geq (1 - c)\,|z_1|^2 + (1 - 1/c)\,|z_2|^2$ is
(a) true for all $z_1, z_2 \in \mathbf{C}$
(b) true if $z_1 = \bar{z}_2$
(c) false for all $z_1, z_2 \in \mathbf{C}$
(d) true if z_1/z_2 is real

59. If $z \in \mathbf{C}$, the minimum value of $|z| + |z - 5|$ is attained at
(a) $z = 0$ (b) $z = 5$
(c) $z = 5/2$ (d) for all $z \in [0, 5]$

60. The curve represented by
$$z = \frac{3}{2 + \cos\theta + i\sin\theta},\ \theta \in \mathbf{R}$$
(a) never meets the imaginary axis
(b) meets the real axis in exactly two points
(c) has maximum value of $|z|$ as 3
(d) has minimum value of $|z|$ as 1

61. Which of the following satisfies $|z + 1| = z + 2 + 2i$
(a) $1 + i$ (b) $-\frac{1}{2} - i$
(c) $\frac{1}{2} - 2i$ (d) 0

62. If complex numbers z_1, z_2, z_3 are the vertices of a triangle, then the complex number z which makes the triangle a parallelogram is given by
(a) $z_1 + z_2 - z_3$ (b) $z_1 + z_3 - z_2$
(c) $z_2 + z_3 - z_1$ (d) $z_1 + z_2 + z_3$

63. If $z_1, z_2, \ldots, z_n$ lie on $|z| = r$ and $\text{Re}\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{z_j}{z_k}\right) = 0$, then
(a) $\sum_{j=1}^{n} z_j = 0$ (b) $\left|\sum_{j=1}^{n} z_j\right| = 0$
(c) $\sum_{j=1}^{n}\frac{1}{z_j} = 0$ (d) none of these

64. Let S denote the set of complex numbers z such that $\log_{1/3}(\log_{1/2}(|z|^2 + 4|z| + 3)) < 0$, then S is contained in
(a) $(0, 1)$ (b) $\{z \mid Re(z) > 0\}$
(c) $\{z \mid Re(z) > 3\}$ (d) none of these

65. If $z \in \mathbf{C}$ and $\log_{|z+i|}|3 + 4i| < \log_{|z+i|}|5 + 12i|$, then z lies.
(a) inside a circle passing through the origin
(b) outside a circle passing through the origin
(c) inside the disc $|z + i| < 5$
(d) outside the curve $z = -i + e^{i\theta},\ \theta \in \mathbf{R}$

SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos. 66 to 70

For $z = x + iy,\ x, y \in \mathbf{R}$, define
$$e^z = e^x(\cos y + i\sin y),$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$
and
$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

66. Number of value of z for which $e^z = 0$ is
(a) 0 (b) 1
(c) two (d) infinite

67. Number of value of z for which $\sin z = 0$ is
(a) 0 (b) 1
(c) two (d) infinite

68. Number of values of z for which $\cos z = 0$ is
(a) 0 (b) 1
(c) two (d) infinite

69. Solution set of $\sin(iy) = 0$ is
(a) $\{n\pi \mid n \in \mathbf{I}\}$ (b) $\{0\}$
(c) $\left\{(2n-1)\frac{\pi}{2}, n \in \mathbf{I}\right\}$ (d) none of these

70. $e^{z_1} = e^{z_2}$ if and only if z_1 and z_2 differ by
(a) $n\pi$ (b) $2n\pi$
(c) $(2n + 1)\pi/2$ (d) none of these
where $n \in \mathbf{I}$.

Paragraph for Question Nos. 71 to 75

For $z \neq 0$, define
$$\log z = \log|z| + i\,(\arg z)$$
where $-\pi < \arg(z) \leq \pi$
i.e. $\arg(z)$ stands for the principal argument of z.

71. $\log z = 1$ if and only if z equals
(a) e (b) $e, -e$
(c) $e + 2k\pi i,\ k \in \mathbf{I}$ (d) $\pm e + 2k\pi i,\ k \in \mathbf{I}$

72. For $z, z' \neq 0$, $\log(zz')$ and $\log z + \log z'$ differ by a multiple of
(a) $n\pi$ (b) $2n\pi$
(c) 0 (d) $(2n + 1)\pi/2$,
where $n \in \mathbf{I}$.

73. $z\log(e^{x+iy})$ equals
(a) $e^x + \arg(x + iy)$
(b) $x + iy$
(c) $x + iy + 2k\pi,\ k \in \mathbf{I}$
(d) none of these

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74. $e^{\log z}$ equals
(a) z (b) $|z|$
(c) $z + 2k\pi i,\ k \in \mathbf{I}$ (d) none of these

75. $\log(-i)$ equals
(a) $\pi i/2$ (b) πi
(c) $-\pi i/2$ (d) 0

Paragraph for Question Nos. 76 to 80

If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the roots of polynomial equation
$f(x) = a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m = 0$,
then
$f(x) = a_0(x - \alpha_1) \ldots (x - \alpha_m)$.

and $$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1} + \cdots + \frac{1}{x-\alpha_m}$$

76. If $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$, then numerical value of
$$1 + \frac{1}{2-\omega} + \frac{1}{2-\omega^2} + \cdots + \frac{1}{2-\omega^{n-1}}$$
equals
(a) $\frac{n(2^{n-1})}{2^n+1}$ (b) $\frac{n(2^{n-1})}{2^n-1}$
(c) 0 (d) 1

77. If $\alpha = \cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)$, then
$(x-1)(x+1)(x-a)(x-\bar{\alpha}) \ldots (x-\alpha^{n-1})(x-\bar{\alpha}^{n-1})$
equals
(a) $x^n - 1$ (b) $x^n + 1$
(c) $x^{2n} + 1$ (d) $x^{2n} - 1$

78. $(x^2 - 2x\cos\frac{\pi}{n} + 1)(x^2 - 2x\cos\frac{2\pi}{n} + 1)$
$\ldots (x^2 - 2x\cos\frac{n-1}{n}\pi + 1)$ equals
(a) $1 + x + \ldots + x^{n-1}$
(b) $1 - x + x^2 - \ldots + (-1)^{n-1} x^{n-1}$
(c) $1 + x + \ldots + x^{2n-2}$
(d) 0

79. $2^{n-1}(\cos\theta - \cos\frac{\pi}{n})(\cos\theta - \cos\frac{2\pi}{n})$
$\ldots (\cos\theta - \cos\frac{n-1}{n}\pi)$ equals
(a) $\frac{\sin\theta}{\sin(n\theta)}$ (b) $\frac{\sin n\theta}{\sin\theta}$
(c) 1 (d) $-n\cos\theta$

80. $2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{n-1}{n}\pi$ equals
(a) n (b) 0
(c) $2n$ (d) $-n$

Paragraph for Question Nos. 81 to 85

Let $\omega = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ and $\alpha = \omega + \omega^2 + \omega^4$
and $\beta = \omega^3 + \omega^5 + \omega^6$.

81. $\alpha + \beta$ equals
(a) 0 (b) −1 (c) −2 (d) 1

82. $\alpha\beta$ equals
(a) −1 (b) 0 (c) 1 (d) 2

83. α and β are roots of the equations
(a) $x^2 + x + 1 = 0$ (b) $x^2 + x + 2 = 0$
(c) $x^2 + 3x + 5 = 0$ (d) none of these

84. 2α equals
(a) $-1 + \sqrt{7}i$ (b) $-1 - \sqrt{7}i$
(c) $1 + 7i$ (d) $1 - 7i$

85. $\sum_{k=0}^{6} \omega^{k^2}$ equals
(a) i (b) $\sqrt{7}i$ (c) $-i$ (d) $-\sqrt{7}i$

SECTION IV

INTEGER ANSWER TYPE

86. If $\frac{3iz_2}{5z_1}$ is purely real, then find $5\left|\frac{3z_1 + 7z_2}{3z_1 - 7z_2}\right|$.

87. If $-3 + x^2yi$ and $x^2 + y + 4i$ are conjugate of each other, then find $|x| + |y|$.

88. If $z + \sqrt{2}\,|z+1| + i = 0$, then find $|z|^2$.

89. If $|z| < 1/2$ and $4|(1+i)z^3 + iz| < k$, find the least integral value of k.

90. If $z = \cos\theta + i\sin\theta$ is a root of
$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0,$$
where $p_1, p_2, \ldots, p_n \in \mathbf{R}$,
find the value of
$22 + 7(p_1\sin\theta + p_2\sin 2\theta + \ldots + p_n\sin(n\theta))$.

91. Find the value of
$$2^{100}\sin\left(\frac{\pi}{100}\right)\sin\left(\frac{2\pi}{100}\right)\cdots\sin\left(\frac{99}{100}\pi\right).$$

92. If $8iz^3 + 12z^2 - 18z + 27i = 0$, find $4|z|^2$.

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93. If $|z + 4| = 3$, find the greatest possible value of $|z - 7|$

94. If $\omega \neq 1$ is a cube root of unity, and

$$\frac{1}{a+\omega}+\frac{1}{b+\omega}+\frac{1}{c+\omega}=\frac{2}{\omega}$$

and $$\frac{1}{a+\omega^2}+\frac{1}{b+\omega^2}+\frac{1}{c+\omega^2}=\frac{2}{\omega^2}$$

find the value of

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}$$

95. If α, β are the roots of $x^2 - 2x + 4 = 0$, Find the value of $\alpha^6 + \beta^6$.

96. If the points $a(\cos \alpha + i \sin \gamma)$, $b(\cos \beta + i \sin \beta)$ and $c(\cos \gamma + i \sin \gamma)$ are collinear, then find $|z|$ where

$$z = bc \sin(\beta - \gamma) + ca \sin(\gamma - \alpha) + ab \sin(\alpha - \beta) + 3i$$

97. If $\omega = \cos\left(\frac{2\pi}{7}\right) + i \sin\left(\frac{2\pi}{7}\right)$ and $\alpha = \omega + \omega^2 + \omega^4$, $\beta = \omega^3 + \omega^5 + \omega^6$, find the value of $\alpha\beta$.

SECTION V

MATRIX MATCH TYPE

98. Match the curve on the right on which the roots of the equation in on the left lie.

(a) $z^n = (1 - z)^n$	(p) $x = 0$
(b) $(1 - z)^n = (1 + z)^n$	(q) $x = 1/2$
(c) $z^n = 2 + \sqrt{21}\,i$	(r) circle with radius 5
(d) $(z - i)^n = \sqrt{21} + 2i$	(s) circle with centre at $z = i$

99. α, β are cube roots of i which are not purely imaginary.

(a) $\alpha^2 + \beta^2$	(p) -1
(b) $\alpha^4 + \beta^4$	(q) 2
(c) $\alpha^6 + \beta^6$	(r) 1
(d) $\alpha^{12} + \beta^{12}$	(s) -2

100. If $a, b \in \mathbf{R}$ and $a^2 - 4b \geq 0$, the equation $z^4 + az^2 + b = 0$ has

(a) four real roots if	(p) $a = 0, b = 0$
(b) two real and two imaginary roots if	(q) $a > 0, b > 0$
(c) four imaginary roots if	(r) $a \leq 0, b \geq 0$
(d) four equal roots	(s) $a \leq 0$

101. Domain of definition of function

(a) $\sin^{-1}\|z\|$	(p) $z \neq 1$
(b) $\tan^{-1}\|z\|$	(q) $0 < \|z\| \leq 1$
(c) $\log\|z\| + \cos^{-1}\|z\|$	(r) $\|z\| \leq 1$
(d) $\log\|z - 1\| + \sec^{-1}\|z\|$	(s) $\|z\| \geq 1$

102. Centre of circle

(a) $\left\|\frac{z-9}{z-1}\right\| = 3$	(p) $5 + 3i$
(b) $z = 5 + 3i + 2\left(\frac{1+it}{1-it}\right)$, $t \in \mathbf{R}$	(q) $1 + 4i$
(c) passing through $3, -\sqrt{3}i, \sqrt{3}i$	(r) 0
(d) having join of $3 + 5i$ and $-1 + 3i$ as diameter	(s) 1

103. If α, β are roots of $z^2 \sin^2\theta - z \sin 2\theta + 1 = 0$ where $0 < \theta < \pi/2$ and n is an integer, then value of

(a) $(\alpha\beta)^n$	(p) $2^n \cot^n \theta$
(b) $(\alpha + \beta)^n$	(q) $\operatorname{cosec}^{2n} \theta$
(c) $\alpha^n + \beta^n$	(r) $2 \operatorname{cosec}^n \theta \cos(n\theta)$
(d) $\|\alpha\|^n + \|\beta\|^n$	(s) $2 \operatorname{cosec}^n \theta$

104. For $z = x + iy$, define $e^z = e^x(\cos y + i \sin y)$. The set of complex numbers z such that

(a) $e^z = 2$	(p) $\{2 + 2n\pi i \mid n \in \mathbf{I}\}$
(b) $e^z = 1 - \sqrt{3}i$	(q) $\{\log 2 + i(2n - 1/3)\pi \mid n \in \mathbf{I}\}$
(c) $e^z = e^2$	(r) $\{(2n + 1)\pi i \mid n \in \mathbf{I}\}$
(d) $e^z = -1$	(s) $\{\log 2 + 2n\pi i \mid n \in \mathbf{I}\}$

SECTION VI

REASONING TYPE

105. Circum-centre of ΔABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is $S(z_0)$.
The altitude from vertex A to side BC meets the circumcircle at $E(z_4)$.

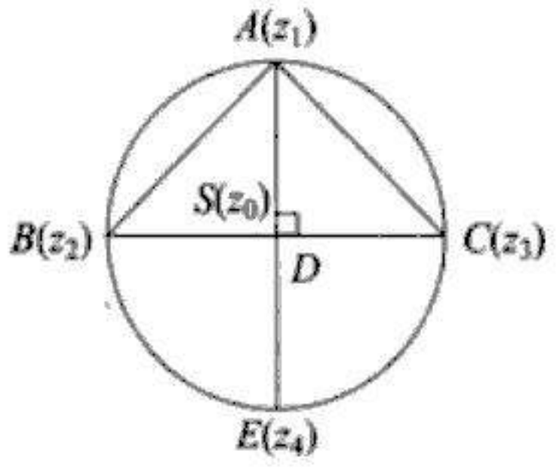

Fig. 1.58

Statement-1: Affix of the orthocentre of ΔABC is $z_1 + z_2 + z_3 - 2z_0$.
Statement-2: z_4 is given by

$$z_0 - \frac{(z_0 - z_2)(z_0 - z_3)}{z_0 - z_1}$$

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106. **Statement-1:** Let $a, b, c \in \mathbf{R}$ and $z_1, z_2 \in$ are complex numbers then $az_1\bar{z}_1 + b(z_1\bar{z}_2 + \bar{z}_1 z_2) + cz_2\bar{z}_2$ is purely real.
Statement-2: If $a, b, c \in \mathbf{R}$ and $\alpha \notin \mathbf{R}$ is a root of $ax^2 + bx + c = 0$, then $a(\alpha + \bar{\alpha}) + b = 0$ and $a\alpha\bar{\alpha} = c$.

107. Let A_1 and A_2 represent z_1 and z_2.
Statement-1: Equation of circle with A_1A_2 as diameter is $|2z - z_1 - z_2| = |z_1 - z_2|$
Statement-2: If z is any point on the circle with A_1A_2 as diameter, then

$$(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) = 0$$

108. **Statement-1:** If z is a complex number such that $\left|\frac{2z+3}{3z+1}\right| = \sqrt{\frac{5}{3}}$, then z lies on a circle.
Statement-2: If a, b are two complex numbers and $k > 0, k \neq 1$, then

$$\left|\frac{z+a}{z+b}\right| = k$$

represents a circle.

109. **Statement-1:** If $z^2 - z + 1 = 0$ and n is a natural number, then

$$\sum_{k=1}^{n}\left(z^k + z^{-k}\right)^2 = n + 3\left[\frac{n}{3}\right]$$

where $[x]$ denotes the greatest integer $\leq x$.
Statement-2: If $w \neq 1$ is a cube root of unity, then

(a) $w^k + \bar{w}^k = \begin{cases} 0 - 1 & \text{if } k \text{ is not a multiple of 3} \\ 2 & \text{if } k \text{ is a multiple of 3} \end{cases}$

110. **Statement-1:** Two lines $a\bar{z} + \bar{a}z + b = 0 = 0$, $a_1\bar{z} + \bar{a}_1 z + b_1 = 0$ (where $a, a_1 \in \mathbf{C}$, $a, a_1 \neq 0$ and $b, b_1 \in \mathbf{R}$) are parallel if and only if $\frac{a}{a_1}$ is real.
Statement-2: Two lines $a\bar{z} + \bar{a}z + b = 0$, $a_1\bar{z} + \bar{a}_1 z + b_1 = 0$ (where $a, a_1 \in \mathbf{C}$, $a, a_1 \neq 0$ and $b, b_1 \in \mathbf{R}$) are perpendicular if and only if $\frac{a}{a_1}$ is purely imaginary.

111. **Statement-1:** The minimum value of

$$f(\theta) = \left|\frac{2i}{3 - ie^{i\theta}}\right| \text{ is } \frac{1}{\sqrt{2}}$$

Statement-2: The maximum value of

$$f(\theta) = \left|\frac{2i}{3 - ie^{i\theta}}\right| \text{ is } 1.$$

112. **Statement-1:** Let m, n be two natural numbers such that $m = a^2 + b^2$ and $n = c^2 + d^2$ for some $a, b, c, d \in \mathbf{I}$, then there exist $e, f \in \mathbf{I}$ such that $mn = e^2 + f^2$
Statement-2: If $z_1, z_2 \in \mathbf{C}$, then $|z_1 + z_2| = |z_1| + |z_2|$ if and only if $z_1z_2 > 0$.

LEVEL 2

SECTION I

SINGLE CORRECT CHOICE TYPE

1. If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 = 0$, then z_1, z_2, z_3 are the vertices of a triangle which is
(a) right angled (b) isosceles
(c) obtuse angled (d) equilateral

2. If z_1, z_2, z_3 are the vertices of an isosceles triangle with right angle at z_2, then value of

$$(z_1 - z_2)^2 + (z_2 - z_3)^2$$

is
(a) 0 (b) 1
(c) -1 (d) $3i$

3. If z_1, z_2, z_3 are the vertices of an isosceles triangle, then value of

$$(z_2 - z_3)^2 + (z_3 - z_1)^2 + (z_1 - z_2)^2$$

is
(a) 0 (b) 1
(c) -1 (d) $3i$

4. If roots of the equation $z^2 + \alpha z + \beta = 0$ ($\alpha, \beta \in \mathbf{C}$) are real, then value of $(\text{Im } \beta)^2 + (\text{Im } \alpha)(\text{Im } (\alpha\bar{\beta}))$ is
(a) 0 (b) 1
(c) -1 (d) i

5. If roots of the equation $z^2 + \alpha z + \beta = 0$ ($\alpha, \beta \in \mathbf{C}$) are purely imaginary, then value of $(\text{Im } \beta)^2 - (\text{Re } \alpha)(\text{Re } (\alpha\bar{\beta}))$ is
(a) 0 (b) 1
(c) -1 (d) i

6. If roots of the equation $z^2 + \alpha z + \beta = 0$ lie on $|z| = 1$, then
(a) $2|\text{Im } \alpha| = 1 - |\beta|^2$ (b) $2|\text{Im } \alpha| = |\beta|^2 - 1$
(c) $\text{Im } \alpha = 0$ (d) none of these

7. If $z_1 = 1 + i$ and $z_2 = 1 - i$ are the two vertices of an equilateral triangle then third vertex is given by
(a) $\pm\sqrt{3}$ (b) $1 \pm \sqrt{3}$
(c) $\sqrt{3} \pm 1$ (d) none of these

8. If $\alpha \neq 0, |\alpha| \neq 1$, then all the circles passing through α and $\frac{1}{\bar{\alpha}}$,
(a) are concentric with the circle $|z| = 1$
(b) co-axal

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(c) are orthogonal to the circle $|z| = 1$
(d) none of these

9. If $\text{Re}(z) > 0$, then value of the $|1 + z + ... + z^n|$ cannot be less than
(a) $|z|^n - \frac{1}{|z|}$ (b) $|z|^n + \frac{1}{|z|}$
(c) $n|z|^n$ (d) $n|z|^n + 1$

10. If $|z| = 1$, then z can be written in the form
(a) $\frac{1+x}{1-x}$ $(x \in \mathbf{R})$ (b) $\frac{1+ix}{1-ix}$ $(x \in \mathbf{R})$
(c) $\frac{i+x}{1-x}$ $(x \in \mathbf{R})$ (d) none of these

11. If $x^2 + x + 1 = 0$, then value of
$$\left(x+\frac{1}{x}\right)^3+\left(x^2+\frac{1}{x^2}\right)^3+\cdots+\left(x^{100}+\frac{1}{x^{100}}\right)^3$$
is
(a) 100 (b) – 100
(c) 50 (d) – 50

12. If z_1 lies in $|z - 3| < 4$, z_2 on $|z - 1| + |z + 1| = 3$ and $A = |z_1 - z_2|$, then
(a) $0 \le A \le 15/2$ (b) $0 < A \le 15/2$
(c) $0 \le A \le 17/2$ (d) $0 \le A < 17/2$

13. If $\omega \ne 1$ is a cube root of unity, then value of
$(x + y)^2 + (x\omega + y\omega^2)^2 + (x\omega^2 + y\omega)^2$
is
(a) $3xy$ (b) $6xy$
(c) $4xy$ (d) xy

14. If $z \in \mathbf{C}$, then $\left|\frac{2z-i}{5z+1}\right| = m$, $m \in \mathbf{R}$ represents a straight line if
(a) $m = 2$ (b) $m = 5$
(c) $m = 2.5$ (d) $m = 0.4$

15. If $\omega \ne 1$ is a cube root of unity, then roots of $(x - 2i)^3 + i = 0$ are
(a) i, i, i (b) $i, 2i + \omega, 2i + \omega^2$
(c) $i, 2i - \omega, 2i - \omega^2$ (d) none of these

16. The minimum value of $|z - 3| + |z + 5|$ is
(a) 1 (b) 4
(c) 6 (d) 8

17. If $z = \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{2009} + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^{2009}$, then
(a) $z = 0$ (b) $\text{Re}(z) > 0$
(c) $\text{Im}(z) > 0$ (d) $\text{Re}(z) < 0, \text{Im}(z) > 0$

18. If $|z| = 3$, then $\frac{9+z}{1+\bar{z}}$ equals
(a) z (b) $\bar{z}$
(c) $3z$ (d) $z + \bar{z}$

19. If p, q, r are positive integers, then remainder when
$$x^{3p} + x^{3q+1} + x^{3r+2}$$
is divided by $x^2 + x + 1$ is
(a) 0 (b) 1
(c) $x + 1$ (d) $x - 1$

20. If z lies on the curve $|z - 3 - 4i| = 3$, then least value of $|z|$ is
(a) 2 (b) 3
(c) 8 (d) none of these

SECTION II

MULTIPLE CORRECT CHOICE TYPE

21. If the imaginary part of the complex number
$(z-1)(\cos\alpha - i\sin\alpha) + (z-1)^{-1}(\cos\alpha + i\sin\alpha)$
is zero, then
(a) $|z| = 1$ (b) $|z - 1| = 1$
(c) $\arg(z) = \alpha$ (d) $\arg(z - 1) = \alpha$

22. Two lines $\bar{a}z + a\bar{z} + c = 0$, $\bar{b}z + b\bar{z} + d = 0$ where $a, b \in \mathbf{C} - \{0\}$ and $c, d \in \mathbf{R}$, are
(a) parallel iff $a\bar{b} = \bar{a}b$
(b) parallel iff $\frac{a}{b}$ is real
(c) perpendicular iff $a\bar{b} + b\bar{a} = 0$
(d) perpendicular iff $\frac{a}{b}$ is imaginary

23. Let $z_1, z_2 \in \mathbf{C}$, $z_1 \ne z_2$.
(a) Equation of straight line through z_1 and z_2 is
$$\text{Re}\left(\frac{z-z_1}{z_2-z_1}\right) = 0$$
(b) Equation of circle with z_1 z_2 as diameter is $|2z - z_1 - z_2| = |z_1 - z_2|$
(c) Equation of any circle passing through z_1 and z_2
$$|2z - z_1 - z_2| - |z_1 - z_2| + \lambda\,\text{Re}\left(\frac{z-z_1}{z_2-z_1}\right) = 0$$
where $\lambda \in \mathbf{R}$.
(d) none of these

24. Let two given circles be
$$C_1 : z\bar{z} + \bar{a}z + a\bar{z} + c = 0$$
and $C_2 : z\bar{z} + \bar{b}z + b\bar{z} + d = 0$
where $a, b \in \mathbf{C}$ and $c, d \in \mathbf{R}$.
(a) Radius of circle C_1 is $\sqrt{|a|^2 - c}$
(b) C_1 and C_2 will touch externally iff
$|a - b| = \sqrt{|a|^2 - c} + \sqrt{|b|^2 - d}$
(c) C_1 and C_2 will intersect orthogonally iff $a\bar{b} + \bar{a}b = c + d$

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(d) radical axis of C_1 and C_2 is

$$(\bar{a} - \bar{b})z + (a - b)\,\bar{z} + c - d = 0$$

25. Let C be the circle

$$|z - z_1| = k|z - z_2|$$

where $z_1, z_2 \in \mathbf{C}$, $z_1 \neq z_2$, and $k > 0, k \neq 1$.

(a) centre of C lies on the line $\text{Re}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$

(b) radius of C is $\frac{k|z_2 - z_1|}{|1 - k^2|}$

(c) mid point of the segment joining z_1 z_2 lies on the circle

(d) none of these

26. Let $c > 0$. For the ellipse

$$|z - ci| + |z + ci| = 4c$$

(a) eccentricity is 1/2

(b) one of the directrix is $z = 4ci$

(c) minor axis is the real axis

(d) major axis is the imaginary axis

27. ABC is an equilateral triangle whose circumcentre is at the origin. Let the point A be represented by $-1 + \sqrt{3}i$.

(a) length of side of ΔABC is $2\sqrt{3}$

(b) centroid of ΔABC is $-1 + i$

(c) orthocentre of ΔABC is O

(d) in-radius of ΔABC is 1.

28. If z_1 and z_2 are fixed complex numbers, the locus of z so that

$$\frac{z - z_1}{z - z_2} = e^{i\theta}, \quad \theta \in \mathbf{R},$$

is

(a) a circle

(b) a semicircle

(c) a straight line

(d) perpendicular bisector of segment joining z_1 and z_2.

29. Which of the following represent a straight line through z_1 and z_2?

(a) $z = z_1 + t\,(z_2 - z_1),\ t \in \mathbf{R}$

(b) $\text{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$

(c) $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

(d) $\frac{z - z_1}{z - z_2} = e^{i\theta},\ \theta \in \mathbf{R}$

30. If $a \neq 0$, then $\text{Re}\left(\frac{z}{a}\right) = 1$ represents

(a) a straight line

(b) rectangular hyperbola

(c) a hyperbola with eccentricity $\sqrt{2}$.

(d) part of circle with centre at a.

SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos. 31 to 34

For $z = x + iy,\ x, y \in \mathbf{R}$,

define $e^z = e^x\,(\cos y + i \sin y)$,

$$\sinh z = \frac{1}{2}\,(e^z - e^{-z})$$

$$\cosh z = \frac{1}{2}\,(e^z + e^{-z})$$

31. For $z \in \mathbf{C}$, conjugate of e^z equals

(a) e^{-z} (b) $e^{\bar{z}}$

(c) $e^{-\bar{z}}$ (d) e^z

32. $(\cosh z)^2 + (\sinh z)^2$ equals

(a) 1 (b) -1

(c) 0 (d) none of these

33. $\cosh z = \sinh z$ holds if z equals

(a) $(2n - 1)\pi/2$ (b) $2n\pi$

(c) $(2n + 1)\pi i$ (d) $(2n - 1)\pi i/2,\ n \in \mathbf{I}$.

34. $\sinh z = 0$ if z equals

(a) $2n\pi i$ (b) $(2n - 1)\pi i$

(c) $(2n - 1)\pi i$ (d) $(2n - 1)\pi i/2,\ n \in \mathbf{I}$

Paragraph for Question Nos. 35 to 38

If z_1, z_2 are two complex numbers representing points A and B, we define complex slope of line AB as $\mu = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

35. Complex slope of line $\bar{a}z + a\bar{z} + b = 0$ where $a \in \mathbf{C}$ and $b \in \mathbf{R}$ is

(a) $\frac{a}{\bar{a}}$ (b) $\frac{\bar{a}}{a}$

(c) $-\frac{a}{\bar{a}}$ (d) $-\frac{\bar{a}}{a}$

36. If μ and μ' are complex slopes of two perpendicular lines, then

(a) $\mu\mu' = -1$ (b) $\mu = \mu'\ e^{\pi i/2}$

(c) $\mu + \mu' = 0$ (d) $\mu\mu' = 1$

37. If μ and μ' are complex slopes of two parallel lines then

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(a) $\mu = \mu'$ (b) $\mu\mu' = -1$

(c) $\mu\mu' = 1$ (d) $\mu = \mu' e^{i\pi/2}$

38. If a line makes an angle $\theta(\neq \pi/2)$ with the positive direction of the x-axis, then its complex slope equals

(a) $e^{2i\theta}$ (b) $e^{-2i\theta}$

(c) $ie^{2i\theta}$ (d) none of these

SECTION IV

INTEGER ANSWER TYPE

39. Find the number of complex numbers which are conjugate of their own cube.

40. If $w \neq 1$ is a cube root of unity, and $a + b = 11$, $a^3 + b^3 = 1001$, find the value of $(aw^2 + bw)(aw + bw^2)$

41. If $|z_1| = |z_2| = |z_3|$ and $\arg(z_2/z_1) = 60°$, find the degree measure of $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$.

42. Let eccentricity of ellipse with focii z_1 and z_2 be 1/3. If $|z_1 - z_2| = 5$ and z lies on the ellipse, find maximum possible value of $|z - z_1|$

SECTION V

MATRIX MATCH TYPE

43. If $\omega \neq 1$ is a cube root of unity, $a, b \in \mathbf{R}$ then value of

(a) $(a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$	(p) $6ab$
(b) $(a + b)^2 + (a\omega + b\omega^2)^2 + (a\omega^2 + b\omega)^2$	(q) $a^3 + b^3 + 1 - 3ab$
(c) $(a + b + 1)(a + b\omega + \omega^2)(a + b\omega^2 + \omega)$	(r) $a^3 + b^3$
(d) $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$	(s) $(2a - b - c)(2b - c - a)(2c - a - b)$

44. If $w = -\frac{1}{4} + \frac{\sqrt{3}i}{4}$, then value of

(a) $1 - w + w^2 - w^3 + \ldots$	(p) $\frac{1}{7}(5 + \sqrt{3}i)$
(b) $1 - w^2 + w^4 - w^6 + \ldots$	(q) $\frac{2}{21}(9 - \sqrt{3}i)$
(c) $1 + w + w^2 + w^3 + \ldots$	(r) $1 - \frac{1}{\sqrt{3}}i$
(d) $1 + w^2 + w^4 + \ldots$	(s) $\frac{2}{13}(7 + \sqrt{3}i)$

45.

(a) G is the greatest and L be the least value of $\|z - 2\|$ if z satisfies $\|z + 2 + 3i\| \leq 1$	(p) $L + G = 10$
(b) G is the greatest and L is least value of $\|z\|$ when z satisfies $\|z - 5i\| \leq 1$	(q) $LG = 624$
(c) G and L are the roots of $t^2 - 10t + 36 = 0$.	(r) $G - L = 2$
(d) G is the sum of infinite geometric series $5 + \frac{5}{2} + \frac{5}{2^2} + \frac{5}{2^3} + \cdots$ and $L = G - 7$.	(s) $\frac{G}{L} = \frac{3}{2}$

SECTION VI

REASONING TYPE

46. **Statement-1:** Let ABC be a triangle and P be a point on the circum-circle of ΔABC, then $PA^2 + PB^2 + PC^2$ is independent of the position of point P.
Statement-2: Let A, B, C be three points in a plane such that $CA^2 + CB^2 = AB^2/2$, then ΔCAB is an isosceles triangle.

47. **Statement-1:** Centre of circle $\left|\frac{z+1}{z-1}\right| = 2$ is $\frac{5}{3}$

Statement-2: Radius of circle $\left|\frac{z-1}{z+1}\right| = 2$ is $\frac{4}{3}$

48. Let ABC be a triangle and P be any point in the complex plane.
Statement-1: $BC^2 + CA^2 + AB^2 \leq 3(PA^2 + PB^2 + PC^2)$
Statement-2: $BC^2 + CA^2 + AB^2 = 3(PA^2 + PB^2 + PC^2)$
$\Leftrightarrow$ P is centroid of ΔABC.

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PAST YEARS' IIT QUESTIONS

1. If $x = a + b$, $y = a\gamma + b\beta$ and $z = a\beta + b\gamma$, where γ and β are the complex cube roots of unity, show that
$$xyz = a^3 + b^3$$ [1978]

2. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, prove that
$$(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$$ [1979]

3. If the cube roots of unity are 1, ω, ω^2, then roots of the equation $(x-1)^3 + 8 = 0$ are
(a) $-1, 1 + 2\omega, 1 + 2\omega^2$
(b) $-1, 1 - 2\omega, 1 - 2\omega^2$
(c) $-1, -1, -1$
(d) none of these [1979]

4. The smallest positive integer n for which
$$\left(\frac{1+i}{1-i}\right)^n = 1$$
is
(a) 8 (b) 12
(c) 16 (d) none of these
[1980]

5. Find the real values of x and y for which the following equation is satisfied
$$\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$$ [1980]

6. It is given that n is an odd integer greater than 3, but not a multiple of 3. Prove that $x^3 + x^2 + x$ is a factor of $(x+1)^n - x^n - 1$. [1980]

7. True/False
For complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$ if $x_1 \le x_2$ and $y_1 \le y_2$. Then for all complex numbers z with $1 \cap z$, we have
$$\frac{1-z}{1+z} \cap 0$$ [1981]

8. The complex numbers $z = x + iy$ which satisfy the equation
$$\left|\frac{z - 5i}{z + 5i}\right| = 1$$
lie on
(a) the x-axis
(b) the straight line $y = 5$
(c) a circle passing through the origin
(d) none of these [1981]

9. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that
$$z_1^2 + z_2^2 + z_3^2 = 3z_0^2$$ [1981]

10. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then
(a) $\mathrm{Re}(z) = 0$
(b) $\mathrm{Im}(z) = 0$
(c) $\mathrm{Re}(z) > 0$, $\mathrm{Im}(z) > 0$
(d) $\mathrm{Re}(z) > 0$, $\mathrm{Im}(z) < 0$ [1982]

11. The inequality $|z - 4| < |z - 2|$ represents the region given by
(a) $\mathrm{Re}(z) > 0$ (b) $\mathrm{Re}(z) < 2$
(c) $\mathrm{Re}(z) > 0$ (d) none of these
[1982]

12. If $z = x + iy$ and $w = (1 - iz)/(z - i)$, then $|w| = 1$ implies that, in the complex plane,
(a) z lies on the imaginary axis
(b) z lies on the real axis
(c) z lies on the unit circle
(d) None of these [1983]

13. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if
(a) $z_1 + z_4 = z_2 + z_3$
(b) $z_1 + z_3 = z_2 + z_4$
(c) $z_1 + z_2 = z_3 + z_4$
(d) None of these [1983]

14. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1 z_2 = 0$. [1983]

15. True/False
If the complex numbers z_1, z_2 and z_3 are the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 \neq 0$. [1984]

16. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, then the two triangles:
(a) have the same area
(b) are similar
(c) are congruent
(d) none of these [1985]

17. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\mathrm{Re}(z_1 \bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies:

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(a) $|w_1| = 1$ (b) $|w_2| = 1$

(c) $\text{Re}(w_1 \bar{w}_2) = 0$ (d) none of these

[1985]

18. True/False
If three complex numbers are in A.P., then they lie on a circle in the complex plane. [1985]

19. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\dfrac{z_1 + z_2}{z_1 - z_2}$, may be

(a) zero (b) real and positive

(c) real and negative (d) purely imaginary

(e) none of these [1986]

20. Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz and $z + iz$ is $\dfrac{1}{2}|z|^2$. [1986]

21. Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$ [1986]

22. If z_1 and z_2 are two nonzero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg z_1 - \arg z_2$ is equal to

(a) $-\pi$ (b) $-\dfrac{\pi}{2}$

(c) 0 (d) $\dfrac{\pi}{2}$

(e) π [1987]

23. The value of $\sum_{k=1}^{6}\left(\sin\dfrac{2\pi k}{7} - i\cos\dfrac{2\pi k}{7}\right)$ is

(a) -1 (b) 0

(c) $-i$ (d) i

(e) none of these [1987]

24. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other, for

(a) $x = n\pi$ (b) $x = \left(n + \dfrac{1}{2}\right)\pi$

(c) $x = 0$ (d) no value of x

[1988]

25. For any two complex numbers z_1, z_2 and any two real numbers a and b,

$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 =$ ——— [1988]

26. True/False
The cube roots of unity when represented on the Argand diagram form the vertices, of an equilateral triangle. [1988]

27. If α, β, γ are the cube roots of p, $p < 0$, then for any x, y, and z,

$$\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} = \text{_____}.$$ [1989]

28. If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then $a =$ _____ and $b =$ _____. [1989]

29. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $\dfrac{z - z_1}{z - z_2} = \dfrac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$ [1990]

30. Column Matching Type
$z \neq 0$ is a complex number.

Column 1	Column 2
(i) $\text{Re}(z) = 0$	(a) $\text{Re}(z^2) = 0$
(ii) $\arg(z) = \pi/4$	(b) $\text{Im}(z^2) = 0$
	(c) $\text{Re}(z^2) = \text{Im}(z^2)$

[1992]

31. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. If the points D and M represent the complex numbers $1 + i$ and $2 - i$ respectively, then A represents the complex number ____ or ____. [1993]

32. Prove that

$$\sum_{r=1}^{k} (-3)^{r-1} ({}^{3n}C_{2r-1}) = 0$$

where $k = (3n/2)$ and n is an even positive integer. [1993]

33. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$ then $z_2 =$ _____ $z_3 =$ _____. [1994]

34. Let α and β be the roots of the equation $x^2 + x + 1 = 0$. The equation whose roots are α^{19}, β^7 is

(a) $x^2 - x - 1 = 0$ (b) $x^2 - x + 1 = 0$

(c) $x^2 + x - 1 = 0$ (d) $x^2 + x + 1 = 0$

[1994]

35. If ω is an imaginary cube root of unity, then the value of

$$\sin\left\{(\omega^{10} + \omega^{23})\pi - \frac{\pi}{4}\right\} \text{ is}$$

(a) $-\dfrac{\sqrt{3}}{2}$ (b) $-\dfrac{1}{\sqrt{2}}$

(c) $\dfrac{1}{\sqrt{2}}$ (d) $\dfrac{\sqrt{3}}{2}$ [1994]

36. If $\omega\ (\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$. Then A and B are respectively the numbers:

(a) 0, 1 (b) 1, 1

(c) 1, 0 (d) -1, 1 [1995]

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37. If $\omega (\neq 1)$ is a cube root of unity, then

$$\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$$

[1995]

equals

(a) 0 (b) 1

(c) i (d) ω [1995]

38. Let z and w be two non-zero complex numbers such that $|z| = |w|$ and Arg z + Arg $w = \pi$. Then z equals

(a) w (b) $-w$

(c) $\bar{w}$ (d) $-\bar{w}$

[1995]

39. Let z and w be two complex numbers such that $|z| \le 1$, $|w| \le 1$ and $|z + iw| = |z - i\bar{w}| = 2$. Then z equals

(a) 1 or i (b) i or $-i$

(c) 1 or -1 (d) i or -1

[1995]

40. If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$. [1995]

41. If $|z| \le 1$, $|w| \le 1$, show that

$|z - w|^2 \le (|z| - |w|)^2 + (\text{Arg } z - \text{Arg } w)^2$ [1995]

42. For positive integers n_1, n_2 the value of the expression

$(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, where $i = \sqrt{-1}$, is a real number if and only if

(a) $n_1 = n_2 + 1$ (b) $n_1 = n_2 - 1$

(c) $n_1 = n_2$ (d) $n_1 > 0, n_2 > 0$

[1996]

43. The value of expression

$1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + \ldots + (n-1)(n-\omega)(n-\omega^2)$,

where ω is an imaginary cube root of unity is ____. [1996]

44. Find all non-zero complex number z satisfying $\bar{z} = iz^2$ [1996]

45. Let $\bar{b}z + b\bar{z} = c$, $b \neq 0$ be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of a point z_2 through the line, then show that

$$c = \bar{z}_1 b + z_2 \bar{b}$$ [1997]

46. Let z_1 and z_2 be the roots of the equation $z^2 + pz + q = 0$, where the coefficients of p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, show that

$$p^2 = 4q \cos^2\left(\frac{\alpha}{2}\right)$$ [1997]

47. Prove that

$$\sum_{k=1}^{n} (n-k) \cos\left(\frac{2k\pi}{n}\right) = -\frac{n}{2}$$

where $n \ge 3$ is an integer. [1997]

48. If ω is a cube root of unity, then $(1 + \omega - \omega^2)^7$ equals

(a) $128\ \omega$ (b) $-128\ \omega$

(c) $128\ \omega^2$ (d) $-128\ \omega^2$

[1998]

49. The value of the sum $\sum_{n=1}^{13} (i^n + i^{n+1})$ where $i = \sqrt{-1}$ equals

(a) i (b) $i - 1$

(c) $-i$ (d) 0

[1998]

50. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is

(a) $\frac{3}{4}$ (b) $3\sqrt{3}$

(c) 3 (d) $\frac{3\sqrt{3}}{2}$

[1998]

51. If $i = \sqrt{-1}$, then

$$4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$$ is equal to

(a) $1 - i\sqrt{3}$ (b) $-1 + i\sqrt{3}$

(c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

[1999]

52. For complex numbers z and w, prove that

$$|z|^2 w - |w|^2 z = z - w$$

if and only if $z = w$ or $z\bar{w} = 1$ [1999]

53. If arg $(z) < 0$, then arg $(-z)$ – arg (z) =

(a) π (b) $-\pi$

(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$ [2000]

54. If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is

(a) equal to 1 (b) less than 1

(c) greater than 3 (d) equal to 3

[2000]

55. Let z_1 and z_2 be n^{th} roots of unity which subtend a right angle at the origin. Then n must be of the form

(a) $4k + 1$ (b) $4k + 2$

(c) $4k + 3$ (d) $4k$ [2001]

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56. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is
(a) of area zero
(b) right-angled isosceles
(c) equilateral
(d) obtuse-angled isosceles [2001]

57. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is
(a) 3ω (b) $3\omega(\omega - 1)$
(c) $3\omega^2$ (d) $3\omega(1 - \omega)$
[2002]

58. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is
(a) 0 (b) 2
(c) 7 (d) 17 [2002]

59. Let a complex number α, $\alpha \neq 1$ be a root of the equation
$$z^{p+q} - z^p - z^q + 1 = 0$$
where p, q are distinct primes. Show that either $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$, but not both together. [2002]

60. If $|z| = 1$ and $w = \frac{z-1}{z+1}$ (where $z \neq -1$), then Re(w) equals
(a) 0 (b) $-\frac{1}{|z+1|^2}$
(c) $\left|\frac{z}{z+1}\right| \cdot \frac{1}{|z+1|^2}$ (d) $\frac{\sqrt{2}}{|z+1|^2}$
[2003]

61. If z_1 and z_2 are two complex numbers such that $|z_1| < 1$ and $|z_2| > 1$ then show that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$.
[2003]

62. Show that there exists no complex number z such that $|z| < 1/3$ and $\sum_{r=1}^{n} a_r z^r = 1$, where $|a_i| < 2$ for $i = 1, 2, \ldots, n$. [2003]

63. If $\omega (\neq 1)$ is a complex cube root of unity, the least value of $n \in \mathbf{N}$ for which $(1 + \omega^2)^n = (1 + \omega^4)^n$ is
(a) 6 (b) 5
(c) 3 (d) 2
[2004]

64. If $k > 0$, $k \neq 1$, show that $\left|\frac{z-\alpha}{z-\beta}\right| = k$ represents a circle. Find its centre and radius. [2004]

65. Let a, b and c be three integers such that at least two of them are unequal and $\omega (\neq 1)$ is a cube root of unity. The least value of the expression $|a + b\omega + c\omega^2|$ is
(a) 0 (b) 1
(c) $\sqrt{3/2}$ (d) 1/2 [2005]

66. The shaded region in Fig. 1.59 is

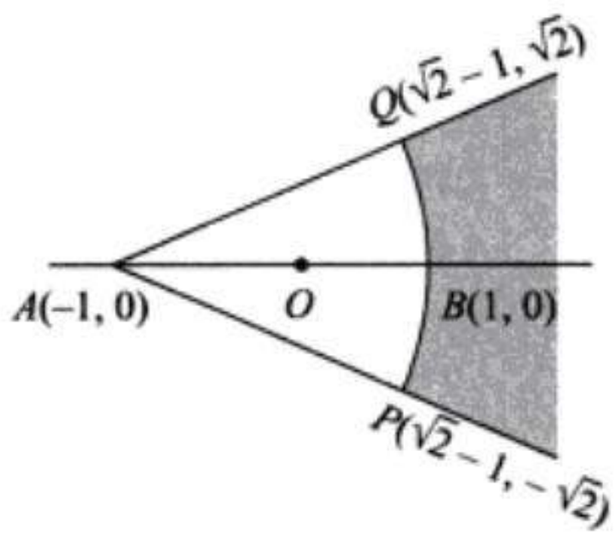

Fig. 1.59

(a) $\{z : |z - 1| < 2, |\arg(z + 1)| < \pi/4\}$
(b) $\{z : |z + 1| < 2, |\arg(z + 1)| < \pi/2\}$
(c) $\{z : |z - 1| > 2, |\arg(z + 1)| < \pi/4\}$
(d) $\{z : |z + 1| > 2, |\arg(z + 1)| < \pi/4\}$

67. Let $w = \alpha + i\beta$, $\beta \neq 0$ and $z \neq 1$. The set of values of z for which $\text{Im}\left(\frac{w - \bar{w}z}{1 - z}\right) = 0$ is
(a) $\{z : |z| = 1\}$ (b) $\{z : z = \bar{z}\}$
(c) $\{z : z \neq 1\}$ (d) $\{z : |z| = 1, z \neq 1\}$
[2006]

68. A man walks a distance of 3 units from the origin towards the north-east ($N\ 45°E$) direction. From there, he walks a distance of 4 units towards the north-west ($N\ 45°W$) direction to reach a point P. Then the position of P in the Argand plane is
(a) $3e^{i\pi/4} + 4i$ (b) $(3 - 4i)e^{i\pi/4}$
(c) $(4 + 3i)e^{i\pi/4}$ (d) $(3 + 4i)e^{i\pi/4}$
[2007]

69. If $|z| = 1$ and $z \neq 1$, then all the values of $\frac{z}{1 - z^2}$ lie on
(a) a line not passing through the origin
(b) $|z| = \sqrt{2}$ (c) the x-axis
(d) the y-axis [2007]

70. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from ori-

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gin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\mathbf{i} + \mathbf{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by

(a) $6 + 7i$ (b) $-7 + 6i$
(c) $7 + 6i$ (d) $-6 + 7i$

[2008]

Paragraph for Question Nos. 71 to 73

Let A, B, C be three sets of complex numbers as defined as follows:

$A = \{z: \text{Im}(z) \geq 1\}$

$B = \{z: |z - 2 - i| = 3$

$C = \{z: \text{Re}((1 - i)z) = \sqrt{2}\}$

71. Then number of elements in the set $A \cap B \cap C$ is
(a) 0 (b) 1
(c) 2 (d) ∞

72. Let z be any point in $A \cap B \cap C$. Then $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between
(a) 25 and 29 (b) 30 and 34
(c) 35 and 39 (d) 40 and 44

73. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then $|z| - |w| + 3$ lies between
(a) -6 and 3 (b) -3 and 6
(c) -6 and 6 (d) -3 and 9

[2008]

74. Let $z = \cos\theta + i \sin\theta$. Then value of

$$\sum_{m=1}^{15} \text{Im}(z^{2m-1})$$

at $\theta = 2°$ is

(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3\sin 2°}$

(c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

[2009]

75. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation

$z\bar{z}^3 + \bar{z}z^3 = 350$

is
(a) 48 (b) 32
(c) 40 (d) 80

[2009]

ANSWERS

LEVEL 1

SECTION I

1. (a)	2. (a)	3. (a)	4. (d)
5. (b)	6. (a)	7. (b)	8. (b)
9. (b)	10. (d)	11. (a)	12. (c)
13. (a)	14. (b)	15. (b)	16. (c)
17. (d)	18. (a)	19. (a)	20. (d)
21. (b)	22. (a)	23. (b)	24. (c)
25. (b)	26. (b)	27. (d)	28. (c)
29. (b)	30. (d)	31. (a)	32. (b)
33. (b)	34. (a)	35. (c)	36. (b)
37. (c)	38. (c)	39. (b)	40. (d)

SECTION II

41. (a), (b), (c) 42. (a), (b), (c), (d)
43. (a), (b), (c) 44. (a), (c), (d)
45. (a), (c) 46. (b), (c), (d)
47. (a), (b), (c), (d) 48. (a), (b), (c), (d)
49. (a), (b), (c) 50. (b), (c), (d)
51. (a), (b), (c), (d) 52. (a), (b), (c), (d)
53. (a), (b) 54. (a), (b), (c), (d)
55. (a), (b), (c), (d) 56. (a), (b), (c), (d)
57. (a), (c) 58. (a)
59. (a), (b), (c), (d) 60. (b), (c), (d)
61. (c) 62. (a), (b), (c)
63. (a), (b), (c) 64. (d)
65. (b), (d)

SECTION III

66. (a)	67. (d)	68. (d)	69. (b)
70. (b)	71. (a)	72. (b)	73. (c)
74. (d)	75. (c)	76. (a)	77. (d)
78. (c)	79. (b)	80. (a)	81. (b)
82. (d)	83. (b)	84. (b)	85. (b)

SECTION III

86. 5	87. 5	88. 5	89. 3
90. 22	91. 100	92. 9	93. 14
94. 2	95. 128	96. 3	97. 2

SECTION IV

98.

	p	q	r	s
a	p	**q**	r	s
b	**p**	q	r	s
c	p	q	**r**	s
d	p	q	r	**s**

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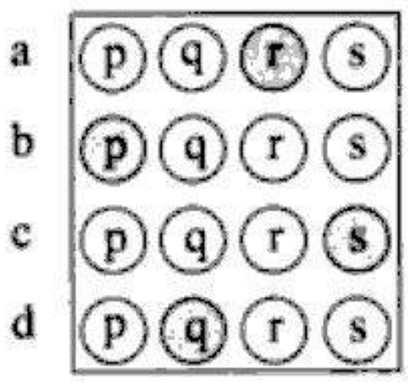
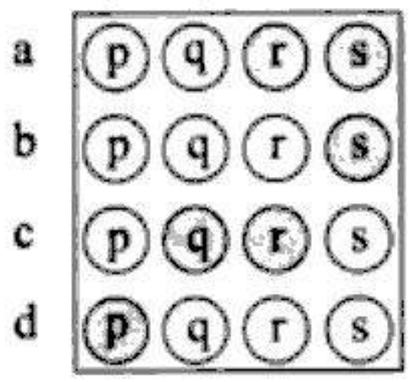
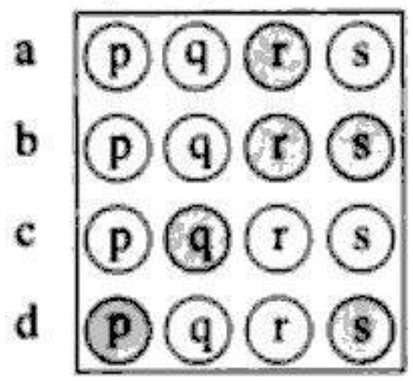
99.



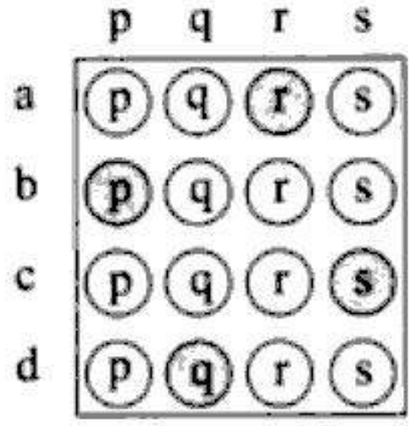
100.



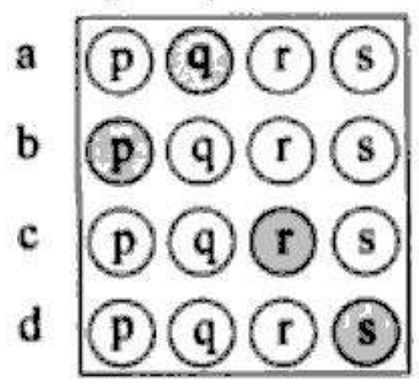
101.



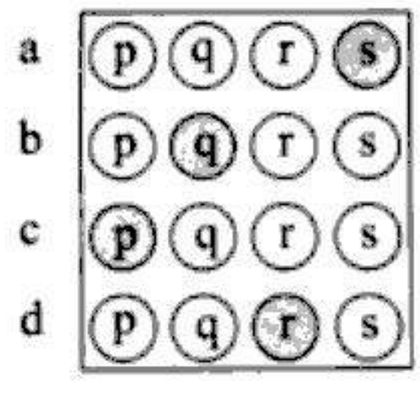
102.



103.



104.



SECTION VI

105. (b) 106. (b) 107. (a)
108. (a) 109. (a) 110. (b)
111. (d) 112. (b)

LEVEL 2

SECTION I

1. (a) 2. (a) 3. (d) 4. (a)
5. (a) 6. (d) 7. (b) 8. (c)
9. (a) 10. (b) 11. (b) 12. (d)
13. (a) 14. (d) 15. (b) 16. (d)
17. (a) 18. (a) 19. (a) 20. (a)

SECTION II

21. (b), (d) 22. (a), (b), (c), (d)
23. (a), (b), (c) 24. (a), (b), (c), (d)
25. (a), (b) 26. (a), (b), (c)
27. (a), (c), (d) 28. (c), (d)
29. (a), (b), (c) 30. (a)

SECTION III

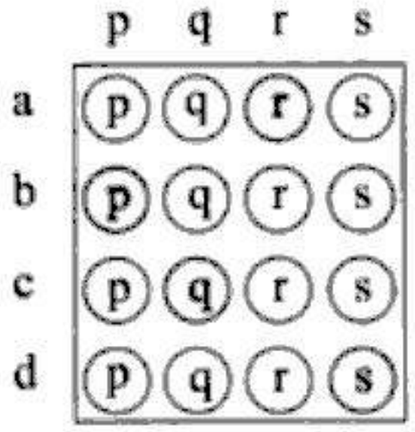
31. (b) 32. (d) 33. (c) 34. (a)
35. (c) 36. (c) 37. (a) 38. (a)

SECTION IV

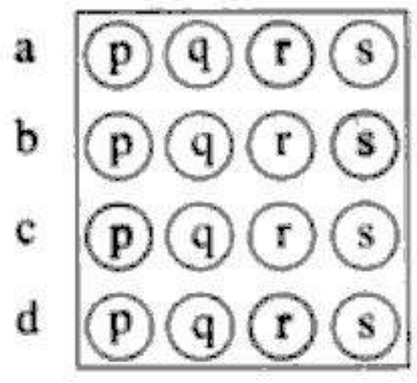
39. 5 40. 91 41. 30 42. 10

SECTION V

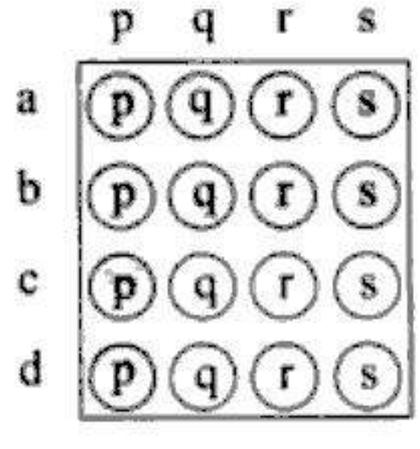
43.



44.



45.



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SECTION VI

46. (c) 47. (b) 48. (b)

PAST YEARS' IIT QUESTIONS

3. (a) 4. (d) 5. $x = 3, y = -1$
7. True 8. (a) 10. (b)
11. (d) 12. (b) 13. (b)
15. False 16. (b) 17. (a), (b), (c)
18. False 19. (a), (d) 20. (c)
23. (d) 24. (d)
25. $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$ 26. True
27. w or w^2 28. $2-\sqrt{3}$, $2-\sqrt{3}$
30. (i) $\leftrightarrow$ (b), (ii) $\leftrightarrow$ (a)
31. $3 - i/2$ or $1 - 3i/2$ 33. $-2, 1-\sqrt{3}i$
34. (d) 35. (c) 36. (b)
37. (a) 38. (d) 39. (c)
42. (d) 43. $\frac{1}{4}(n+1)^2(2n-1)$
44. $i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}$ 48. (d)
49. (b) 50. (c) 51. (c)
53. (a) 54. (a) 55. (d)
56. (c) 57. (b) 58. (b)
60. (a) 63. (c)
64. centre $= \frac{\alpha - k^2\beta}{1-k^2}$, radius $= \frac{k|\alpha-\beta|}{|1-k^2|}$
65. (b) 66. (d) 67. (d)
68. (d) 69. (d) 70. (d)
71. (b) 72. (c) 73. (b)
74. (d) 75. (a)

HINTS AND SOLUTIONS

LEVEL 1

1. $|z_1 - z_2|^2 = |1 - \bar{z}_1 z_2|^2$

$\Rightarrow |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2$

$= 1 + |\bar{z}_1|^2 |z_2|^2 - \bar{z}_1 z_2 - z_1\bar{z}_2$

$\Rightarrow |z_1|^2 |z_2|^2 - |z_1|^2 - |z_2|^2 + 1 = 0$

$\Rightarrow (|z_1|^2 - 1)(|z_2|^2 - 1) = 0$

As $|z_1| \neq 1$, we get $|z_2|^2 = 1$ or $|z_2| = 1$

2. $\begin{vmatrix} 1 & 1 & 1 \\ az_1 & az_2 & az_3 \\ \bar{a}\bar{z}_1 & \bar{a}\bar{z}_2 & \bar{a}_3\bar{z}_3 \end{vmatrix} = |a|^2 \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{vmatrix}$

As $|a|^2 > 0$ we get

$f(z_1, z_2, z_3) = f(az_1, az_2, az_3)$

3. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ z_1 + a & z_2 + a & z_3 + a \\ \bar{z}_1 + a & \bar{z}_2 + \bar{a} & \bar{z}_3 + \bar{a} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{vmatrix}$

[using $R_2 \to R_2 - aR_1$ and $R_3 \to R_3 - aR_1$]

4. $|z_2 + z_3|^2 + |z_3 + z_1|^2 + |z_1 + z_2|^2$

$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2) + (z_2\bar{z}_3 + \bar{z}_2 z_3 + z_3\bar{z}_1 + \bar{z}_3 z_1 + z_1\bar{z}_2 + \bar{z}_1 z_2)$

None of the expressions in (a), (b), (c) equals this expression.

5. $2i \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} z_1 + \bar{z}_1 & z_1 - \bar{z}_1 \\ z_2 + \bar{z}_2 & z_2 - \bar{z}_2 \end{vmatrix}$

$= \frac{1}{2} \begin{vmatrix} 2\bar{z}_1 & z_1 - \bar{z}_1 \\ 2\bar{z}_2 & z_2 - \bar{z}_2 \end{vmatrix} = \begin{vmatrix} \bar{z}_1 & z_1 \\ \bar{z}_2 & z_2 \end{vmatrix} = \bar{z}_1 z_2 - z_1\bar{z}_2$

[using $C_1 \to C_1 - C_2$ and $C_2 \to C_2 + C_1$]

6. $|a - b|^2 - |\bar{a} + b|^2 = -\bar{a}b - a\bar{b} - \bar{a}\bar{b} - ab$

$= -(a + \bar{a})(b + \bar{b}) = -4(Re(a))(Re(b)) < 0$

$\Rightarrow x[|a - b| + |\bar{a} + b|] < 0$

7. $x = \frac{1}{(1 - a/z)(1 - z\bar{a})}$

$= \frac{1}{(1 - a\bar{z})(1 - z\bar{a})} = \frac{1}{|1 - a\bar{z}|^2} > 0$

8. $x = |z_1 z_2| - \frac{1}{2}(z_1 z_2 + \bar{z}_1\bar{z}_2)$

$-\frac{1}{2}\left[|z_1|^2 + |z_2|^2 - \bar{z}_1\bar{z}_2 - z_1 z_2 - |z_1|^2 - |z_2|^2 + 2|z_1||z_2|\right] = 0$

9. $0 = \frac{1 - z}{1 + z} + \frac{1 - \bar{z}}{1 + \bar{z}}$

$= \frac{(1 - z)(1 + \bar{z}) + (1 - \bar{z})(1 + z)}{(1 + z)(1 + \bar{z})} = \frac{2(1 - |z|^2)}{|1 + z|^2}$

10. $az + b\bar{z} + c = 0$ (1)

$\Rightarrow \bar{a}\bar{z} + \bar{b}z + \bar{c} = 0$ (2)

Multiplying (1) by $\bar{a}$ and (2) by b and subtracting,

we get $z = \frac{b\bar{c} - \bar{a}c}{|a|^2 - |b|^2}$

11. From problem 10, we get

$(|a|^2 - |b|^2)z = b\bar{c} - \bar{a}c$

L.H.S. $= 0$ and R.H.S. $\neq 0$

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12. $az + b\bar{z} + c = 0$ (1)

$\Rightarrow \quad a\bar{a}z + \bar{a}b\bar{z} + \bar{a}c = 0$

$\Rightarrow \quad b\bar{b}z + \bar{a}b\bar{z} + b\bar{c} = 0$

$\Rightarrow \quad \bar{b}z + \bar{a}\,\bar{z} + c = 0$ (2)

From (1) and (2) we get

$$(a + \bar{b})z + (\bar{a} + b)\bar{z} + c + \bar{c} = 0$$

This is of the form

$$\bar{A}z + A\bar{z} + \alpha = 0$$

where $A = \bar{a} + b$ and $\alpha = c + \bar{c} \in \mathbf{R}$ and thus represents a straight line.

13. $A = \dfrac{\bar{z} - \bar{z}_0}{z - z_0} \quad \Rightarrow |A| = 1$

Also, $Az - \bar{z} + \bar{z}_0 - Az_0 = 0$

$\Rightarrow A(z_0 - \bar{A}\bar{z}_0)\,z - (z_0 - \bar{A}z_0)\bar{z} + |\bar{z}_0 - Az_0|^2 = 0$

$\Rightarrow (Az_0 - \bar{z}_0)z + (\bar{A}\bar{z}_0 - z_0)\bar{z} + |\bar{z}_0 - Az_0|^2 = 0$ (1)

As it is of the form

$$\bar{\alpha}z + \alpha\bar{z} + c = 0$$

where $\alpha = \bar{A}\bar{z}_0 - z_0$ and $c \in \mathbf{R}$, the given equation represents a straight line.

14. If m is slope of the line (1) in problem 13, then

$$-m = \frac{\mathrm{Re}(\alpha)}{\mathrm{Im}(\alpha)} \quad \Rightarrow \quad im = \frac{\alpha + \bar{\alpha}}{\alpha - \bar{\alpha}}$$

$$\Rightarrow \quad \frac{1 + im}{1 - im} = -\frac{\alpha}{\bar{\alpha}} = \frac{\bar{A}\bar{z}_0 - z_0}{\bar{z}_0 - Az_0} = \frac{1}{A}$$

$[\because |A| = 1]$

15. $A = \dfrac{1 - i\tan\theta}{1 + i\tan\theta} = \dfrac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}$

$= (\cos\theta - i\sin\theta)^2 = e^{-2i\theta}$

16. As $|z - a|^2 = r^2$ or $|z - a| = r \Rightarrow$ z lies on a circle with centre at a and radius r.

17. $z\bar{z} + az + b\bar{z} + c = 0$ (1)

$\Rightarrow \quad \bar{z}z + \bar{a}\,\bar{z} + \bar{b}z + \bar{c} = 0$ (2)

Subtracting (2) from (1) we get

$$(a - \bar{b})z + (b - \bar{a})\bar{z} + c - \bar{c} = 0$$

$$\Rightarrow \quad \bar{z} = \frac{(\bar{c} - c) - (a - \bar{b})z}{b - \bar{a}}$$

Substituting this in (1) we get a quadratic equation in z.

18. Putting $\bar{a} = b$ and $\bar{c} = c$ in (2) above we get

$$z\bar{z} + b\bar{z} + \bar{b}z + c = 0$$

$$\Rightarrow \quad |z + b|^2 = |b|^2 - c \qquad (3)$$

This is a circle with centre at $-b$ and radius $\sqrt{|b|^2 - c}$.

19. If $|b|^2 < c$, (3) has no solution.

20. $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$

$= 2[|z_1|^2 + |z_2|^2 + |z_3|^2] - z_1(\bar{z}_2 + \bar{z}_3)$

$- z_2\,(\bar{z}_3 + \bar{z}_1) - z_3\,(\bar{z}_1 + \bar{z}_2)$

$= 2[|z_1|^2 + |z_2|^2 + |z_3|^2] + z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3$

$= 3(|z_1|^2 + |z_2|^2 + |z_3|^2)$

21. $E = 2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)$

$- \bar{z}_1\,(z_2 + z_4) - \bar{z}_2\,(z_1 + z_3)$

$- \bar{z}_3\,(z_2 + z_4) - \bar{z}_4\,(z_1 + z_3)$

$= 8 + |z_1 + z_3|^2 + |z_2 + z_4|^2 \geq 8$

22. From problem 21,

$E - 2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)$

$= |z_1 + z_3|^2 + |z_2 + z_4|^2 = 0$

$\Leftrightarrow \quad z_1 + z_3 = 0$ and $z_2 + z_4 = 0$

23. $2\mathrm{Re}(z) = \dfrac{u - v}{1 + uv} + \dfrac{\bar{u} - \bar{v}}{1 + \bar{u}\,\bar{v}}$

$= \dfrac{u - v}{1 + uv} + \dfrac{v - u}{1 + uv} = 0$

24. Let $u = r_1\,e^{i\phi}$, $v = r_2\,e^{i\theta}$ where $r_1 < 1$, $r_2 < 1$.

$1 - \bar{u}\,v = 1 - r_1\,r_2\,e^{i(\theta - \phi)}$

$|1 - \bar{u}\,v|^2 = (1 - r_1\,r_2\cos(\theta - \phi))^2$

$+ r_1^2\,r_2^2\sin^2(\theta - \phi)$

$= 1 - 2r_1r_2\cos(\theta - \phi) + r_1^2\,r_2^2$

$\leq (1 - r_1\,r_2)^2 = (1 - |u|\,|v|)^2$

$\Rightarrow \quad |1 - \bar{u}\,v| \leq 1 - |u|\,|v|$

$\Rightarrow \quad \dfrac{1}{1 - |u||v|} \leq \dfrac{1}{|1 - \bar{u}\,v|}$

Also, $\quad ||u| - |v|| < |u - v|$

Thus, $\dfrac{||u| - |v||}{1 - |u||v|} \leq \left|\dfrac{u - v}{1 - \bar{u}v}\right|$

25. Let $\alpha = \alpha + i\beta$, $z = x + iy$, so that

$$\bar{\alpha}z + \alpha\bar{z} + c = 0$$

becomes $2(\alpha x + \beta y) + c = 0$ (1)

Let $\quad m_1 =$ slope of (1) $= -\dfrac{\alpha}{\beta}$

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Also, $m_2 = \text{slope } OA = \dfrac{\beta}{\alpha}$

As $m_1 m_2 = -1$, $OA \perp L$

26. $$0 = \begin{vmatrix} z & \bar{z} & 1 \\ i & -i & 1 \\ iz & -i\bar{z} & 1 \end{vmatrix} = \begin{vmatrix} z-i & \bar{z}+i & 1 \\ 0 & 0 & 1 \\ iz-i & -i\bar{z}+i & 1 \end{vmatrix}$$

$\Rightarrow (z-i)(\bar{z}-1) + (z-1)(\bar{z}+i) = 0$

$\Rightarrow 2z\bar{z} - (1-i)z - (1+i)\bar{z} = 0$

$\Rightarrow z$ lies on a circle

27. $|z|^2 - 5|z| + 1 = 0$

$$\Rightarrow \qquad |z| = \frac{5 \pm \sqrt{21}}{2}$$

$$\Rightarrow \qquad z = \frac{1}{2}(5 \pm \sqrt{21})e^{i\theta}$$

where $\theta \in \mathbf{R}$.

28. $0 < \arg(z) < \pi$

$$\Rightarrow \qquad \text{Im}(z) > 0$$

29. As $1, \alpha_1, \alpha_2, \ldots \alpha_{n-1}$ are roots of $x^n - 1 = 0$, sum of the products taken two at a time is 0.

$$\Rightarrow \qquad \sum_{k=1}^{n-1} 1(\alpha_k) + \sum_{i<j} \alpha_i \alpha_j = 0$$

$$\Rightarrow \qquad \sum_{i<j} \alpha_i \alpha_j = -\left(\sum_{k=1}^{n-1} \alpha_k\right) = 1$$

30. The point O(0), $A(z)$, $B(z+iz)$, $C(iz)$ are vertices of a square as

(i) the mid point of OC = the mid point of AB

(ii) $OC = AB = \sqrt{2}\,|z|$

(iii) $OA = OC = |z|$

31. $|z-2|^2 = 4|z-1|^2$

$\Rightarrow |z|^2 - 2z - 2\bar{z} + 4 = 4(|z|^2 - z - \bar{z} + 1)$

$\Rightarrow 3|z|^2 - 2(z + \bar{z}) = 0$

$\Rightarrow 3|z|^2 - 4\text{Re}(z) = 0$

32. $(z+1)(z^2 - z + 1) + 2z(z+1) = 0$

$$\Rightarrow \qquad (z+1)(z^2 + z + 1) = 0$$

$$\Rightarrow \qquad z = -1, \omega, \omega^2$$

out of these ω and ω^2 satisfy

$$z^{2006} + z^{100} + 1 = 0$$

33. Produce the join of z_1 and z_2 to meet $a\bar{z} + a\bar{z} + b = 0$ in z. Suppose z divides the join of z_1 and z_2 externally in the ratio $k : 1$ where $k > 0$, so that

$$z = \frac{z_1 - kz_2}{1-k}$$

As z lies on $\bar{a}z + a\bar{z} + b = 0$,

$$\bar{a}(z_1 - kz_2) + a(\bar{z}_1 - k\bar{z}_2) + b(1-k) = 0$$

$$\Rightarrow \qquad \frac{\bar{a}z_1 + a\bar{z}_1 + b}{\bar{a}z_2 + a\bar{z}_2 + b} = k > 0$$

34. $z_1 + z_2 = -a$, $z_1 z_2 = b$

Now, 0, z_1, z_2 form an equilateral triangle

$\Leftrightarrow 0^2 + z_1^2 + z_2^2 = 0(z_1 + z_2) + z_1 z_2$

$\Leftrightarrow (z_1 + z_2)^2 = 3z_1 z_2$

$\Leftrightarrow a^2 = 3b$

35. $z(|z| + a) = -2a$

$$\Rightarrow \qquad z = \frac{-2a}{|z| + a}$$

As $a > 0$, $|z| \geq 0$, z is real and negative.

36. $$(a-2) + ib = -\frac{(1 + 2\cos\theta) + 2i\sin\theta}{(2 + \cos\theta) + i\sin\theta}$$

$$\Rightarrow (a-2)^2 + b^2 = \frac{(1 + 2\cos\theta)^2 + 4\sin^2\theta}{(2 + \cos\theta)^2 + \sin^2\theta}$$

$$= \frac{5 + 4\cos\theta}{5 + 4\cos\theta} = 1$$

37. $(z-4)^2 = -7$

$$\Rightarrow \qquad z^2 - 8z + 23 = 0$$

Dividing $z^3 - 4z^2 - 9z + 91$ by $z^2 - 8z + 23$ we get

$$z^3 - 4z^2 - 9z + 91$$
$$= (z+4)(z^2 - 8z + 23) - 1$$
$$= -1$$

38. $|z| = |z - 1|$ represent the straight $z = 1/2$.

This line meets the circle $|z| = 2$ in exactly two points viz. $\dfrac{1}{2} \pm \dfrac{\sqrt{15}}{2}i$.

39. The line joining $z_1 + 1$ and $z_2 + 1$ is parallel to the line joining z_1 and z_2 and is at distance 1 from it.

$\therefore$ z is at a distance one from the line joining $z_1 + 1$ and $z_2 + 1$

40. As $z_2 = \dfrac{4z_1 + 3z_3}{7}$

z_2 divides the join of z_1 and z_3 in the ratio 3 : 4.

41. $z = \cos(\pi/3) + i\sin(\pi/3)$

$$z^2 = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2}(1 - \sqrt{3}i)$$

$\therefore \quad z^2 + z = \sqrt{3}i$

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42. Let $z_1 = \cos\theta + i\sin\theta$

and $z_2 = \cos\phi + i\sin\phi$

$a = \cos\theta,\ b = \sin\theta$

$c = \cos\phi,\ d = \sin\phi$

$$z_1\bar{z}_2 = \cos(\theta-\phi) + i\sin(\theta-\phi)$$

$$\text{Re}(z_1\bar{z}_2) = 0 \Rightarrow \cos(\theta-\phi) = 0$$

$\Rightarrow \theta - \phi = \pi/2$

$\therefore z_1 = -\sin\phi + i\cos\phi$

$\Rightarrow a = -\sin\phi = -d,$

$b = \cos\phi = c$

Now, $w_1 = a + ic = -\sin\phi + i\cos\phi$

$w_2 = b + id = \cos\phi + i\sin\phi$

$|w_1| = |w_2| = 1 \Rightarrow |w_1w_2| = 1$

Also, $\text{Re}(\bar{w}_1 w_2) = 0$

43. $2\cos\theta = x + 1/x$

$\Rightarrow x = \cos\theta \pm i\sin\theta$

Similarly $y = \cos\phi \pm i\sin\phi$

$$x^n + \frac{1}{x^n} = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$$

$$= 2\cos(n\theta)$$

Next $$\frac{x}{y} = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$\therefore \frac{x}{y} + \frac{y}{x} = 2\cos(\theta-\phi)$$

Lastly, $xy = \cos(\theta+\phi) + i\sin(\theta+\phi)$

$$\Rightarrow xy + \frac{1}{xy} = 2\cos(\theta+\phi)$$

44. $|z_1/z_2 + 1| = |z_1/z_2 - 1|$

$\Rightarrow z_1/z_2$ lies on the perpendicular bisector of -1 and 1

$\Rightarrow z_1/z_2$ lies on the imaginary axis

$\Rightarrow z_1 = ikz_2$ for some $k \in \mathbf{R}$

Also, the join of O and z_1 makes an angle of $\pi/2$ with the join of O and z_2.

45. $AC = |-iz_1 + iz_2| = |z_1 - z_2| = AB$

Also, $BC = |(1-i)(z_1 - z_2)| = \sqrt{2}\ |AB.$

Thus, A, B, C are vertices of an isosceles right triangle.

46. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|$

$\Rightarrow |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1z_2 = |z_1|^2 + |z_2|^2 + 2|z_1z_2|$

$\Rightarrow z_1\bar{z}_2 + \bar{z}_1z_2 \le 0$

Also, since

$$|z_1 - z_2| = |z_1 - 0| + |z_2 - 0| \quad (1)$$

we get z_1, z_2, O are collinear.

In view of (1), z_1 and z_2 lie on a straight line on a straight line opposite side of O and

$$\arg\left(\frac{z_1}{z_2}\right) = \pi$$

47. $z_1 = \cos\theta + i\sin\theta$

$z_2 = 2(\cos\phi + i\sin\phi)$

$\Rightarrow |z_1 - 2z_2|^2 = (\cos\theta - 4\cos\phi)^2 + (\sin\theta - 4\sin\phi)^2$

$= 1 + 16 - 8\cos(\theta-\phi)$

$\Rightarrow 9 \le |z_1 - 2z_2|^2 \le 25$

Similarly, $1 \le |z_1 + z_2| \le 3$, $|z_1 - z_2| \ge 1$ and $|z_1 - 3z_2| \ge 5$.

48. Refer Fig. 1.60

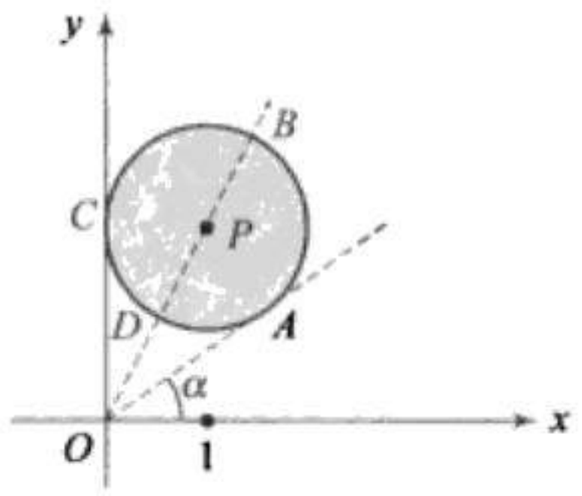

Fig. 1.60

Min (arg z) = α

where $y = (\tan\alpha)x$ is tangent to the circle

$$(x-1)^2 + (y-2)^2 = 1 \quad (1)$$

$$\therefore \frac{|\tan\alpha - 2|}{\sqrt{1+\tan^2\alpha}} = 1$$

$\Rightarrow (\tan\alpha - 2)^2 = \sec^2\alpha$

$\Rightarrow \tan^2\alpha - 4\tan\alpha + 4 = \sec^2\alpha \Rightarrow \tan\alpha = 3/4$

$\therefore$ Min (arg z) = $\tan^{-1}(3/4)$

Clearly, max (arg z) = $\pi/2$

Also, Min ($|z|$) = OD and max ($|z|$) = OB where ODB pass through the centre of (1).

Now, $OD = OP - 1 = \sqrt{5} - 1$

and $OB = OP + 1 = \sqrt{5} + 1$

49. (a) is true as sides an equilateral of triangle are equal.

For (b) note that

$$0^2 + z_1^2 + z_2^2 = 0(z_1 + z_2) + z_1z_2$$

$$\Rightarrow (z_1 + z_2)^2 = 3z_1z_2$$

As $\angle z_1\,Oz_2 = \pi/3$, (c) is true.

50. For all values of k except $k = 3$. In case $k = 3$, the equation represents a straight line.

51. $x = |1 - z| + |z + 1| + |z|$

$\ge |(1 - z) + (z + 1)| + |z| \ge 2 + |z| \ge 2$

This value is attained at $z = -1, 0, 1$.

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52. In an equilateral triangle orthocentre, circumcentre and centroid coincide.
For (c), recall for an equilateral triangle
$$(z_1 + z_2 + z_3)^2 = 3(z_2z_3 + z_3z_1 + z_1z_2)$$

53. If z is any point $\bar{b}z + b\bar{z} = c$, then
$$|z - z_1| = |z - z_2|$$
$$\Rightarrow \quad (\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} + |z_1|^2 - |z_2|^2 = 0$$
This line is identical to the line
$$\bar{b}z + b\bar{z} - c = 0$$
$$\therefore \quad \frac{\bar{z}_2 - \bar{z}_1}{\bar{b}} = \frac{z_2 - z_1}{b} = \frac{|z_2|^2 - |z_1|^2}{c}$$

54. Each of the given line is tangent to $x^2 + y^2 = 1/2$

55. $|z_1 - z_2|^2 = |1 - z_1\,z_2|^2$
$$\Leftrightarrow \quad |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1z_2 = 1 + |\bar{z}_1z_2|^2 - \bar{z}_1z_2 - z_1\bar{z}_2$$
$$\Leftrightarrow \quad (|z_1|^2 - 1)(|z_2|^2 - 1) = 0$$

56. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$
$$\Leftrightarrow \quad z_1\bar{z}_2 + \bar{z}_1z_2 = 0 \quad \Leftrightarrow \quad z_1\bar{z}_2 \text{ is purely imaginary}$$
$$\Leftrightarrow \quad \frac{z_1}{z_2}|z_2|^2 \text{ is purely imaginary}$$
As $\frac{z_1}{z_2}$ is purely imaginary, Oz_1, z_2 is a right triangle.

57. $\dfrac{z^2}{z-1} = \dfrac{\bar{z}^2}{\bar{z}-1}$
$$\Leftrightarrow \quad z^2(\bar{z} - 1) = \bar{z}^2(z - 1)$$
$$\Leftrightarrow \quad z\bar{z}(z - \bar{z}) - (z^2 - \bar{z}^2) = 0$$
$$\Leftrightarrow \quad (z\bar{z} - z - \bar{z})(z - \bar{z}) = 0$$
$$\Leftrightarrow \quad z \text{ lies on circle } z\bar{z} - z - \bar{z} = 0 \text{ or } z \text{ lies on the real axis.}$$

58. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$ (1)
But $\text{Re}(z_1\bar{z}_2) \le |z_1\bar{z}_2| = \left(\sqrt{c}\,|z_1|\right)\left(\frac{1}{\sqrt{c}}|z_2|\right)$
$$\le c|z_1|^2 + (1/c)\,|z_2|^2$$
Thus, from (1)
$$|z_1 - z_2|^2 \ge (1 - c)\,|z_1|^2 + (1 - 1/c)\,|z_2|^2$$

59. Since $|z| + |5 - z| \ge |z + (5 - z)| = 5$, minimum value of $|z| + |z - 5|$ is 5.

60. $\cos\theta + i\sin\theta = \dfrac{3}{z} - 2$
$$\Rightarrow \quad \frac{|3 - 2z|^2}{|z|^2} = 1$$
$$\Rightarrow \quad 4|z|^2 - 6z - 6\bar{z} + 9 = |z|^2$$
$$\Rightarrow \quad |z|^2 - 2z - 2\bar{z} + 3 = 0$$
$$\Rightarrow \quad |z - 2|^2 = 1 \quad \text{or} \quad |z - 2| = 1$$

61. As L.H.S. is real, $\text{Im}(z) = -2$

62. Complete parallelogram taking BC, CA and AB as diagonals.

63. Let
$$S = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{z_j}{z_k} = \left(\sum_{k=1}^{n}z_k\right)\left(\sum_{j=1}^{n}\frac{1}{z_j}\right)$$
$$= \left(\sum_{k=1}^{n}z_k\right)\left(\sum_{j=1}^{n}\frac{\bar{z}_j}{r^2}\right)$$
$$[\because \; z\bar{z}_j = r]$$
$$= \frac{1}{r^2}\left(\sum_{k=1}^{n}z_k\right)\left(\overline{\sum_{j=1}^{n}z_j}\right)$$
$$= \frac{1}{r^2}\left|\sum_{k=1}^{n}z_k\right|^2$$
Thus, S is a real number, so
$$\text{Re } S = S = 0 \Leftrightarrow \sum_{k=1}^{n}z_k = 0$$
$$\Rightarrow \quad \sum_{k=1}^{n}z_k = 0 \Rightarrow \sum_{k=1}^{n}\bar{z}_k = 0$$
$$\Rightarrow \quad \sum_{k=1}^{n}\frac{r^2}{z_k} = 0 \Rightarrow \sum_{k=1}^{n}\frac{1}{z_k} = 0$$

64. $\log_{1/2}(|z|^2 + 4|z| + 3) > 1$
$\Rightarrow \quad |z|^2 + 4|z| + 3 < 1/2$. Not possible

65. $\log_{|z+i|}(5) < \log_{|z+i|}(13)$
$\Rightarrow \quad |z + i| > 1$

66. $e^z = 0 \Rightarrow e^x\cos y = 0, e^x\sin y = 0$
As $e^x \ne 0$, $\cos y = \sin y = 0$. Not possible

67. $\sin z = 0 \quad \Rightarrow \quad e^{2iz} = 1$
$$\Rightarrow \quad e^{-2y}\cos 2x = 1, e^{-2y}\sin 2x = 0$$
$$\Rightarrow \quad \sin 2x = 0 \quad \Rightarrow \quad x = n\pi/2, n \in \mathbf{I}.$$
$$\therefore \quad e^{-2y}\cos(ny) = 1$$
$$\Rightarrow \quad (-1)^n e^{-2y} = 1 \quad \Rightarrow \quad n = 2m$$
and $\quad e^{-2y} = 1$

68. Similar to 67.

69. $\sin(iy) = 0 \quad \Rightarrow \quad e^{-2y} = 1 \quad \Rightarrow \quad y = 0$

70. $e^{x_1}\cos y_1 = e^{x_2}\cos y_2$
$e^{x_1}\sin y_1 = e^{x_2}\sin y_2$
$$\Rightarrow \quad \tan y_1 = \tan y_2 \quad \Rightarrow \quad y_2 = n\pi + y_1$$
As in problem 67, $y_2 = 2n\pi + y_1$.
Thus, $\quad e^{x_1} = e^{x_2} \quad \Rightarrow \quad x_1 = x_2$
$\therefore \quad z_1 = z_2 + 2n\pi$

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71. $\log z = 1 \quad \Leftrightarrow \quad \log |z| = 1, \arg z = 0$

72. Can be checked easily by taking
$$z = -1, \quad z' = -2$$

73. $|e^{x+iy}| = e^x$

74. $e^{\log z} = e^{\log|z|}\, e^{i(\arg z)}$
$= |z| e^{i\arg(z)}$

75. $\log(-i) = \log|-i| + i \arg(-i) = -\pi i/2$

76. Note that $\omega^n = 1$ and $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are roots of $x^n - 1 = 0$. Thus,
$$\frac{1}{x-1} + \frac{1}{x-\omega} + \cdots + \frac{1}{x-\omega^{n-1}} = \frac{nx^{n-1}}{x^n - 1}$$
Put $x = 2$

77. Note that $1, -1, \alpha, \bar{\alpha}, \ldots \alpha^{n-1}, \bar{\alpha}^{n-1}$ are the 2n roots of $x^{2n} - 1 = 0$. Thus,
$x^{2n} - 1 = (x-1)(x+1)(x-\alpha)(x-\bar{\alpha}) \ldots (x-\alpha^{n-1})$
$(x-\bar{\alpha}^{n-1})$

78. Use $(x-\alpha)(x-\bar{\alpha}) = x^2 - 2x\cos\frac{\pi}{n} + 1$ etc.
and $\frac{x^{2n}-1}{x^2-1} = x^{2n-2} + x^{2n-3} + \ldots + x + 1$

79. In 78, divide both the sides by x^{n-1} to obtain
$$\left(x + \frac{1}{x} - 2\cos\frac{\pi}{n}\right)\left(x + \frac{1}{x} - 2\cos\frac{2\pi}{n}\right)$$
$$\cdots\left(x + \frac{1}{x} - 2\cos\frac{n-1}{n}\pi\right) = \frac{x^n - x^{-n}}{x - x^{-1}}$$
Now, put $x = \cos\theta + i\sin\theta$

80. From 79,
$$\frac{\sin(n\theta)}{\sin\theta} = 2^{n-1}(\cos\theta - \cos\frac{\pi}{n}) \cdots (\cos\theta - \cos\frac{n-1}{n}\pi)$$
Take limit as $\theta \to 0$ to obtain
$$n = 2^{n-1}\left(2\sin^2\frac{\pi}{2n}\right)\left(2\sin^2\frac{2\pi}{2n}\right)\cdots\left(2\sin^2\frac{n-1}{2n}\pi\right)$$
$$= 2^{n-1}\left(2\sin\frac{\pi}{2n}\sin\frac{n-1}{2n}\pi\right)\left(2\sin\frac{2\pi}{2n}\sin\frac{n-2}{2n}\pi\right)$$
$$\cdots\left(2\sin\frac{n-1}{2n}\pi\sin\frac{\pi}{n}\right)$$
Now, use
$$2\sin\frac{k\pi}{2n}\sin\left(\frac{n-k}{2n}\pi\right) = \sin\left(\frac{k\pi}{n}\right)$$

81. $\alpha + \beta = \sum_{k=1}^{n} \omega^k = \frac{\omega(1-\omega^6)}{1-\omega} = -1$
as $\omega^7 = 1$

82. $\alpha\beta = (\omega + \omega^2 + \omega^4)(\omega^3 + \omega^5 + \omega^6)$
$= 3 + (\alpha + \beta) = 2$

83. Desired equation is
$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

84. $\alpha = \frac{1}{2}(-1 \pm \sqrt{7}i)$
But $\text{Im}\,\alpha = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{\pi}{7} > 0$
$\therefore \quad \alpha = \frac{1}{2}(-1 + \sqrt{7}i)$

85. (b) $\sum_{k=0}^{6} w^{k^2} = 1 + 2\alpha = \sqrt{7}i$

86. Let $\frac{3iz_2}{5z_1} = k$ where $k \in \mathbf{R}$
$\Rightarrow \frac{z_2}{z_1} = -\frac{5i}{3}k = ai\,(say)$
Now, $5\left|\frac{3z_1 + 7z_2}{3z_1 - 7z_2}\right| = 5\left|\frac{3 + 7ai}{3 - 7ai}\right| = 5$

87. $-3 + x^2yi = x^2 + y - 4i$
$\Rightarrow \quad x^2 + y = -3, \; x^2y = -4$
$\Rightarrow \quad -\frac{4}{y} + y = -3 \quad \Rightarrow \quad -4 + y^2 = -3y$
$\Rightarrow \quad y^2 + 3y - 4 = 0 \quad \Rightarrow \quad y = 1, -4$
$y = 1 \quad \Rightarrow \quad x^2 = -4,$
not possible.
$y = -4 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$
Thus, $|x| + |y| = 5$

88. Let $z = x + iy$
$$x + iy + \sqrt{2(x+1)^2 + 2y^2} + i = 0$$
$\Rightarrow \quad x + \sqrt{2(x+1)^2 + 2y^2} = 0, \; y = -1$
$\therefore \quad x + \sqrt{2(x+1)^2 + 2} = 0$
$\Rightarrow \quad 2(x+1)^2 + 2 = x^2$
$\Rightarrow \quad x^2 + 4x + 4 = 0 \quad \Rightarrow \quad x = -2$
Thus, $|z|^2 = 4 + 1 = 5$

89. $4|(1+i)z^3 + iz| \le 4|1+i|\,|z|^3 + 4|i|\,|z|^3$
$$< 4\sqrt{2}\left(\frac{1}{8}\right) + 2 < 3$$

90. As $p_1, p_2, \ldots p_n \in \mathbf{R}$, $\cos\theta - i\sin\theta = 1/z$ a root of the given equation.
$\Rightarrow 1 + p_1 z + p_2 z^2 + \ldots + p_n z^n = 0$
$\Rightarrow p_1 \sin\theta + p_2 \sin 2\theta + \ldots + p_n \sin(n\theta) = 0$

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91. 100th roots of unity are $\alpha_0, \alpha_1, \ldots, \alpha_{99}$

where $\alpha_j = \cos\left(\frac{2\pi j}{100}\right) + i\sin\left(\frac{2\pi j}{100}\right)$

Thus,

$(x - \alpha_1) \ldots (x - \alpha_{99}) = x^{99} + \ldots + x + 1$

Putting $x = 1$,

$(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{99}) = 100$

$\Rightarrow \quad (1 - \bar{\alpha}_1)(1 - \bar{\alpha}_2) \ldots (1 - \bar{\alpha}_{99}) = 100$

$\therefore \quad (1 - \alpha_1)(1 - \bar{\alpha}_1)(1 - \alpha_2)(1 - \bar{\alpha}_2) \ldots$

$(1 - \alpha_{99})(1 - \bar{\alpha}_{99}) = 10^4$

But $(1 - \alpha_k)(1 - \bar{\alpha}_k)$

$= 2\left(1 - \cos\frac{2k\pi}{100}\right) = 2^2 \sin^2\left(\frac{k\pi}{100}\right)$

Thus,

$2^{198} \sin^2\left(\frac{\pi}{100}\right) \ldots \sin^2\left(\frac{99\pi}{100}\right) = 10^4$

$\Rightarrow \quad 2^{99} \sin\left(\frac{\pi}{100}\right) \ldots \sin\left(\frac{99\pi}{100}\right) = 100$

92. The given equation can be written as

$8z^3 - 12iz^2 + 18iz + 27 = 0$

$\Rightarrow \quad 4z^2(2z - 3i) + 9i(2z - 3i) = 0$

$\Rightarrow \quad (4z^2 + 9i)(2z - 3i) = 0$

$\Rightarrow \quad z^2 = -9i/4 \quad \text{or} \quad z = 3i/2$

$\Rightarrow \quad |z|^2 = 9/4$

93. If $|z + 4| = 3$, z lies on a circle with centre at -4 and radius 3. Furtherst point on this circle from 7 is -7. Therefore maximum value of $|z - 7|$ is 14

94. Note that ω, ω^2 are roots of

$$\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} = \frac{2}{x} \qquad (1)$$

$\Rightarrow \quad x^3 + (bc + ca + ab)x + 2abc = 0$

If α is the third root of this equation, then $\alpha + w + \omega^2 = 0 \quad \text{or} \quad \alpha = 1$.

From (1), we get

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$$

95. $x = 1 \pm \sqrt{3}i = 2\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right)$

$\alpha^6 + \beta^6 = 2^7 \cos 2\pi = 128$

96. $\begin{vmatrix} a\cos\alpha & a\sin\alpha & 1 \\ b\cos\beta & b\sin\beta & 1 \\ c\cos\gamma & c\sin\gamma & 1 \end{vmatrix} = 0$

$\Rightarrow \quad bc\sin(\gamma - \beta) + ca\sin(\alpha - \gamma) + ab\sin(\beta - \alpha) = 0$

$\therefore \quad |z| = 3$

97. $\alpha\beta = (\omega + \omega^2 + \omega^4)\left(\frac{1}{\omega} + \frac{1}{\omega^2} + \frac{1}{\omega^4}\right)$

$= 3 + \sum_{k=1}^{6} w^k = 3 + (-1) = 2$

98. (a) $|z|^n = |1 - z|^n \quad \Rightarrow \quad |z| = |1 - z|$

$\Rightarrow$ z lies on $x = 1/2$

Similarly for (b)

(c) $|z|^n = 5 \quad \Rightarrow \quad |z| = 5^{1/n}$

(d) $|z - i| = 5^{1/n}$

99. $z^3 = i = (-i)^3$

$\Rightarrow \quad z = -i, -i\omega, -i\omega^2$

Let $\alpha = -i\omega, \beta = -i\omega^2$

100. (a) Both the roots of $x^2 + ax + b = 0$ are non-negative if $a \le 0, b \ge 0$

(b) $x^2 + ax + b = 0$ has one non-negative and one negative root if $b \le 0$

(c) $x^2 + ax + b = 0$ has both negative roots if $a > 0, b > 0$

101. (a) $\sin^{-1}|z|$ is defined for all z with $|z| \le 1$.

(b) $\tan^{-1}|z|$ is defined for each $z \in \mathbf{C}$.

(c) $\log|z|$ is defined for $|z| > 0$ and $\cos^{-1}|z|$ for $|z| \le 1$

(d) $\log|z - 1|$ is defined for $|z - 1| > 0$ and $\sec^{-1}|z|$ for $|z| \ge 1$.

102. (a) $\left|\frac{z-9}{z-1}\right| = 3 \quad \Leftrightarrow \quad |z| = 3$

(b) $|z - (5 + 3i)| = \frac{2\sqrt{1+t^2}}{\sqrt{1+t^2}} = 2$

(c) $3, \sqrt{3}i$ and $\sqrt{3}i$ are equidistant from 1.

(d) Mid point of the diameter is the centre

103. $\alpha, \beta = \cot\theta \pm i$

Also, $\alpha\beta = \text{cosec}^2\theta, \quad \alpha + \beta = 2\cot\theta$

$\alpha^n + \beta^n = 2\,\text{cosec}^n\theta\cos\theta$

and $|\alpha|^n + |\beta|^n = 2\,\text{cosec}^n\theta$

104. (a) $e^x\cos y = 2, e^x\sin y = 0$

$\Rightarrow \quad \sin y = 0$ and $\cos y > 0 \quad \Rightarrow \quad y = 2n\pi, n \in \mathbf{I}$

Also, $e^x = 2 \quad \Rightarrow \quad x = \log 2$

Thus, $z = \log 2 + 2n\pi i, n \in \mathbf{I}$

(b) $e^x\cos y = 1, e^x\sin y = -\sqrt{3}$

Square and add to obtain

$e^{2x} = 4 \quad \Rightarrow \quad e^x = 2 \quad \Rightarrow \quad x = \log 2$

$\therefore \quad \cos y = 1/2, \sin y = -\sqrt{3}/2$

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$\Rightarrow \quad y = 2n\pi - \pi/3$

$\therefore \quad z = \log 2 \pm i\,(2n\pi - \pi/3),\ n \in \mathbf{I}.$

(c) $e^x \cos y = e^2,\ e^x \sin y = 0$

Squaring and adding

$$e^{2x} = e^4 \Rightarrow x = 2$$

$\therefore \quad \cos y = 1,\ \sin y = 0 \quad \Rightarrow \quad y = 2n\pi$

Thus, $\quad z = 2 + 2n\pi i,\ n \in \mathbf{I}$

(d) $e^x \cos y = -1,\ e^x \sin y = 0$

Square and add to obtain $x = 0$

$\cos y = -1,\ \sin y = 0 \quad \Rightarrow \quad y = (2n+1)\pi,\ n \in \mathbf{I}$

105. Centroid G of ΔABC is $\frac{1}{3}(z_1 + z_2 + z_3)$. Let Orthocentre H of ΔABC be z. As G divides the join of H and S in the ratio 2 : 1, we get

$$\frac{1}{3}(z_1 + z_2 + z_3) = \frac{2z_0 + z}{2+1}$$

$$\Rightarrow \quad z = z_1 + z_2 + z_3 - 2z_0.$$

As $AD \perp BC$, $\dfrac{z_4 - z_1}{z_2 - z_3}$ is purely imaginary.

$$\Rightarrow \quad \frac{z_4 - z_1}{z_2 - z_3} + \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_3} = 0$$

$$\Rightarrow \quad \frac{w_4 - w_1}{w_2 - w_3} + \frac{\bar{w} - \bar{w}_1}{\bar{w}_2 - \bar{w}_3} = 0$$

where $w_i = z_i - z_0$ for $i = 1, 2, 3, 4$

As z_1, z_2, z_3, z_4 lie on a circle with centre at z_0

$$|z_i - z_0| = r$$

$$\Rightarrow \quad w_i \bar{w}_i = r^2$$

for $i = 1, 2, 3, 4$.

Now, from (1), we get

$$\frac{w - w_1}{w_2 - w_3} + \frac{r^2/w_4 - r^2/w_1}{r^2/w_2 - r^2/w_3} = 0$$

$$\Rightarrow \quad \frac{w - w_1}{w_2 - w_3}\left[1 + \frac{w_2 w_3}{w w_1}\right] = 0$$

$$\Rightarrow \quad w = -\frac{w_2 w_3}{w_1}$$

106. Take conjugate of the given expression to see that $az_1\bar{z}_1 + b(z_1\bar{z}_2 + \bar{z}_1 z_2) + cz_2\bar{z}_2$ is real.
For statement-2 see Theory in chapter 2.

107. Use Pythagoras theorem and statement-2.

108. $$\sqrt{\frac{5}{3}} = \left|\frac{2z+3}{3z+1}\right| = \frac{2}{3}\left|\frac{z+3/2}{z+1/3}\right|$$

$$\Rightarrow \quad \left|\frac{z+3/2}{z+1/3}\right| = \frac{3}{2}\sqrt{\frac{5}{3}} = \frac{\sqrt{15}}{2} \neq 1$$

represents a circle.
For statement-2, see Theory.

109. For statement-2, note that $\omega^k = \bar{\omega}^k = 1$ if k is a multiple of 3, and $i - k$ is not a multiple of 3, then one of ω^k, $\bar{\omega}^k$ equals w and other equals ω^2.
Next, note that roots of $z^2 - z + 1 = 0$ are $-\omega$ and $-\omega^2 = -\bar{\omega} - 1/\omega$. Now, use

$$(z^k + z^{-k})^2 = [(-\omega) + (-\bar{\omega})^k]^2$$

$$= \begin{cases} 1 & \text{if } k \text{ is not a multiple of 3} \\ 4 & \text{if } k \text{ is a multiple of 3} \end{cases}$$

110. Let $a = \alpha + i\beta$ and $a_1 = \alpha_1 + i\beta_1$
Now, the two lines are given by

$$2(\alpha x + \beta y) + b = 0 \qquad (1)$$

and $\quad 2(\alpha_1 x + \beta_1 y) + b_1 = 0 \qquad (2)$

The lines (1) and (2) are parallel if and only if

$$-\frac{\alpha}{\beta} = -\frac{\alpha_1}{\beta_1} \Leftrightarrow \frac{\alpha}{i\beta} = \frac{\alpha_1}{i\beta}$$

$$\Leftrightarrow \quad \frac{\alpha + i\beta}{\alpha - i\beta} = \frac{\alpha_1 + i\beta_1}{\alpha_1 - i\beta_1} \Leftrightarrow \frac{a}{\bar{a}} = \frac{a_1}{\bar{a}_1}$$

$$\Leftrightarrow \quad \frac{a}{a_1} = \overline{\left(\frac{a}{a_1}\right)} \Leftrightarrow \frac{a}{a_1} \text{ is real}$$

Next, (1) and (2) are perpendicular to each other if and only if

$$\left(-\frac{\alpha}{\beta}\right)\left(-\frac{\alpha_1}{\beta_1}\right) = -1$$

$$\Leftrightarrow \quad \frac{\alpha}{i\beta} = \frac{-\beta_1}{i\alpha_1}$$

$$\Leftrightarrow \quad \frac{\alpha + i\beta}{\alpha - i\beta} = \frac{-\beta_1 + i\alpha_1}{-\beta_1 - i\alpha_1} = \frac{i(\alpha_1 + i\beta_1)}{(-i)(\alpha_1 - i\beta_1)}$$

$$\Leftrightarrow \quad \frac{a}{\bar{a}} = -\frac{a_1}{\bar{a}_1} \Leftrightarrow \frac{a}{a_1} \text{ is purely imaginary}$$

111. $$f(\theta)^2 = \frac{4}{(3+\sin\theta)^2 + \cos^2\theta} = \frac{4}{10 + 6\sin\theta}$$

$$= \frac{2}{5 + 3\sin\theta}$$

$\therefore f(\theta)^2$ is maximum when $\cos\theta = -1$, and maximum when $\cos\theta = 1$.

Thus, $1/2 \le f(\theta) \le 1$.

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112. $m = |a + ib|^2$ and $n = |c + id|^2$,

$$mn = (|a + ib|\,|c + id|)^2$$
$$= |(ac - bd) + i(ad + bc)|^2$$
$$= e^2 + f^2$$

where $e = ac - bd$, $f = ad + bc \in \mathbf{I}$.

Statement-2 is false, since

$$|z_1 + z_2|^2 = (|z_1| + |z_2|)^2$$
$$\Leftrightarrow \quad |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\,\bar{z}_2) = |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2|$$
$$\Leftrightarrow \quad \mathrm{Re}(z_1\,\bar{z}_2) = |z_1\,\bar{z}_2)$$
$$\Leftrightarrow \quad z_1\,\bar{z}_2 \geq 0.$$

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CHAPTER 2

THEORY OF EQUATIONS

2.1 QUADRATIC EQUATIONS

An equation of the form

$$ax^2 + bx + c = 0 \qquad (1)$$

where $a \neq 0$, $a, b, c \in \mathbf{C}$, the set of complex numbers, is called a *quadratic equation*. The numbers a, b, c are called the *coefficients* of the equation.

A *root* of the quadratic equation (1) is a complex number α such that

$$a\alpha^2 + b\alpha + c = 0$$

The quantity $D = b^2 - 4ac$ is known as the *discriminant* of the equation (1).

Roots of (1) are given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

2.2 NATURE OF ROOTS

1. If $a, b, c \in \mathbf{R}$ and $a \neq 0$. Then the followings hold good:
   (a) The equation (1) has real and distinct roots if and only if $D > 0$.
   (b) The equation (1) has real and equal roots if and only if $D = 0$.
   (c) The equation (1) has complex roots with non-zero imaginary parts if and only if $D < 0$.
   (d) $p + iq$ $(p, q \in \mathbf{R}, q \neq 0)$ is a root of (1) if and only if $p - iq$ is a root of (1).
2. If $a, b, c \in \mathbf{Q}$ and D is a perfect square of a rational number, then (1) has rational roots.
3. If $a, b, c \in \mathbf{Q}$ and $p + \sqrt{q}$ $(p, q \in \mathbf{Q})$ is an irrational root of (1) then $p - \sqrt{q}$ is also a root of (1).
4. If $a = 1$, $b, c \in \mathbf{I}$ and the roots of (1) are rational numbers, then these roots must be integers.
5. If (1) is satisfied by more than two distinct complex numbers, then (1) becomes an identity, that is $a = b = c = 0$.

2.3 RELATION BETWEEN ROOTS AND COEFFICIENTS

If α and β are roots of the quadratic equation (1), then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

A quadratic equation whose roots are α and β is given by

$$(x - \alpha)(x - \beta) = 0$$

2.4 QUADRATIC EXPRESSION AND ITS GRAPH

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{R}$ and $a \neq 0$.

We have $f(x) = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \qquad (2)$$

When is a Quadratic Expression Always Positive (Negative)?

It follows from (2) that $f(x) > 0\ (< 0)\ \forall x \in \mathbf{R}$ if and only if $a > 0 (< 0)$ and $D = b^2 - 4ac < 0$. See Fig. 2.1 (Fig. 2.2).

Also, it follows from (2) that $f(x) \geq 0\ (\leq 0)\ \forall x \in \mathbf{R}$ if and only if $a > 0$ (< 0) and $D = b^2 - 4ac = 0$. In this case $f(x) > 0$ (< 0) for each $x \in \mathbf{R}$, $x \neq -b/2a$. and the graph of $y = f(x)$ touches the x-axis at $x = -b/2a$.

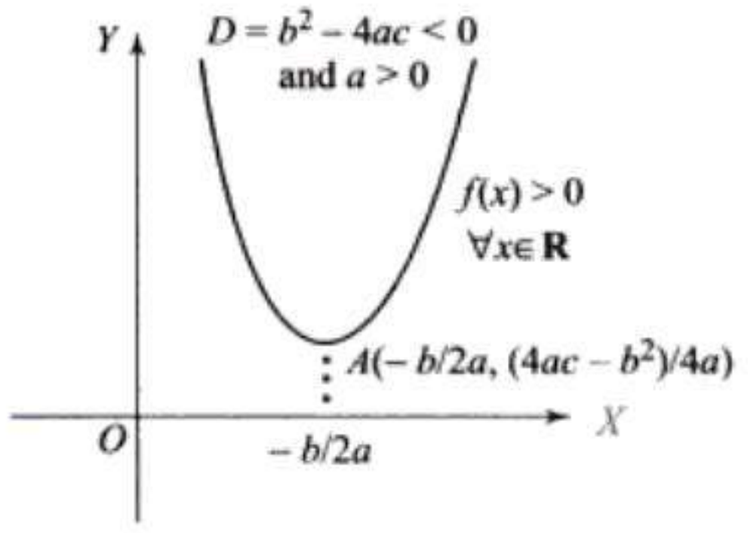

Fig. 2.1

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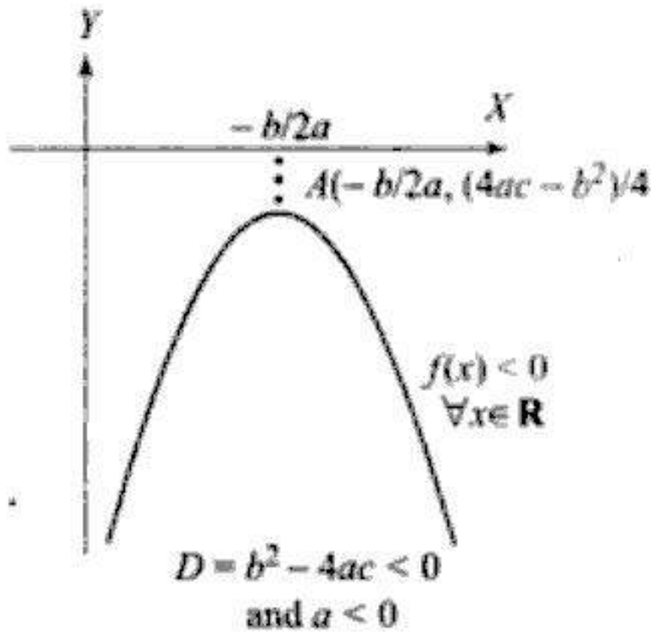

Fig. 2.2

Sign of a Quadratic Expression

If $D = b^2 - 4ac > 0$, then (2) can be written as

$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x + \frac{b-\sqrt{b^2-4ac}}{2a}\right)\right]$$

$$= a\,(x - \alpha)\,(x - \beta)$$

where $\alpha = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $\beta = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$

If $D = b^2 - 4ac > 0$ and $a > 0$ then

$$f(x) \begin{cases} > 0 & \text{for } x < \alpha \text{ or } x > \beta \\ < 0 & \text{for } \alpha < x < \beta \\ = 0 & \text{for } x = \alpha, \beta \end{cases}$$

See Fig. 2.3.

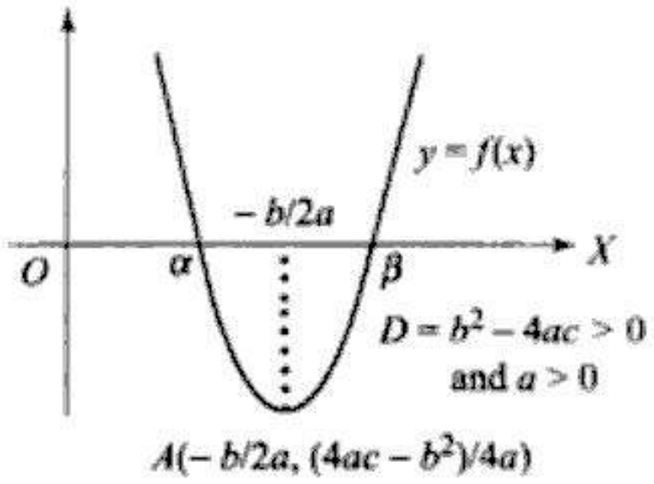

Fig. 2.3

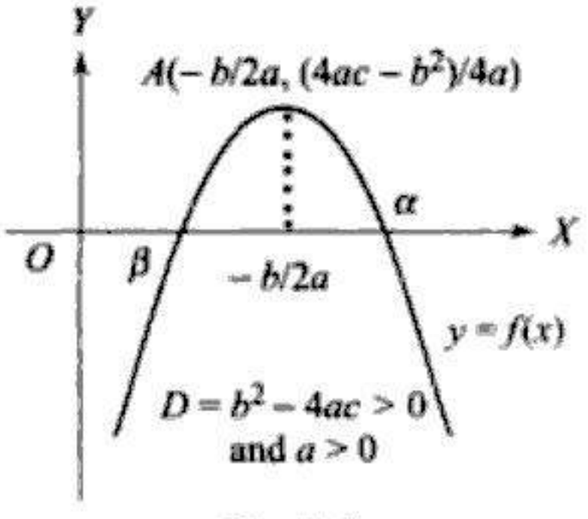

Fig 2.4

If $D = b^2 - 4ac > 0$ and $a < 0$ then

$$f(x) \begin{cases} < 0 & \text{for } x < \beta \text{ or } x > \alpha \\ > 0 & \text{for } \beta < x < \alpha \\ = 0 & \text{for } x = \alpha, \beta \end{cases}$$

See Fig. 2.4.

Note that if $a > 0$, then $f(x)$ attains the least value at $x = -b/2a$. This least value is given by

$$f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}$$

If $a < 0$, then $f(x)$ attains the greatest value at $x = -b/2a$. This greatest value is given by

$$f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}$$

2.5 POSITION OF ROOTS OF A QUADRATIC EQUATION

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{R}$ be a quadratic expression and let a real number.

Conditions for Both the Roots to be More than a Real Number k.

If $a > 0$, then the parabola $y = ax^2 + bx + c$ opens upwards and intersects the x-axis in α and β where

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case both the roots α and β will be more than k if k lies to left of both α and β. See Fig. 2.5.

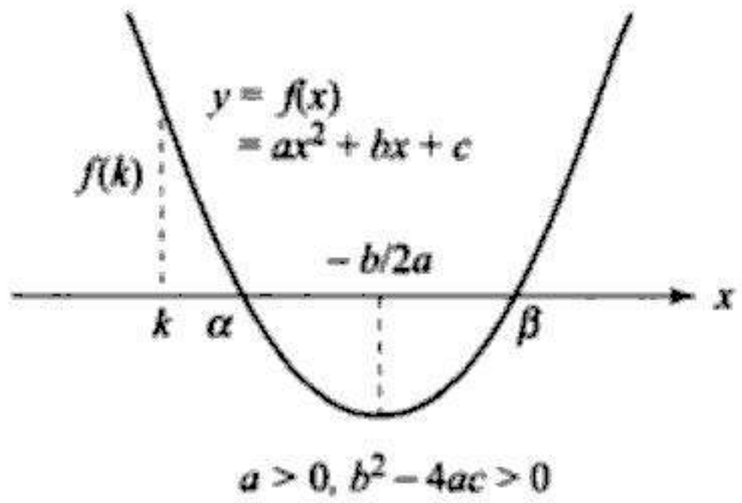

Fig. 2.5

From the Fig. 2.5, we note that both the roots are more than k if and only if

(i) $D > 0$ (ii) $k < \dfrac{-b}{2a}$ (iii) $f(k) > 0$

In case $a < 0$, both the roots will be more than k (refer Fig. 2.6) if and only if

(i) $D > 0$ (ii) $k < \dfrac{-b}{2a}$ (iii) $f(k) < 0$

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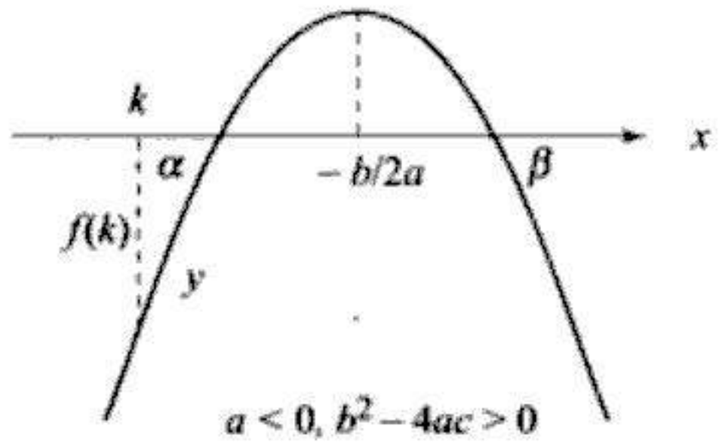

Fig. 2.6

Combining the above two sets of conditions, we get both the roots of $ax^2 + bx + c = 0$ are more than a real number k if only if

(i) $D > 0$ (ii) $k < \frac{-b}{2a}$ (iii) $af(k) > 0$

Conditions for Both the Roots to be Less Than a Real Number k

Both the roots of $ax^2 + bx + c = 0$ are less than a real number k if and only if

(i) $D > 0$ (ii) $k > \frac{-b}{2a}$ (iii) $af(k) > 0$

2.6 CONDITIONS FOR A NUMBER k TO LIE BETWEEN THE ROOTS OF A QUADRATIC EQUATION

The real number k lies between the roots of the quadratic equation $f(x) = ax^2 + bx + c = 0$ if and only if a and $f(k)$ are of opposite signs, that is, if and only if

(i) $a > 0$ (ii) $D > 0$ (iii) $f(k) < 0$

[See Fig. 2.7]

or

(i) $a < 0$ (ii) $D > 0$ (iii) $f(k) > 0$

[See Fig. 2.8]

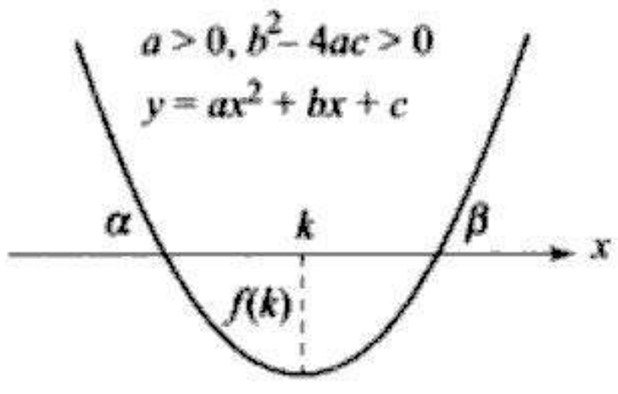

Fig. 2.7

Combining, we may say k lies between the roots of $f(x) = ax^2 + bx + c = 0$ if and only if

(i) $D > 0$ (ii) $af(k) < 0$

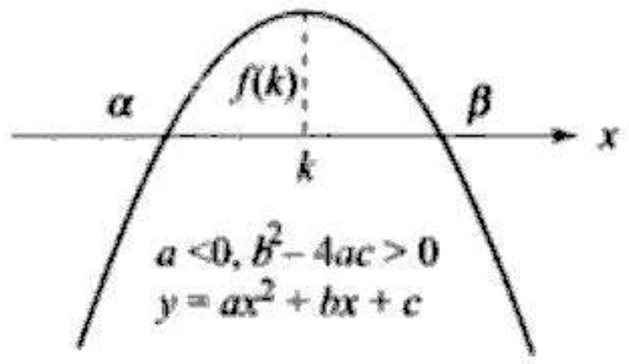

Fig. 2.8

2.7 CONDITIONS FOR EXACTLY ONE ROOT OF A QUADRATIC EQUATION TO LIE IN THE INTERVAL (k_1, k_2) WHERE $k_1 < k_2$

If $a>0$, then exactly one root of $f(x) = ax^2+bx+c=0$ lies in the interval (k_1, k_2) if and only if $f(k_1) > 0$ and $f(k_2) < 0$. Also, exactly one root lies in the interval (k_1, k_2) if and only if $f(k_1) < 0$ and $f(k_2) > 0$. See Fig. 2.9. Thus, if $a>0$, exactly one root of $f(x)=ax^2+bx+c=0$ lies in the interval (k_1, k_2) if and only if $f(k_1)\,f(k_2) < 0$.

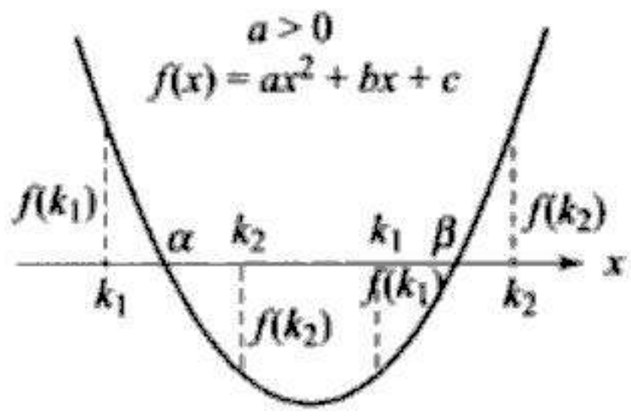

Fig. 2.9

Similarly, if $a < 0$, exactly one of the roots of $f(x) = ax^2 + bx+c=0$ lies in the interval (k_1, k_2) if $f(k_1)\,f(k_2)<0$.

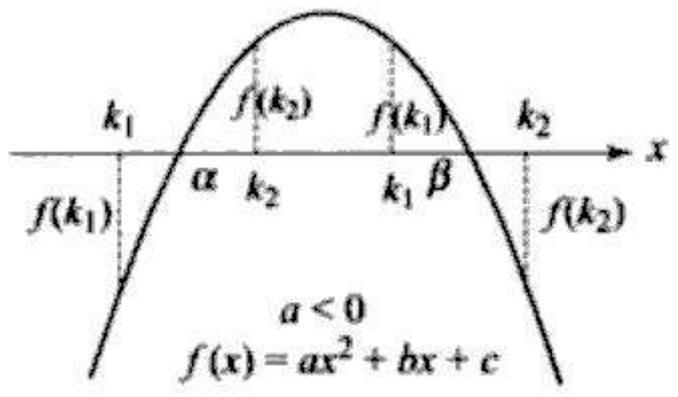

Fig. 2.10

2.8 CONDITIONS FOR BOTH THE ROOTS OF A QUADRATIC EQUATION TO LIE IN THE INTERVAL (k_1, k_2) WHERE $k_1 < k_2$

If $a > 0$, both the roots of $f(x) = ax^2 + bx + c = 0$ lie in the interval (k_1, k_2) if and only if

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(i) $D > 0$ (ii) $k_1 < -\frac{b}{2a} < k_2$

(iii) $f(k_1) > 0$ and $f(k_2) > 0$

See Fig. 2.11.

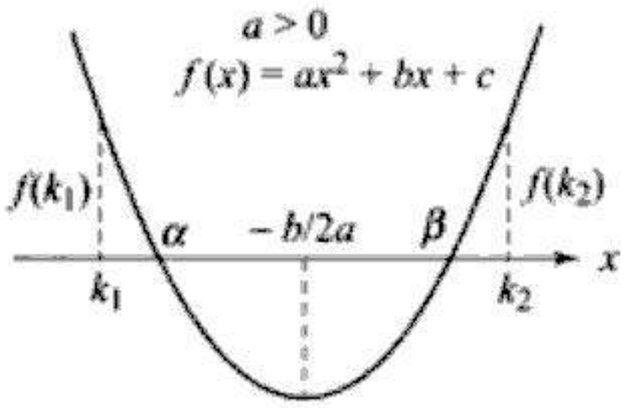

Fig. 2.11

In case $a < 0$, the conditions read as

(i) $D > 0$ (ii) $k_1 < -\frac{b}{2a} < k_2$

(iii) $f(k_1) < 0$ and $f(k_2) < 0$

See Fig. 2.12

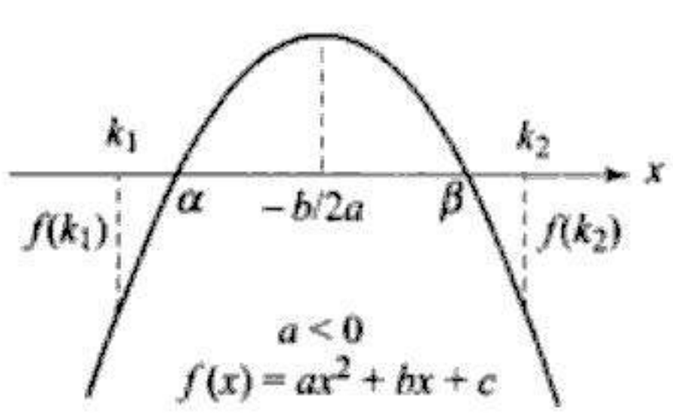

Fig. 2.12

2.9 CONDITIONS FOR A QUADRATIC EQUATION TO HAVE A REPEATED ROOT

The quadratic equation $f(x) = ax^2 + bx + c = 0, a \neq 0$ has α as a repeated if and only if $f(\alpha) = 0$ and $f'(\alpha) = 0$. In this case $f(x) = a(x - \alpha)^2$. In fact $\alpha = -b/2a$. See Fig. 2.13 and Fig. 2.14.

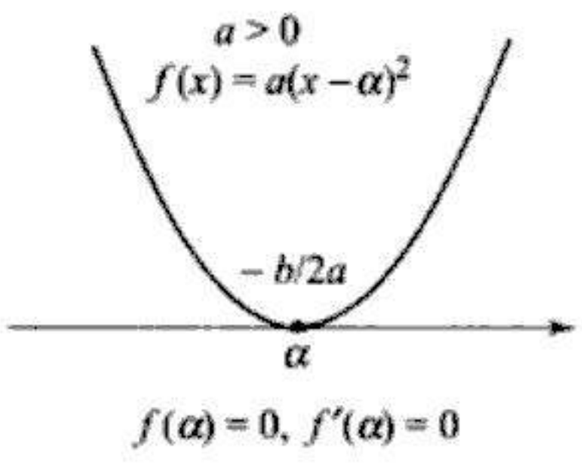

Fig. 2.13

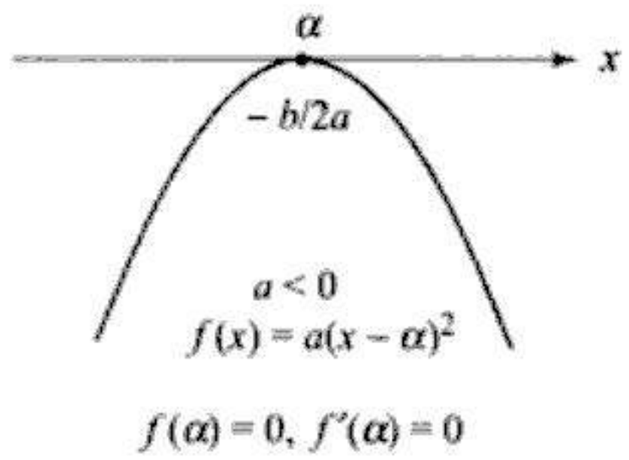

Fig. 2.14

2.10 CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE A COMMON ROOT

Suppose that the quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ (where $a, a' \neq 0$ and $ab' - a'b \neq 0$) have a common root. Let this common root be α. Then

$$a\alpha^2 + b\alpha + c = 0 \quad \text{and} \quad a'\alpha^2 + b'\alpha + c' = 0$$

Solving the above equations, we get

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

$$\Rightarrow \quad \alpha^2 = \frac{bc' - b'c}{ab' - a'b} \text{ and } \alpha = \frac{a'c - ac'}{ab' - a'b}$$

Eliminating α we get

$$\frac{(a'c - ac')^2}{(ab' - a'b)^2} = \frac{bc' - b'c}{ab' - a'b}$$

$$\Rightarrow \quad (a'c - ac')^2 = (bc' - b'c)(ab' - a'b)$$

This is the required condition for two quadratic equation to have a common root.

See Fig. 2.15

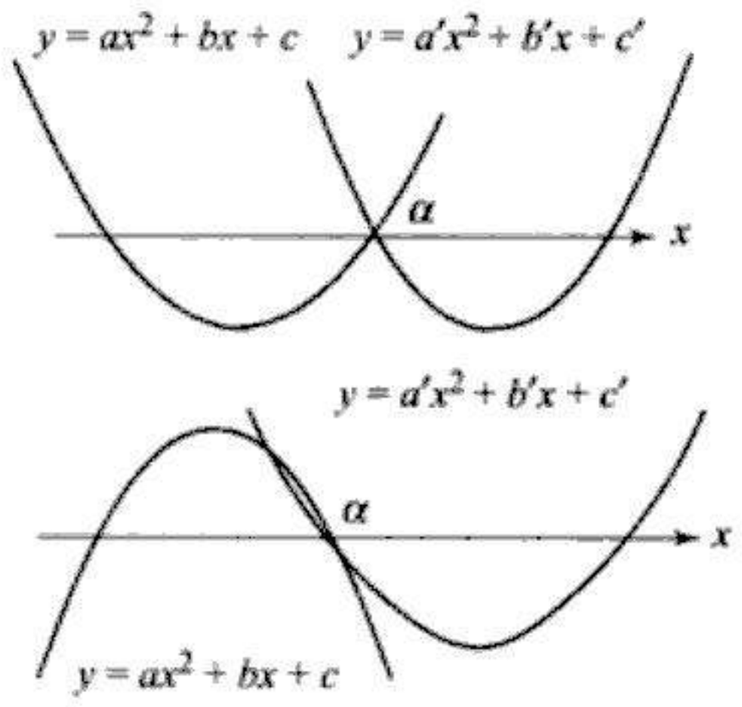

Fig. 2.15

How to Obtain the Common Root?

Make coefficients of x^2 in both the equations same and subtract one equation from the other to obtain a linear equation in x. Solve it for x to obtain the common root.

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2.11 CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE THE SAME ROOTS

In case $a'b - ab' = 0$, then

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

In this case both the equations have the same roots.

2.12 EQUATIONS OF HIGHER DEGREE

The equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

where $a_0, a_1, \cdots, a_{n-1}, a_n \in \mathbf{C}$, the set of complex numbers, and $a_0 \neq 0$, is said to be an equation of degree n. An equation of degree n has exactly n roots. Let $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbf{C}$ be the n roots of (1). Then

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

Also $\alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_1}{a_0}$ and

$$\alpha_1\alpha_2 \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

2.13 CUBIC AND BIQUADRATIC EQUATIONS

If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a},\ \beta\gamma + \gamma\alpha + \alpha\beta = \frac{c}{a} \text{ and } \alpha\beta\gamma = -\frac{d}{a}.$$

Also, if $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$\alpha+\beta+\gamma+\delta = -b/a, (\alpha+\beta)(\gamma+\delta) + \alpha\beta+\gamma\delta = c/a$

$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -d/a,\ \alpha\beta\gamma\delta = e/a.$

2.14 TRANSFORMATION OF EQUATIONS

We now list some of the methods to form an equation whose roots are given in terms of the roots of some other equation.

Let the given equation be

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \qquad (1)$$

Method 1: To form an equation whose roots are $k(\neq 0)$ times roots of the equations in (1), replace x by x/k in (1).

Method 2: To form an equation whose roots are the negatives of the roots in equation (1), replace x by $-x$ in (1). Alternatively, change the sign of the coefficients of $x^{n-1}, x^{n-3}, x^{n-5}, \cdots$ etc. in (1).

Method 3: To form an equation whose roots are k more than the roots of equation in (1), replace x by $x - k$ in (1).

Method 4: To form an equation whose roots are reciprocals of the roots in equation (1), replace x by $1/x$ in (1) and then multiply both the sides by x^n.

Method 5: To form an equation whose roots are square of the roots of the equation in (1) proceed as follows:

Step 1 Replace x by $\sqrt{x}$ in (1)

Step 2 Collect all the terms involving $\sqrt{x}$ on one side.

Step 3 Square both the sides and simplify.

For instance, to form an equation whose roots are squares of the roots of $x^3 + 2x^2 - x + 2 = 0$, replace x by $\sqrt{x}$ to obtain

$$x\sqrt{x} + 2x - \sqrt{x} + 2 = 0$$

$$\Rightarrow \quad \sqrt{x}(x - 1) = -2(x + 1)$$

Squaring we get

$$x(x - 1)^2 = 4(x + 1)^2$$

$$x^3 - 6x^2 - 7x - 4 = 0$$

Method 6: To form an equation whose roots are cubes of the roots of the equation in (1) proceed as follows:

Step 1 Replace x by $x^{1/3}$.

Step 2 Collect all the terms involving $x^{1/3}$ and $x^{2/3}$ on one side.

Step 3 Cube both the sides and simplify.

2.15 DESCARTES RULE OF SIGNS FOR THE ROOTS OF A POLYNOMIAL

Rule 1: The maximum number of positive real roots of a polynomial equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

is the number of changes of the signs of coefficients from positive to negative and negative to positive. For instance, in the equation $x^3 + 3x^2 + 7x - 11 = 0$ the signs of coefficients are

$$+ + + -$$

As there is just one change of sign, the number of positive roots of $x^3 + 3x^2 + 7x - 11 = 0$ is at most 1.

Rule 2: The maximum number of negative roots of the polynomial equation $f(x) = 0$ is the number of changes from positive to negative and negative to positive in the signs of coefficients of the equation $f(-x) = 0$.

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2.16 SOME HINTS FOR SOLVING POLYNOMIAL EQUATIONS

1. To solve an equation of the form
$$(x-a)^4 + (x-b)^4 = A$$
put $y = x - \frac{a+b}{2}$
In general to solve an equation of the form
$$(x-a)^{2n} + (x-b)^{2n} = A$$
where n is a positive integer, we put
$y = x - \frac{a+b}{2}$.

2. To solve an equation of the form
$$a_0 (f(x))^{2n} + a_1 (f(x))^n + a_2 = 0 \qquad (1)$$
we put $(f(x))^n = y$ and solve $a_0 y^2 + a_1 y + a_2 = 0$ to obtain its roots y_1 and y_2.
Finally, to obtain solutions of (1) we solve,
$$(f(x))^n = y_1$$
and $$(f(x))^n = y_2$$

3. An equation of the form
$$(ax^2 + bx + c_1)(ax^2 + bx + c_2) \cdots (ax^2 + bx + c_n) = A$$
where $c_1, c_2, \cdots, c_n, A \in \mathbf{R}$, can be solved by putting $ax^2 + bx = y$.

4. An equation of the form
$$(x-a)(x-b)(x-c)(x-d) = Ax^2$$
where $ab = cd$, can be reduced to a product of two quadratic polynomials by putting $y = x + \frac{ab}{x}$.

5. An equation of the form
$$(x-a)(x-b)(x-c)(x-d) = A$$
where $a < b < c < d$, $b - a = d - c$
can be solved by change of variable
$$y = \frac{(x-a)+(x-b)+(x-c)+(x-d)}{4}$$
$$= x - \frac{1}{4}(a+b+c+d)$$

6. A polynomial $f(x, y)$ is said to be symmetric if $f(x, y) = f(y, x) \vee x, y$.

All symmetric polynomials can be represented as a function of $x + y$ and xy.

2.17 USE OF CONTINUITY AND DIFFERENTIABILITY TO FIND ROOTS OF A POLYNOMIAL EQUATION

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$, then f is continuous on $\mathbf{R}$.

Since f is continuous on $\mathbf{R}$, we may use the intermediate value theorem to find whether or not f has a real root.
If there exists a and b such that $a < b$ and $f(a) f(b) < 0$, then there exists at least one $c \in (a, b)$ such that $f(c) = 0$.
Also, if $f(-\infty)$ and $f(a)$ are of opposite signs, then at least one root of $f(x) = 0$ lies in $(-\infty, a)$. Also, if $f(a)$ and $f(\infty)$ are of opposite signs, then at least one root of $f(x) = 0$ lies in (a, ∞).

Result 1 If $f(x) = 0$ has a root α of multiplicity r (where $r > 1$), then we can write
$$f(x) = (x-\alpha)^r g(x)$$
where $g(\alpha) \neq 0$.
Also, $f'(x) = 0$ has α as a root with multiplicity $r - 1$.

Result 2 If $f(x) = 0$ has n real roots, then $f'(x) = 0$ has $(n-1)$ real roots.
It follows immediately using Result 1 and Rolle's Theorem.

Result 3 If $f(x) = 0$ has n distinct real roots, we can write
$$f(x) = a_0 (x-\alpha_1)(x-\alpha_2) \cdots (x-\alpha_n)$$
where $\alpha_1, \alpha_2, \cdots \alpha_n$ are n distinct roots of $f(x) = 0$.
We can also write
$$\frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{1}{x-\alpha_k}$$

SOLVED EXAMPLES

SECTION I

SINGLE CORRECT CHOICE TYPE

Example 1 The number of real roots of
$$(x+3)^4 + (x+5)^4 = 16 \qquad (1)$$
is
(a) 0 (b) 2
(c) 4 (d) none of these

Ans. (b)

Solution Put $y = x + \frac{3+5}{2} = x + 4$ [See 2.16 in Theory]
The equation (1) becomes
$$(y-1)^4 + (y+1)^4 = 16$$
$\Rightarrow$ $2\{y^4 + 6y^2 + 1\} = 16 \Rightarrow y^4 + 6y^2 - 7 = 0$
$\Rightarrow$ $(y^2 + 7)(y^2 - 1) = 0$
$\Rightarrow$ $y^2 = -7$ or $y^2 = 1$
$\Rightarrow$ $y = \pm\sqrt{7}i$ or $y = \pm 1$
$\Rightarrow$ $x = -4 \pm \sqrt{7}i$ or $x = -4 \pm 1 = -5$ or -3.
Thus, the given equation has two real roots.

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Example 2 The product of real roots of the equation

$$|x|^{6/5} - 26|x|^{3/5} - 27 = 0 \qquad (1)$$

is

(a) -3^{10} (b) -3^{12} (c) $-3^{12/5}$ (d) $-3^{21/5}$

Ans. (a)

Solution Put $|x|^{3/5} = y$ [See 2.16 in Theory], so that (1) becomes

$$y^2 - 26y - 27 = 0$$

$$\Rightarrow \quad (y + 1)(y - 27) = 0 \Rightarrow y = -1 \text{ or } 27$$

As $y = |x|^{3/5} \geq 0 \ \forall x$, we get $y = 27$

$$\Rightarrow \quad |x|^{3/5} = 3^3 \Rightarrow |x| = 3^5 \Rightarrow x = \pm 3^5$$

$\therefore$ product of roots of (1) is $(3^5)(-3^5) = -3^{10}$.

Example 3 Sum of the non-real roots of

$$(x^2 + x - 2)(x^2 + x - 3) = 12 \qquad (1)$$

is

(a) 1 (b) -1 (c) -6 (d) 6

Ans. (b)

Solution Put $x^2 + x = y$ [See 2.16 in Theory], so that the equation (1) becomes

$$(y - 2)(y - 3) = 12$$

$$\Rightarrow \quad y^2 - 5y - 6 = 0$$

$$\Rightarrow \quad (y - 6)(y + 1) = 0 \Rightarrow y = 6, -1.$$

When $y = 6$, we get $x^2 + x - 6 = 0$

$$\Rightarrow \quad (x + 3)(x - 2) = 0 \text{ or } x = -3, 2$$

When $y = -1$, we get $x^2 + x + 1 = 0$

$\Rightarrow$ $x = w, w^2$ and their sum is -1.

Example 4 If $\tan A$ and $\tan B$ are the roots of the quadratic equation $x^2 - px + q = 0$, then value of $\sin^2(A + B)$ is

(a) $\dfrac{p^2}{p^2 + q^2}$ (b) $\dfrac{p^2}{(q + p)^2}$

(c) $1 - \dfrac{p}{(1 - q)^2}$ (d) $\dfrac{p^2}{(1 - q)^2 + p^2}$

Ans. (d)

Solution We have $\tan A + \tan B = p$, and $\tan A \tan B = q$.

$$\therefore \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{p}{1 - q}$$

Now, $\sin^2(A + B) = \dfrac{\tan^2(A + B)}{\sec^2(A + B)}$

$$= \frac{p^2/(1 - q)^2}{1 + p^2/(1 - q)^2} = \frac{p^2}{(1 - q)^2 + p^2}$$

Example 5 If $x \in \mathbf{R}$, the number of solutions of $\sqrt{2x+1} - \sqrt{2x-1} = 1$ is

(a) 1 (b) 2 (c) 4 (d) infinite

Ans. (a)

Solution For the equation to be defined, we have $2x + 1 \geq 0$ and $2x - 1 \geq 0 \Rightarrow 2x \geq -1$ and $2x \geq 1 \Rightarrow x \geq 1/2$.

We rewrite the equation as

$$\sqrt{2x+1} = 1 + \sqrt{2x-1}$$

Squaring both the sides, we get

$$2x + 1 = 1 + 2x - 1 + 2\sqrt{2x-1}$$

$$\Rightarrow \quad 1 = 2\sqrt{2x-1} \Rightarrow 2x - 1 = 1/4 \Rightarrow x = 5/8$$

Since $x = 5/8$ satisfies the given equation, the given equation has just one solution.

Example 6 If l, m, n are real, $l + m \neq 0$, then the roots of the equation

$$(l + m)x^2 - 3(l - m)x - 2(l + m) = 0 \text{ are}$$

(a) real and equal (b) complex
(c) real and unequal (d) none of these

Ans. (c)

Solution Discriminant of the given equation is

$$D = 9(l - m)^2 + 8(l + m)^2$$

As $l + m \neq 0$, $(l + m)^2 > 0$. Also, $(l - m)^2 \geq 0$.

Thus, $D > 0$.

Hence, roots of the given equation are real and unequal.

Example 7 The real value of a for which the sum of the squares of the roots of the equation

$x^2 - (a - 2)x - a - 1 = 0$ assume the least value is

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (b)

Solution Discriminant of the equation is

$$(a - 2)^2 + 4(a + 1)$$
$$= a^2 - 4a + 4 + 4a + 4$$
$$= a^2 + 8 > 0 \text{ as } a \in \mathbf{R}.$$

$\therefore$ Roots of the given equation are real. Let these roots be α and β.

Then $\alpha + \beta = a - 2$, $\alpha\beta = -(a + 1)$.

We have

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (a - 2)^2 + 2(a + 1)$$
$$= a^2 - 4a + 4 + 2a + 2 = a^2 - 2a + 6$$
$$= (a - 1)^2 + 5$$

Thus, $\alpha^2 + \beta^2$ is least when $a = 1$.

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Example 8. If p and q are distinct primes and $x^2 - px + q = 0$ has distinct positive integral roots, then $p + q$ equals

(a) 5 (b) 7 (c) 19 (d) 40

Ans. (a)

Solution. Let α and β be two roots of $x^2 - px + q = 0$ and assume $\alpha < \beta$. We have $\alpha + \beta = p$, $\alpha\beta = q$

As q is prime, we must have $\alpha = 1$, $\beta = q$.

Thus, $p = 1 + q \Rightarrow p - q = 1$.

As p and q are prime, this is possible if $p = 3$, $q = 2$.

Example 9 The real values of a for which the quadratic equation $3x^2 + 2(a^2 + 1)x + (a^2 - 3a + 2) = 0$ possesses roots of opposite signs lie in

(a) $(-\infty, 1)$ (b) $(-\infty, 0)$

(c) $(1, 2)$ (d) $\left(\frac{3}{2}, 2\right)$

Ans. (c)

Solution The quadratic equation $3x^2 + 2(a^2 + 1)\, x + a^2 - 3a + 2 = 0$ will have two roots of opposite signs if it has real roots and the product of the roots is negative, that is, if

$$4(a^2 + 1)^2 - 12\,(a^2 - 3a + 2) \geq 0$$

and $$\frac{a^2 - 3a + 2}{3} < 0.$$

Both of these conditions are met if

$a^2 - 3a + 2 < 0$ i.e. if $(a - 1)(a - 2) < 0$ or $1 < a < 2$.

Example 10 If $a, b, c \in \mathbf{R}$ and $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has

(a) at least one root in [0, 1]
(b) at least one root in [1, 2]
(c) at least one root in [3/2, 2]
(d) none of these.

Ans. (a)

Solution Let $f(x) = ax^3 + bx^2 + cx$. Note that f is continuous and derivable on $\mathbf{R}$. Also $f(0) = 0$ and $f(1) = a + b + c = 0$. By the Rolle's theorem, there exists at least one $\alpha \in (0, 1)$ such that

$$f'(\alpha) = 0 \Rightarrow 3a\alpha^2 + 2b\alpha + c = 0$$

Thus, $3ax^2 + 2bx + c = 0$ has at least one root in [0, 1].

Example 11 Let $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbf{R}$ and $a \neq 0$. Suppose $f(x) > 0$ for all $x \in \mathbf{R}$.

Let $g(x) = f(x) + f'(x) + f''(x)$. Then

(a) $g(x) > 0\ \forall\ x \in \mathbf{R}$
(b) $g(x) < 0\ \forall\ x \in \mathbf{R}$
(c) $g(x) = 0\ \forall\ x \in \mathbf{R}$
(d) $g(x) = 0$ has real roots.

Ans. (a)

Solution Since $f(x) > 0\ \forall\ x \in \mathbf{R}$, $a > 0$ and $b^2 - 4ac < 0$.

We have $f'(x) = 2ax + b$ and $f''(x) = a$. Thus

$$g(x) = ax^2 + bx + c + 2ax + b + 2a$$
$$= ax^2 + (2a + b)x + (2a + b + c)$$

We have $a > 0$ and $D = (2a + b)^2 - 4a(2a + b + c)$

$$= 4a^2 + 4ab + b^2 - (8a^2 + 4ab + 4ac)$$
$$= b^2 - 4ac - 4a^2 < 0,$$

since $b^2 - 4ac < 0$.

Thus $g(x) > 0\ \forall\ x \in \mathbf{R}$.

Example 12 If $b < 0$, then the roots x_1 and x_2 of the equation $2x^2 + 6x + b = 0$, satisfy the condition $(x_1/x_2) + (x_2/x_1) < k$ where k is equal to.

(a) -3 (b) -5 (c) -6 (d) -2

Ans. (d)

Solution The discriminant of the quadratic equation $2x^2 + 6x + b = 0$ is given by $D = 36 - 8b > 0$. Therefore, the given equation has real roots.

We have $$\frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{x_1^2 + x_2^2}{x_1 x_2} = \frac{(x_1 + x_2)^2 - 2x_1 x_2}{x_1 x_2}$$

$$= \frac{(-3)^2 - 2(b/2)}{b/2} = \frac{18}{b} - 2 < -2$$

$[\because\ b < 0]$

Example 13 If $ax^2 + bx + c$, $a, b, c \in \mathbf{R}$ has no real zeros, and if $c < 0$, then,

(a) $a < 0$ (b) $a + b + c > 0$
(c) $a > 0$ (d) none of these

Ans. (a)

Solution Let $f(x) = ax^2 + bx + c$. Since $f(x)$ has no real zeros, either $f(x) > 0$ or $f(x) < 0$ for all $x \in \mathbf{R}$. Since $f(0) = c < 0$, we get $f(x) < 0\ \forall\ x \in \mathbf{R}$. Therefore, $a < 0$ as the parabola $y = f(x)$ must open downwards.

Example 14 If x is real, then the maximum value of

$$y = 2(a - x)\left(x + \sqrt{x^2 + b^2}\right) \text{ is}$$

(a) $a^2 + b^2$ (b) a^2

(c) $\sqrt{a^2 + b^2}$ (d) $a\sqrt{a^2 + b^2}$

Ans. (a)

Solution Let $t = x + \sqrt{x^2 + b^2}$

$$\Rightarrow \frac{1}{t} = \frac{1}{x + \sqrt{x^2 + b^2}} = \frac{\sqrt{x^2 + b^2} - x}{b^2}$$

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$$\therefore \qquad t - \frac{b^2}{t} = 2x \quad \text{and} \quad t + \frac{b^2}{t} = 2\sqrt{x^2 + b^2}$$

Thus $2(a - x)\left(x + \sqrt{x^2 + b^2}\right) = \left(2a - t + \frac{b^2}{t}\right) \cdot (t)$

$$= 2at - t^2 + b^2 \quad = a^2 + b^2 - (a^2 - 2at + t^2)$$
$$= a^2 + b^2 - (a - t)^2$$

Therefore, $y = 2(a - x)\left(x + \sqrt{x^2 + b^2}\right) \le a^2 + b^2$. Hence, the maximum value of y is $a^2 + b^2$. This value is attained at $t = a$ or $x = (a^2 - b^2)/2a$.

Example 15 If both the roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ exceed 3, then

(a) $a < 1$ (b) $a > 11/9$
(c) $a > 3/2$ (d) $a < 5/2$

Ans. (b)

Solution We can write the given equation

$$(x - 3a)^2 = 2a - 2$$

Note that $a \ge 1$ and

$$x = 3a \pm \sqrt{2a - 2}$$

Both the roots will exceed 3 if smaller of the two roots exceed 3, that is, if

$$3a - \sqrt{2a - 2} > 3$$

$$\Rightarrow \qquad 3(a - 1) > \sqrt{2}\sqrt{a - 1}$$

$$\therefore \qquad a > 1 \text{ and } \sqrt{a - 1} > \sqrt{2}/3$$

$$\Rightarrow \qquad a > 1 + \frac{2}{9} = \frac{11}{9} \qquad (4)$$

Example 16 If the equation $\sqrt{x+1} - \sqrt{x} = a$ has a solution, then

(a) $0 < a < 1$ (b) $a > 1$
(c) $0 < a \le 1$ (d) $a \le 1$

Ans. (c)

Solution. $a = \dfrac{1}{\sqrt{x+1} + \sqrt{x}}$

Thus, $0 < a \le 1$.

Example 17 Let α, β be the roots of the equation $x^2 - ax + b = 0$ and $A_n = \alpha^n + \beta^n$. Then

$A_{n+1} - aA_n + bA_{n-1}$ is equal to.

(a) $-a$ (b) b (c) 0 (d) $a - b$

Ans. (c)

Solution We have $\alpha + \beta = a$ and $\alpha\beta = b$. Now,

$$aA_n - bA_{n-1} = (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$$
$$= \alpha^{n+1} + \alpha^n\beta + \alpha\beta^n + \beta^{n+1} - \alpha^n\beta - \alpha\beta^n$$
$$= \alpha^{n+1} + \beta^{n+1} = A_{n+1}$$

Example 18 Let $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbf{R}$. If $f(x)$ takes real values for real values of x and non-real values for non-real values of x, then.

(a) $a = 0$ (b) $b = 0$ (c) $c = 0$
(d) nothing can be said about a, b, c.

Ans. (a)

Solution Suppose $a \ne 0$. We rewrite $f(x)$ as follows:

$$f(x) = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$f\left(-\frac{b}{2a} + i\right) = a\left\{\left(-\frac{b}{2a} + i + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$= a\left\{-1 + \frac{4ac - b^2}{4a^2}\right\}$$

which is a real number

This is against the hypothesis. Therefore, $a = 0$.

Example 19 If α, β, γ are such that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 6$, $\alpha^3 + \beta^3 + \gamma^3 = 8$, then $\alpha^4 + \beta^4 + \gamma^4$ is

(a) 5 (b) 18 (c) 12 (d) 36

Ans. (b)

Solution We have

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\beta\gamma + \gamma\alpha + \alpha\beta)$$
$$\Rightarrow \qquad 4 = 6 + 2(\beta\gamma + \gamma\alpha + \alpha\beta)$$
$$\Rightarrow \qquad \beta\gamma + \gamma\alpha + \alpha\beta = -1.$$

Also, $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$

$$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta)$$

$$\Rightarrow \quad 8 - 3\alpha\beta\gamma = 2(6 + 1)$$
$$\Rightarrow \qquad 3\alpha\beta\gamma = 8 - 14 = -6 \quad \text{or} \quad \alpha\beta\gamma = -2.$$

Now, $(\alpha^2 + \beta^2 + \gamma^2)^2 = \sum \alpha^4 + 2\sum \beta^2\gamma^2$

$$= \sum \alpha^4 + 2\left[\left(\sum \beta\gamma\right)^2 - 2\alpha\beta\gamma\left(\sum \alpha\right)\right]$$

$$\Rightarrow \qquad \sum \alpha^4 = 36 - 2[(-1)^2 - 2(-2)(2)] = 18$$

Example 20 The condition that the equation

$$\frac{1}{x} + \frac{1}{x + b} = \frac{1}{m} + \frac{1}{m + b}$$

has real roots that are equal in magnitude but opposite in sign is

(a) $b^2 = m^2$ (b) $b^2 = 2m^2$
(c) $2b^2 = m^2$ (d) none of these

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Ans. (b)

Solution Clearly $x = m$ is a root of the equation. Therefore, the other root must be $-m$. That is,

$$\frac{1}{-m}+\frac{1}{-m+b}=\frac{1}{m}+\frac{1}{m+b}$$

$$\Rightarrow \quad \frac{1}{b-m}-\frac{1}{b+m}=\frac{2}{m}$$

$$\Rightarrow \quad \frac{b+m-b+m}{b^2-m^2}=\frac{2}{m}$$

$$\Rightarrow \quad 2m^2 = 2b^2 - 2m^2 \quad \text{or} \quad 2m^2 = b^2.$$

Example 21 If x is real, and

$$k=\frac{x^2-x+1}{x^2+x+1} \text{ then}$$

(a) $1/3 \le k \le 3$ (b) $k \ge 5$

(c) $k \le 0$ (d) $2/3 \le k \le 1$

Ans. (a)

Solution

We have $k=\dfrac{x^2-x+1}{x^2+x+1} \Rightarrow kx^2+kx+k=x^2-x+1$

$$\Rightarrow \quad (k-1)x^2+(k+1)x+k-1=0$$

As x is real, the discriminant

$$D = (k+1)^2 - 4(k-1)^2 \ge 0$$

$$\Rightarrow \quad (3k-1)(-k+3) \ge 0$$

$$\Rightarrow \quad (k-1/3)(k-3) \le 0 \quad \Rightarrow \quad 1/3 \le k \le 3$$

These extreme values are attained at $x = 1$ and $x = -1$.

Example 22 Let $a > 0$, $b > 0$ and $c > 0$. Then both the roots of the equation $ax^2 + bx + c = 0$

(a) are real and negative

(b) have negative real parts

(c) are rational numbers

(d) none of these.

Ans. (b)

Solution We have $D = b^2 - 4ac$. If $D \ge 0$, then the roots of the equation are given by

$$x=\frac{-b\pm\sqrt{D}}{2a}$$

As $D = b^2 - 4ac < b^2$ ($\because a > 0, c > 0$), it follows that the roots of the quadratic equation are negative. In case $D < 0$, then the roots of the equation are given by

$$x=\frac{-b\pm i\sqrt{-D}}{2a}$$

which have negative real parts.

Example 23 Let a, b, c be non-zero real numbers such that

$$\int_0^1 (e^{-x}+e^{x})\ (ax^2+bx+c)dx$$

$$=\int_0^2 (e^{-x}+e^{x})\ (ax^2+bx+c)\,dx$$

Then the quadratic equation $ax^2 + bx + c = 0$ has

(a) no root in (0, 1)

(b) at least one root in (1, 2)

(c) a double root in (0, 1)

(d) none of these.

Ans. (b)

Solution Let $f(x)= (e^{-x} + e^{x})(ax^2 + bx + c)$

We have $\displaystyle\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$

$$\Rightarrow \quad \int_1^2 f(x)dx = 0$$

If $f(x) > 0\ (<0)\ \forall\, x \in [1, 2]$, then $\displaystyle\int_1^2 f(x)dx > 0\ (<0)$. Thus, $f(x) = (e^{-x} + e^{x})(ax^2 + bx + c)$ must be positive for some value of x in $[1, 2]$ and must be negative for some value of x in $[1, 2]$. As $e^{-x} + e^{x} \ge 2$, it follows that if $g(x) = ax^2 + bx + c$, then there exists some $\alpha, \beta \in [1, 2]$ such that $g(\alpha) > 0$ and $g(\beta) < 0$. Since g is continuous on **R**, there exists some γ between α and β such that $g(\gamma) = 0$. Thus, $ax^2 + bx + c = 0$ has at least one root in $(1, 2)$.

Example 24 The equation

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1$$

has

(a) no solution

(b) only one solution

(c) only two solution

(d) more than two solutions.

Ans. (d)

Solution Put $\sqrt{x-1} = t$ or $x = t^2 + 1$ so that the given equation becomes

$$\sqrt{t^2+4-4t}+\sqrt{t^2+9-6t}=1 \quad \text{or}$$

$$\sqrt{(t-2)^2}+\sqrt{(t-3)^2}=1$$

or $$|t-2|+|t-3|=1$$

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This equation is satisfied for all values of t lying between 2 and 3, i.e., $2 \le t \le 3$.
Thus, the given equation is satisfied for all values of x lying between 5 and 10.

Example 25 The equation $3^{x-1} + 5^{x-1} = 34$ has

(a) no solution (b) one solution
(c) two solutions (d) more than two solutions

Ans. (b)

Solution It is quite clear that $x = 3$ satisfies the given equation we now show that there is no other solution. Note that $y = 3^{x-1}$ and $y = 5^{x-1}$ are both increasing functions of x (exponential functions with base greater than 1). Therefore their sum $y = 3^{x-1} + 5^{x-1}$ is also an increasing

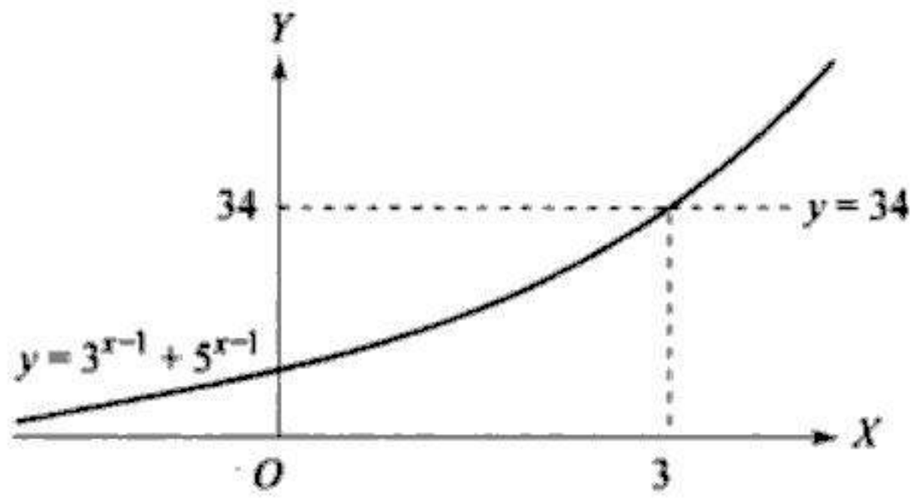

Fig. 2.16

function of x. This means for $x < 3$, $3^{x-1} + 5^{x-1} < 34$ and for $x > 3$, $3^{x-1} + 5^{x-1} > 34$. Thus, the equation has no other solution. (Fig. 2.16)

Example 26 If the harmonic mean between roots of

$$(5 + \sqrt{2})x^2 - bx + 8 + 2\sqrt{5} = 0 \qquad (1)$$

is 4 then b equals

(a) 2 (b) $4 - \sqrt{5}$
(c) 3 (d) $4 + \sqrt{5}$

Ans. (d)

Solution Let α, β be the roots of (1), then

$$4 = \frac{2\alpha\beta}{\alpha+\beta} \Rightarrow \frac{b}{5+\sqrt{2}} = \frac{8+2\sqrt{5}}{2(5+\sqrt{2})}$$

$$\Rightarrow \quad b = 4 + \sqrt{5}.$$

Example 27 In a triangle PQR, $\angle R = \pi/4$. If $\tan(P/3)$ and $\tan(Q/3)$ are the roots of the equation $ax^2 + bx + c = 0$, then

(a) $a + b = c$ (b) $b + c = 0$
(c) $a + c = b$ (d) $b = c$

Ans. (a)

Solution

$$\tan\left(\frac{P}{3}\right) + \tan\left(\frac{Q}{3}\right) = -\frac{b}{a}$$

$$\tan\left(\frac{P}{3}\right)\tan\left(\frac{Q}{3}\right) = \frac{c}{a}.$$

Now, $\angle R = \pi/4 \Rightarrow P + Q = 3\pi/4$

$$\therefore \qquad 1 = \tan\left(\frac{\pi}{4}\right) = \frac{\tan(P/3) + \tan(Q/3)}{1 - \tan(P/3)\tan(Q/3)}$$

$$= \frac{-b/a}{1 - c/a} = \frac{-b}{a-c}$$

$$\Rightarrow \qquad a + b = c.$$

Example 28 If $a \le 0$, then number of real roots of

$$x^2 + 2a|x - a| + 3a^2 = 0$$

is

(a) 0 (b) 1 (c) 2 (d) infinite

Ans. (b)

Solution Put $x - a = t$ and rewrite (1) as

$$(a + t)^2 + 2a|t| + 3a^2 = 0$$

$$\Rightarrow \qquad t^2 + 2a(t + |t|) + 4a^2 = 0 \qquad (2)$$

If $t < 0$, we get $t^2 + 4a^2 = 0$.
Not possible. Therefore $t \ge 0$.
Thus, (2) gives $t^2 + 4at + 4a^2 = 0$

$$\Rightarrow \quad t = -2a.$$

$$\Rightarrow \quad x = -a.$$

Example 29 Let $(a_1, a_2, a_3, a_4, a_5)$ denote a rearrangement of $(3, -5, 7, 4, -9)$, then the equation

$$a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 = 0$$

has

(a) at least two real roots
(b) all four real roots
(c) only imaginary roots
(d) none of these

Ans. (a)

Solution $x = 1$ is always a root of the equation.

Example 30 If three distinct real number a, b and c satisfy

$$a^2(a + p) = b^2(b + p) = c^2(c + p)$$

where $p \in \mathbf{R}$, then value of $bc + ca + ab$ is

(a) $-p$ (b) p (c) 0 (d) $p^2/2$

Ans. (c)

Solution If value of each relation is k, then a, b, c are roots of $x^3 + px^2 - k = 0$.

Example 31 The number of integral roots of the equation

$$x^4 + \sqrt{x^4 + 20} = 22$$

is

(a) 0 (b) 2 (c) 4 (d) 8

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Ans. (b)

Solution Put $x^4 = y$ and write

$$y + 20 = (22 - y)^2$$

$$\Rightarrow \qquad y = 16,\ 29.$$

But $y = 29$ is not possible and $x^4 = y = 16$

$$\Rightarrow \qquad x = \pm 2 \text{ as } x \text{ is an integer.}$$

Example 32 The product of roots of

$$\sqrt[3]{8+x} + \sqrt[3]{8-x} = 1 \qquad (1)$$

is

(a) -21 (b) -189 (c) -9 (d) -5

Ans. (b)

Solution Cubing (1) we obtain

$$8 + x + 8 - x + 3(64 - x^2)^{1/3} = 1$$

$$\Rightarrow \quad (64 - x^2)^{1/3} = -5 \quad \Rightarrow 64 - x^2 = -125$$

$$\Rightarrow \quad x^2 = 189 \quad \Rightarrow \quad x = \pm 3\sqrt{21}$$

Example 33 Let S denote the set of all values of the parameter a for which

$$x + \sqrt{x^2 + a} = a \qquad (1)$$

has no solution, then S equals

(a) $(-1, 1)$ (b) $(-\infty, -1)$
(c) $[-1, \infty)$ (d) $(0, \infty)$

Ans. (b)

Solution If $a = 0$, (1) holds for all $x \le 0$. Write (1) as

$$x^2 + a = a^2 - 2ax + x^2$$

$$\Rightarrow \qquad 2ax = a(a - 1) \Rightarrow x = (a - 1)/2$$

Substituting this (1) we get $a - 1 + |a + 1| = 2a$

This holds if $a \ge -1$.

$\therefore$ The equation has no solution if $a < -1$.

Example 34 The number of roots of the equation

$$\frac{1}{x} + \frac{1}{\sqrt{1-x^2}} = \frac{35}{12} \qquad (1)$$

is

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (d)

Solution Put $x = 1/u$, $\sqrt{1-x^2} = 1/v$ (2)

(1) and (2) give

$$u + v = 35/12,\ u^2 + v^2 = u^2v^2$$

$$\Rightarrow \qquad uv = 25/12,\ -49/12$$

When $uv = 25/12$, we get $x = 3/5,\ 4/5$

When $uv = -49/12$, we get $x = -(5 + \sqrt{73})/14$.

Example 35 If all the roots of $x^3 + px + q = 0$ $p, q \in \mathbf{R}$, $q \ne 0$ are real, then

(a) $p < 0$ (b) $p = 0$ (c) $p > 0$ (d) $p > q$

Ans. (a)

Solution If α, β, γ are the roots of

$$x^3 + px + q = 0, \text{ then}$$

$$\alpha + \beta + \gamma = 0,\ \beta\gamma + \gamma\alpha + \alpha\beta = p,\ \alpha\beta\gamma = -q$$

As $q \ne 0$, none of α, β, γ is zero. Now, use

$$2p = (\alpha + \beta + \gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2) < 0$$

Example 36 Let S denote the set of all real values of a for which the roots of the equation

$$x^2 - 2ax + a^2 - 1 = 0 \qquad (1)$$

lie between 5 and 10, then S equals

(a) $(-1, 2)$ (b) $(2, 9)$
(c) $(4, 9)$ (d) $(6, 9)$

Ans. (d)

Solution We can write (1) as

$$(x - a)^2 = 1 \Rightarrow x = a \pm 1$$

Now, $5 < a \pm 1 < 10$

$\Rightarrow 4 < a < 9,\ 6 < a < 11$

$\therefore\ a \in (6, 9)$.

Example 37 Let S denote the set of all values of a for which the roots of the equation

$$(1 + a)x^2 - 3ax + 4a = 0 \qquad (1)$$

exceed 1, then S equals

(a) $[-16/7, 0)$ (b) $(-16/7, -1)$
(c) $(-1, 2)$ (d) $(-1, 3)$

Ans. (b)

Solution Clearly $a \ne 0, -1$. Both the roots of (1) will exceed 1 if and only if

$$9a^2 - 4(4a)(1 + a) \ge 0,$$

$$\frac{3a}{1+a} > 1,$$

and $(1 + a)[(1 + a) - 3a + 4a] > 0$

$$\Leftrightarrow \qquad a(7a + 16) \le 0,$$

$$(2a - 1)(1 + a) > 0$$

and $(2a + 1)(1 + a) > 0$

$$\Leftrightarrow \quad -\frac{16}{7} \le a \le 0;\ a < -1 \text{ or } a > 1/2$$

$$\text{and } a < -1 \text{ or } a > -1/2$$

$$\Leftrightarrow \qquad -\frac{16}{7} \le a < -1$$

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Example 38 Let S denote the set of all values of S for which the equations

$$2x^2 - 2(2a + 1)x + a(a + 1) = 0$$

has one root less than a and other root greater than a, then S equals

(a) $(0, 1)$ (b) $(-1, 0)$
(c) $(0, 1/2)$ (d) none of these

Ans. (d)

Solution The required a satisfies the inequality

$$2a^2 - 2(2a + 1)a + a(a + 1) < 0$$
$$\Leftrightarrow \quad a(a + 1) > 0 \Leftrightarrow a \in (-\infty, -1) \cup (0, \infty)$$

Example 39 Let $a, b, p, q \in \mathbf{Q}$ and suppose that

$f(x) = x^2 + ax + b = 0$ and $g(x) = x^3 + px + q = 0$

have a common irrational root, then

(a) $f(x)$ divides $g(x)$
(b) $g(x) \equiv xf(x)$
(c) $g(x) \equiv (x - b - q) f(x)$
(d) none of these.

Ans. (a)

Solution Let $\alpha \in \mathbf{R} - \mathbf{Q}$ be a common root of $f(x) = 0$ and $g(x) = 0$. Then $\alpha^2 = -a\alpha - b$. Substituting this in $\alpha^3 + p\alpha + q = 0$, we get

$$(a^2 - b + p)\alpha + ab + q = 0$$

As α is irrational and $a, b, p, q \in \mathbf{Q}$, $p = b - a^2$, $q = -ab$. This gives, $g(x) = (x - a) f(x)$.

Example 40 The number of irrational solutions of the equation

$$\sqrt{x^2 + \sqrt{x^2 + 11}} + \sqrt{x^2 - \sqrt{x^2 + 11}} = 4 \qquad (1)$$

is

(a) 0 (b) 2
(c) 4 (d) 11

Ans. (c)

Solution Put $\sqrt{x^2 + 11} = t$ so that (1) can be written as

$$\sqrt{t^2 + t - 11} + \sqrt{t^2 - t - 11} = 4 \qquad (2)$$

We also have the identity

$$(t^2 + t - 11) - (t^2 - t - 11) = 2t \qquad (3)$$

Dividing (2) by (3) we get

$$\sqrt{t^2 + t - 11} - \sqrt{t^2 - t - 11} = \frac{t}{2} \qquad (4)$$

Adding (2) and (4), we get

$$2\left(\sqrt{t^2 + t - 11}\right) = 4 + t/2$$

$$\Rightarrow \quad t^2 + t - 11 = 4 + t + t^2/16$$

$$\Rightarrow \quad t = 4 \Rightarrow x = \pm\sqrt{5}.$$

SECTION II

MULTIPLE CORRECT CHOICE TYPE

Example 41 Let a, b, c, p, q be five different non-zero real numbers and x, y, z be three numbers satisfying the system of equations

$$\frac{x}{a} + \frac{y}{a - p} + \frac{z}{a - q} = 1$$

$$\frac{x}{b} + \frac{y}{b - p} + \frac{z}{b - q} = 1$$

$$\frac{x}{c} + \frac{y}{c - p} + \frac{z}{c - q} = 1$$

then

(a) $x + y + z = a + b + c - p - q$

(b) $x = \dfrac{abc}{pq}$

(c) $y = \dfrac{(a - p)(b - p)(c - p)}{p(p - q)}$

(d) $z = \dfrac{(a - q)(b - q)(c - q)}{q(q - p)}$

Ans. (a), (b), (c), (d)

Solution Note that a, b, c are roots of

$$\frac{x}{t} + \frac{y}{t - p} + \frac{z}{t - q} = 1$$

$$\Leftrightarrow \quad x(t - p)(t - q) + yt(t - q) + zt(t - p) = t(t - p)(t - q)$$

$$\Leftrightarrow \quad t^3 - (x + y + z + p + q)t^2 + [(p + q)x + qy + pz]t - pqx = 0$$

From here the answers can be given easily.

Example 42 Let a, b, c, d be four integers such that ad is odd and bc is even, then $ax^3 + bx^2 + cx + d = 0$ (1) has

(a) at least one irrational root
(b) all three rational roots
(c) all three integral roots
(d) none of these.

Ans. (a)

Solution Putting $ax = y$ (1) can be written as

$$y^3 + by^2 + acy + a^2d = 0 \qquad (2)$$

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If (1) has all rational roots then (2) has all integral roots.
If α, β, γ are roots of (2) then α, β, γ are divisors of a^2d and as such must be odd integers. Now

$$-b = \alpha + \beta + \gamma \Leftrightarrow b \text{ is odd}$$

and $$ac = \beta\gamma + \gamma\alpha + \alpha\beta \Leftrightarrow c \text{ is odd}$$

$$\Rightarrow bc \text{ is odd. A contradiction.}$$

Example 43 Let α be a repeated root of

$$p(x) = x^3 + 3ax^2 + 3bx + c = 0, \text{ then}$$

(a) α is a root of $x^2 + 2ax + b = 0$

(b) $\alpha = \dfrac{c - ab}{2(a^2 - b)}$

(c) $\alpha = \dfrac{ab - c}{a^2 - b}$

(d) α is a root of $ax^2 + 2bx + c = 0$

Ans. (a), (b), (d)

Solution α is also a root of $p'(x) = 0$ or $x^2 + 2ax + b = 0$
Now

$$\alpha^3 + 3a\alpha^2 + 3b\alpha + c = 0$$

$$\alpha^2 + 2a\alpha + b = 0 \quad (1)$$

$$\Rightarrow \quad a\alpha^2 + 2b\alpha + c = 0 \quad (2)$$

Eliminating α^2 from (1) and (2), we get

$$2(a^2 - b)\alpha + ab - c = 0$$

$$\Rightarrow \alpha = \frac{c - ab}{2(a^2 - b)}$$

Example 44 Let α, β, γ be roots of

$$x^3 - px^2 + qx - r = 0, \text{ then}$$

(a) equation whose $\alpha^2 - \beta\gamma, \beta^2 - \gamma\alpha, \gamma^2 - \alpha\beta$ is $x^3 + (3q - p^2)x^2 + q(3q - p^2)x + q^3 - p^3r = 0$

(b) a permutation of α, β, γ is in G.P. if $q^3 = p^3 r$

(c) Square of one of the roots will be additive inverse of the product of the other two if $q^3 = p^3 r$

(d) none of these

Ans. (a), (b)

Solution $y = \alpha^2 - \beta\gamma = \alpha^2 - \dfrac{r}{\alpha} = \dfrac{\alpha^3 - r}{\alpha}$

$$= \frac{p\alpha^2 - q\alpha}{\alpha} = p\alpha - q$$

$$\Rightarrow \qquad \alpha = \frac{1}{p}(y + q)$$

As α is a root of the given equation

$$\frac{1}{p^3}(y + q)^3 - \frac{p}{p^2}(y + q)^2 + \frac{q}{p}(y + q) - r = 0$$

$$\Rightarrow \quad (y + q)^3 - p^2(y + q)^2 + p^2q(y + q) - p^3r = 0$$

$$\Rightarrow \quad y^3 + (3q - p^2)y^2 + q(3q - p^2)y + q^3 - p^3r = 0$$

If $q^3 - p^3 r = 0$, then $y = 0$ is a root of the above equation.

Example 45 Let α, β, γ be roots of $x^3 + px + q = 0$, then

(a) an equation whose roots are $\alpha^3, \beta^3, \gamma^3$ is $x^3 + 3qx^2 + (3q^2 + p^3)y + q^3 = 0$

(b) $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$

(c) an equation whose roots are $\beta^3 + \gamma^3, \gamma^3 + \alpha^3, \alpha^3 + \beta^3$ is $x^3 + 6qx^2 + (p^3 + 12q^2)x + 3p^3q + 8q^3 = 0$

(d) $\alpha^4 + \beta^4 + \gamma^4 = 2p^2$

Ans. (a), (b), (c), (d)

Solution $y = \alpha^3 \Rightarrow \alpha = y^{1/3}$

$$\therefore \qquad y + py^{1/3} + q = 0$$

$$\Rightarrow \qquad (y + q)^3 = -p^3y$$

$$\Rightarrow \quad y^3 + 3qy^2 + (3q^2 + p^3)y + q^3 = 0 \quad (1)$$

$$\alpha^3 + \beta^3 + \gamma^3 = 3q = 3\alpha\beta\gamma$$

Next $$y = \beta^3 + \gamma^3 = -3q - \alpha^3$$

$$\Rightarrow \qquad \alpha^3 = -3q - y$$

As α^3 is a root of (1)

$$(-3q - y)^3 + 3q(-3q - y)^2 + (3q^2 + p^3)(-3q - y) + q^3 = 0$$

$$\Rightarrow \quad y^3 + 6qy^2 + (p^3 + 12q^2)y + 3p^3q + 8q^3 = 0$$

Lastly, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta)$

$$= -2p$$

$$\alpha^4 + \beta^4 + \gamma^4 = (\alpha^2 + \beta^2 + \gamma^2)^2 - 2(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2)$$

But $\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$

$$= (\beta\gamma + \gamma\alpha + \alpha\beta)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$= p^2$$

Example 46 If $a \in \mathbf{I}$ and

$$(x - a)(x - 12) + 2 = (x + b)(x + c) \quad (1)$$

for $b, c \in \mathbf{I}$, then

(a) $a = 9, b, c = -10, -11$

(b) $a = 15, b, c = -13, -14$

(c) $a = 15, b, c = -3, -4$

(d) $a = 9, b, c = -13, -14$

Ans. (a), (b)

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Solution For (1) to hold

$$(x - a)(x - 12) + 2 = 0$$

must have integral zeros.

If $x \in \mathbf{I}$, then

$$(x - a)(x - 12) = -2$$

$\Rightarrow$ $x - a = 1, x - 12 = -2,$

or $x - a = -1, x - 12 = 2,$

or $x - a = 2, x - 12 = -1,$

or $x - a = -2, x - 12 = 1$

$\Rightarrow$ $a = 9, 15$

When $a = 9, b, c = -10, -11$

When $a = 15, b, c = -13, -14$

Example 47 If the equations $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then

(a) $b + c + 1 = 0$
(b) $b^2 + c^2 + 1 = bc$
(c) $(b - c)^2 + (b - 1)^2 + (c - 1)^2 = 0$
(d) $b + c + 1 = bc$

Ans. (a), (c)

Solution If α is a common root of the two equations,

$$\alpha^2 + b\alpha + c = 0,\ b\alpha^2 + c\alpha + 1 = 0$$

Multiply the first by b and subtracting we get

$$(b^2 - c)\alpha + bc - 1 = 0 \Rightarrow \alpha = \frac{1 - bc}{b^2 - c}$$

Thus,

$$(1 - bc)^2 + b(1 - bc)(b^2 - c) + c(b^2 - c^2) = 0$$

$\Rightarrow$ $b^3 + c^3 + 1 - 3bc = 0$

$\Rightarrow$ $(b + c + 1)(b^2 + c^2 + 1 - bc - b - c) = 0$

$\Rightarrow \frac{1}{2}(b + c + 1)[(b - c)^2 + (b - 1)^2 + (c - 1)^2] = 0.$

Example 48 Let α and β be two distinct real numbers and $p(x)$ be a quadratic polynomial such that

$$p(\alpha) = \alpha \text{ and } p(\beta) = \beta,$$

then

(a) $p(p(x)) - x = 0$ has at least two real roots.
(b) α and β are roots of $p(p(x)) - x = 0$
(c) $p(p(x)) = x$ for each $x \in \mathbf{R}$
(d) none of these.

Ans. (a), (b)

Solution Use $p(p(\alpha)) = p(\alpha) = \alpha$

and $p(p(\beta)) = p(\beta) = \beta$

Example 49 The quadratic equation

$$x^2 - (m - 3)x + m = 0 \qquad (1)$$

has

(a) real distinct roots if and only if $m \in (-\infty, 1) \cup (9, \infty)$
(b) both positive roots if and only if $m \in [9, \infty)$
(c) both negative roots if and only if $m \in (0, 1]$
(d) at least one positive root if and only if $m \in (9, \infty)$

Ans. (a), (b), (c)

Solution (1) will have real and distinct roots

$\Leftrightarrow$ $D = (m - 3)^2 - 4m > 0$

$\Leftrightarrow$ $m^2 - 10m + 9 > 0$

$\Leftrightarrow$ $m \in (-\infty, 1) \cup (9, \infty)$

Roots of (1) will be both positive if

$$D \geq 0,\ m - 3 > 0,\ m > 0$$

$\Rightarrow$ $m \geq 9,\ m > 3.$

$\therefore$ $m \in [9, \infty)$

Roots of (1) will be negative if

$$D \geq 0,\ m - 3 < 0,\ m > 0 \Rightarrow m \leq 1,\ 0 < m < 3$$

Thus, $m \in (0, 1]$

For $m < 0$, equation has one positive and one negative root.

Example 50 Let α, β be the roots of $ax^2 + 2bx + c = 0$ and γ, δ be the roots of $px^2 + 2qx + r = 0$.

If $\alpha, \beta, \gamma, \delta$ are in

(a) A.P. then $\frac{b^2 - ac}{a^2} = \frac{q^2 - pr}{p^2}$

(b) G.P. then $\frac{b^2}{ac} = \frac{q^2}{pr}$

(c) H.P. then $\frac{b^2 - ac}{c^2} = \frac{q^2 - pr}{r^2}$

(d) none of these

Ans. (a), (b), (c)

Solution $\alpha, \beta, \gamma, \delta$ are in A.P.

$\Rightarrow$ $\beta - \alpha = \delta - \gamma$

$\Rightarrow$ $(\beta + \alpha)^2 - 4\beta\alpha = (\delta + \gamma)^2 - 4\delta\gamma$

$\Rightarrow$ $\frac{b^2 - ac}{a^2} = \frac{q^2 - pr}{p^2}$

Next, $\alpha, \beta, \gamma, \delta$ are in G.P.

$\Rightarrow$ $\frac{\alpha}{\beta} = \frac{\gamma}{\delta}$

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$$\Rightarrow \quad \frac{\alpha-\beta}{\alpha+\beta} = \frac{\gamma-\delta}{\gamma+\delta}$$

$$\Rightarrow \quad 1-\frac{4\alpha\beta}{(\alpha+\beta)^2} = 1-\frac{4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad \frac{b^2}{ac} = \frac{q^2}{pr}$$

Finally, $1/\alpha$, $1/\beta$ are roots of $cx^2 + 2bx + a = 0$ and $1/\gamma$, $1/\delta$ are roots of $rx^2 + 2qx + p = 0$.

Next, $1/\alpha$, $1/\beta$, $1/\gamma$, $1/\delta$ are in A.P.

$$\therefore \quad \frac{b^2-ac}{c^2} = \frac{q^2-pr}{r^2}.$$

Example 51 If $a(p+q)^2 + 2bpq + c = 0$ and $a(p+r)^2 + 2bpr + c = 0$, then

(a) $q + r = -\dfrac{2(a+b)\,p}{a}$

(b) $qr = p^2 + \dfrac{c}{a}$

(c) $|q - r| = \dfrac{2}{|a|}\sqrt{(2a+b)bp^2 - ac}$

(d) $|q - r| = \dfrac{2}{|a|}\sqrt{p^2 - 4ac}$

Ans. (a), (b), (c)

Solution q and r are roots of

$$a(p + x)^2 + 2bpx + c = 0$$

or $$ax^2 + 2(a + b)px + ap^2 + c = 0$$

$$\therefore\ q + r = -\frac{2(a+b)\,p}{a},\ qr = p^2 + \frac{c}{a}$$

Next

$$(q - r)^2 = (q + r)^2 - 4qr$$

$$= \frac{4}{a^2}\,[(a + b)^2p^2 - a^2p^2 - ac]$$

$$= \frac{4}{a^2}\,[(2ab + b^2)\,p^2 - ac]$$

Example 52 Let a, b, c be three distinct real numbers such that each of the expression $ax^2 + bx + c$, $bx^2 + cx + a$ and $cx^2 + ax + b$ is positive for each $x \in \mathbf{R}$ and

let $\alpha = \dfrac{bc+ca+ab}{a^2+b^2+c^2}$, then

(a) $\alpha < 4$ (b) $\alpha < 1$

(c) $\alpha > 1/4$ (d) $\alpha > 1$

Ans. (b), (c)

Solution According to the given conditions

$a > 0, b^2 < 4ac;\ b > 0, c^2 < 4ab;\ c > 0, a^2 < 4bc$

$$\therefore \quad a^2 + b^2 + c^2 < 4(bc + ca + ab)$$

$$\Rightarrow \quad \frac{1}{4} < \alpha.$$

Also $a^2 + b^2 + c^2 - (bc + ca + ab)$

$$= \frac{1}{2}\,[(b - c)^2 + (c - a)^2 + (a - b)^2] > 0$$

$$\Rightarrow \quad \alpha < 1.$$

Example 53 Suppose a, b, c are positive integers and $f(x) = ax^2 - bx + c = 0$ has two distinct roots in (0, 1), then

(a) $a \geq 5$ (b) $b \geq 5$

(c) $abc \geq 25$ (d) $abc \geq 250$

Ans. (a), (b), (c)

Solution $f(0)$ and $f(1)$ are of the same sign, thus,

$c\,(a - b + c) > 0 \Rightarrow c\,(a - b + c) \geq 1.$

Suppose $f(x) = a(x - \alpha)\,(x - \beta)$

$\therefore \quad a^2\,\alpha\beta(1 - \alpha)\,(1 - \beta) \geq 1$

But $\alpha(1 - \alpha) = 1/4 - (1/2 - \alpha)^2 \leq 1/4$

Thus, $a^2\alpha\,(1 - \alpha)\,\beta(1 - \beta) < a^2/16$ as α, β are distinct

As $a^2/16 > 1$, we get $a > 4$

As $a \in \mathbf{I}$, $a \geq 5$.

Also, $b^2 \geq 4ac \geq 20 \Rightarrow b \geq 5$.

Next as $a \geq 5$, $b \geq 5$, $c \geq 1$ we get $abc \geq 25$.

Example 54 Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{R}$.

Suppose $|f(x)| \leq 1\ \forall\ x \in [0, 1]$, then

(a) $|a| \leq 8$ (b) $|b| \leq 8$

(c) $|c| \leq 1$ (d) $|a| + |b| + |c| \leq 17$

Ans. (a), (b), (c), (d)

Solution Putting $x = 0, 1, 1/2$, we obtain

$$|c| \leq 1,\ |a + b + c| \leq 1,\ \left|\frac{1}{4}a + \frac{1}{2}b + c\right| \leq 1$$

$$\Rightarrow \quad -1 \leq c \leq 1,\ -1 \leq a + b + c \leq 1,$$

$$-4 \leq a + 2b + 4c \leq 4$$

$$\Rightarrow \quad -4 \leq 4a + 4b + 4c \leq 4$$

and $$-4 \leq -a - 2b - 4c \leq 4$$

Adding we get

$$-8 \leq 3a + 2b \leq 8$$

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Also $\quad -8 \le a + 2b \le 8$

$\therefore \quad -16 \le 2a \le 16 \Rightarrow |a| \le 8$

Since $\quad -1 \le -c \le 1, -8 \le -a \le 8,$

we get $\quad -16 \le 2b \le 16 \Rightarrow |b| \le 8$

Thus, $\quad |a| + |b| + |c| \le 17.$

Example 55 Let $a, b, c > 0$ and a, b, c be in G.P. and $\omega \neq 1$ be a cube root unity. The equation $ax^2 + bx + c = 0$ has

(a) real roots
(b) imaginary roots
(c) roots with negative real parts
(d) roots are $\alpha\omega^2$, $\alpha\omega^2$ where $\alpha = b/a$

Ans. (b), (c), (d)

Solution $b^2 = ac$ and

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\sqrt{ac} \pm i\sqrt{3ac}}{2a}$$

$$= \left(\frac{-1 \pm \sqrt{3}i}{2}\right)\sqrt{\frac{c}{a}} = \frac{b}{a}\omega, \frac{b}{a}\omega^2$$

Example 56 If the equation $ax^2 + bx + c = 0$ $(a > 0)$ has two roots α and β such that $\alpha < -2$ and $\beta > 2$, then

(a) $b^2 - 4ac > 0$ (b) $c < 0$
(c) $a + |b| + c < 0$ (d) $4a + 2|b| + c < 0.$

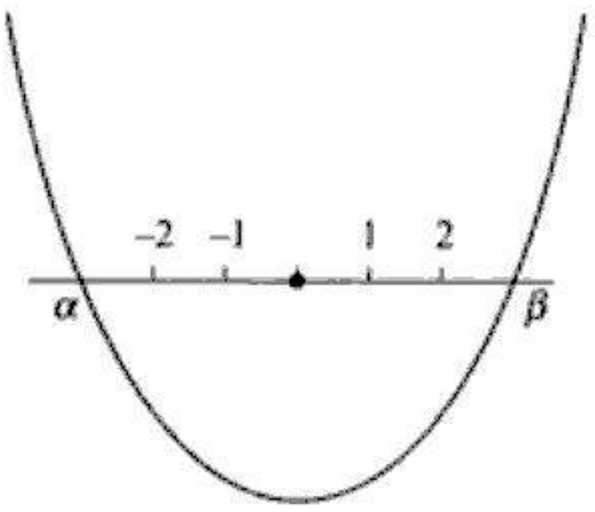

Fig. 2.17

Ans. (a), (b), (c), (d)

Solution Since the equation has two distinct roots α and β, the discriminant $b^2 - 4ac > 0$, we must have

$$f(x) = ax^2 + bx + c < 0 \text{ for } \alpha < x < \beta$$

Since $\quad \alpha < 0 < \beta$ we must have, $f(0) = c < 0.$

Also, as $\alpha < -1$, $1 < \beta$ we get, $f(-1) = a - b + c < 0$ and $\quad f(1) = a + b + c < 0.$

$\Rightarrow \quad a + |b| + c < 0.$

Next, since $\quad \alpha < -2, 2 < \beta,$

$$f(-2) = 4a - 2b + c < 0$$

and $\quad f(2) = 4a + 2b + c < 0$, i.e., $4a + 2|b| + c < 0.$

Example 57 If a, b, c are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root then

(a) $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ are in H.P.

(b) $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$, are in G.P.

(c) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$, are in A.P.

(d) $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$, are in A.P.

Ans. (a), (c)

Solution As a, b, c are in G.P., $b^2 = ac$. Now, the equation $ax^2 + 2bx + c = 0$ can be written as

$$ax^2 + 2\sqrt{ac}\,x + c = 0 \Rightarrow \left(\sqrt{a}\,x + \sqrt{c}\right)^2 = 0$$

$$\Rightarrow \quad x = -\sqrt{\frac{c}{a}}, -\sqrt{\frac{c}{a}}$$

If the two given equations have a common root, then this root must be $-\sqrt{(c/a)}$.

Thus $\quad d\frac{c}{a} - 2e\sqrt{\frac{c}{a}} + f = 0$

$$\Rightarrow \quad \frac{d}{a} + \frac{f}{c} = \frac{2e}{c}\sqrt{\frac{c}{a}} = \frac{2e}{\sqrt{ac}} = \frac{2e}{b}$$

$\Rightarrow \quad \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

Example 58 If $ax + by = 1$, $cx^2 + dy^2 = 1$ have only one solution, then

(a) $\frac{a^2}{c} + \frac{b^2}{d} = 1$ (b) $x = \frac{a}{c}$

(c) $y = \frac{b}{d}$ (d) none of these.

Ans. (a), (b), (c)

Solution We have

$$ax + by = 1 \Rightarrow y = \frac{1 - ax}{b}$$

Putting this value in the second equation, we get

$$cx^2 + \frac{d}{b^2}(1 - ax)^2 = 1$$

$$\Rightarrow \quad (b^2c + a^2d)x^2 - 2adx + d - b^2 = 0 \quad (1)$$

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This quadratic equation will have equal roots if

$$D = 4a^2d^2 - 4(b^2c + a^2d)(d - b^2) = 0$$

$$\Rightarrow \quad a^2d^2 + (b^2c + a^2d)b^2 - b^2cd - a^2d^2 = 0$$

$$\Rightarrow \quad b^2[b^2c + a^2d - cd] = 0$$

$$\Rightarrow \quad b^2c + a^2d = cd \Rightarrow \frac{b^2}{d} + \frac{a^2}{c} = 1$$

Also, in this case

$$x = \frac{2ad}{2(b^2c + a^2d)} = \frac{ad}{cd} = \frac{a}{c}$$

$$y = \frac{1 - ax}{b} = \frac{1}{b}\left(1 - \frac{a^2}{c}\right) = \frac{1}{b} \cdot \frac{b^2}{d} = \frac{b}{d}$$

Example 59 For real x, the expression $\dfrac{(x-a)(x-b)}{x-c}$ will assume all real values provided

(a) $a > b > c$ (b) $a < b < c$

(c) $a > c > b$ (d) $a < c < b$

Ans. (c), (d)

Solution Let $y = \dfrac{(x-a)(x-b)}{x-c}$

$$\Rightarrow \quad x^2 - (a + b + y)x + ab + cy = 0$$

Since x is real, the discriminant

$$(a + b + y)^2 - 4(ab + cy) \geq 0$$

$$\Rightarrow \quad y^2 + 2(a + b - 2c)y + (a + b)^2 - 4ab \geq 0$$

$$\Rightarrow \quad y^2 + 2(a + b - 2c)\, y + (a - b)^2 \geq 0 \qquad (1)$$

Since y takes all real values, (1) is possible if and only if

$$4(a + b - 2c)^2 - 4(a - b)^2 < 0$$

$$\Rightarrow \quad (a + b - 2c + a - b)(a + b - 2c - a + b) < 0$$

$$\Rightarrow \quad 4(a - c)(b - c) < 0 \Rightarrow (a - c)(b - c) < 0$$

This is possible if c lies between a and b, that is, if $a < c < b$ or $a > c > b$.

Example 60 If $\sin\alpha, \cos\alpha$ are the roots of the equation $ax^2 + bx + c = 0, (c \neq 0)$ then

(a) $a^2 - b^2 + 2ac = 0$ (b) $(a + c)^2 = b^2 + c^2$

(c) $a^2 + b^2 - 2ac = 0$ (d) $(a - c)^2 = b^2 + c^2$.

Ans. (a), (b)

Solution We have

$$\sin\alpha + \cos\alpha = -\frac{b}{a}, \sin\alpha\cos\alpha = \frac{c}{a}$$

Now,

$$1 = \sin^2\alpha + \cos^2\alpha$$

$$= (\sin\alpha + \cos\alpha)^2 - 2\sin\alpha\cos\alpha$$

$$1 = \frac{b^2}{a^2} - \frac{2c}{a}$$

$$\Rightarrow \quad a^2 = b^2 - 2ac \Rightarrow a^2 - b^2 + 2ac = 0$$

$$\Rightarrow \quad a^2 + c^2 + 2ac = b^2 + c^2 \Rightarrow (a + c)^2 = b^2 + c^2$$

Example 61 If $x^2 + mx + 1 = 0$ (1)

and $(b - c)x^2 + (c - a)x + (a - b) = 0$ (2)

have both the roots common, then

(a) $m = -2$ (b) $m = -1$

(c) a, b, c are in A.P. (d) a, b, c are in H.P.

Ans. (a), (c)

Solution Note $x = 1$ satisfies (2)

$\therefore \quad x = 1$ satisfies (1).

Thus $1 + m + 1 = 0 \Rightarrow m = -2$

Therefore, (1) becomes

$$x^2 - 2x + 1 = 0 \Rightarrow (x - 1)^2 = 0.$$

Thus, two roots of (1) are 1, 1.

$\therefore$ product of roots of (2) = 1

$$\Rightarrow \quad \frac{a - b}{b - c} = 1 \Rightarrow a - b = b - c$$

$$\Rightarrow \quad a, b, c \text{ are in A.P.}$$

Example 62 If $b^2 \geq 4ac$ for the equations $ax^4 + bx^2 + c = 0$ then all the roots of the equation will be real if

(a) $b > 0, a < 0, c > 0$ (b) $b < 0, a > 0, c > 0$

(c) $b > 0, a > 0, c > 0$ (d) $b > 0, a < 0, c < 0$

Ans. (c), (d)

Solution Put $x^2 = y$. The given equation becomes

$$f(y) = ay^2 + by + c = 0 \qquad (1)$$

The given equation will have four real roots if (1) has two non-negative roots. This can happen if

$$\frac{-b}{a} \geq 0, af(0) \geq 0; b^2 - 4ac \geq 0$$

$$\Rightarrow \quad -ab \geq 0, ac \geq 0 \qquad [\because b^2 - 4ac \geq 0 \text{ is given}]$$

Thus a and b must have opposite sign whereas $a, -b$ and c must have the same sign.

$$\Rightarrow \quad a > 0, b < 0, c > 0 \quad \text{or} \quad a < 0, b > 0, c < 0.$$

Example 63 If $a, b \in \mathbf{R}$ and the quadratic equation $x^2 + (a - b)x - a - b + 1 = 0$ has unequal real roots for all $b \in \mathbf{R}$, then

(a) $a > 0$ (b) $a > 1$

(c) $a < 0$ (d) a can take any value.

Ans. (a), (c)

Solution The quadratic equation

$$x^2 + (a - b)x - a - b + 1 = 0 \qquad (1)$$

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will have unequal real roots if

$$D = (a-b)^2 + 4(a+b-1) > 0$$

$$\Rightarrow \qquad b^2 + (4-2a)b + a^2 + 4a - 4 > 0 \qquad (2)$$

This inequality will hold for all $b \in \mathbf{R}$ if and only if discriminant of the quadratic expression on LHS of $(2) < 0$.

That is, if

$$(4-2a)^2 - 4(a^2 + 4a - 4) < 0$$

$$\Rightarrow \qquad 16 - 16a + 4a^2 - 4a^2 - 16a + 16 < 0$$

$$\Rightarrow \qquad 32 - 32a < 0 \quad \Rightarrow \quad a > 1.$$

Example 64 If α, β are the roots of the equation $ax^2 + 2bx + c = 0$ and $\alpha + h$, $\beta + h$ are the roots of the equations $Ax^2 + 2Bx + C = 0$, then

(a) $h = \dfrac{b}{a} - \dfrac{B}{A}$ (b) $\dfrac{b^2 - ac}{B^2 - AC} = \dfrac{a^2}{A^2}$

(c) $h = \dfrac{Ac + aC}{A + a}$ (d) $\dfrac{b^2 - ac}{B^2 - AC} = \dfrac{a}{A}$

Ans. (a), (b)

Solution We have

$$\alpha + \beta = \frac{2b}{a}$$

and $(\alpha + h) + (\beta + h) = \dfrac{2B}{A}$

$$\Rightarrow \qquad 2h = -\frac{2B}{A} + \frac{2b}{a} \quad \Rightarrow \quad h = \frac{b}{a} - \frac{B}{A}$$

Also

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \frac{4b^2}{a^2} - \frac{4c}{a} = \frac{4(b^2 - ac)}{a^2}$$

and $(\alpha - \beta)^2 = [(\alpha + h) - (\beta + h)]^2 = \dfrac{4(B^2 - AC)}{A^2}$

Thus, $\dfrac{4(b^2 - ac)}{a^2} = \dfrac{4(B^2 - AC)}{A^2}$

$$\Rightarrow \qquad \frac{b^2 - ac}{B^2 - AC} = \frac{a^2}{A^2}$$

Example 65. Let α, β be the roots of $x^2 - 4x + A = 0$ and γ, δ be the roots of $x^2 - 36x + B = 0$. If α, β, γ, δ forms an increasing G.P., then

(a) $B = 81\ A$ (b) $A = 3$

(c) $B = 243$ (d) $A + B = 251$

Ans. (a), (b), (c)

Solution Let common ratio be r, then

$$\alpha(1 + r) = \alpha + \beta = 4,\ \alpha r^2 (1 + r) = 36$$

$\therefore\ r^2 = 9 \Rightarrow r = 3$. Thus, $\alpha = 1$

Therefore $A = \alpha\beta = \alpha^2 r = 3$

and $B = \gamma\delta = \alpha^2 r^5 = 243$

SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos. 66 to 70

A polynomial $p(x)$ of degree two or less which takes values y_0, y_1, y_2 at three distinct values x_0, x_1, x_2 respectively, is given by

$$p(x)^* = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\ y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\ y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\ y_2$$

Example 66 A polynomial of degree 2 which takes values y_0, y_0, y_1 at points x_0, $x_0 + t$, x_1 $(t \neq 0)$ is given by

(a) $\dfrac{(x-x_1)(2x_0 - x_1 - x)}{(x_0 - x_1)(x_0 - x_1 + t)}\ y_0 + \dfrac{(x-x_0)(x-x_0-t)}{(x_1-x_0)(x_1-x_0-t)}\ y_1$

(b) $\dfrac{(x-x_1)(2x_0 - x_1 - x)}{(x_0 - x_1)(x_0 - x_1 - t)}\left\{y_0 + \dfrac{(x_1 - x_0)y_1}{x_1 - x_0 - t}\right\}$

(c) $\dfrac{(x-x_0)(2x_0 - x_1 - x)}{(x_0 - x_1)(x_0 - x_1 + t)}\left\{y_0 + \dfrac{(x_0 - x_1)y_1}{x_1 + t - x_0}\right\}$

(d) none of these

Example 67 Let p be a polynomial of degree 2 and $a \neq 0, 1$, then

(a) $p(x) = x[(x-1)\,p(0) + xp(a)] + x(x-1)\,p(a)$

(b) $p(x) = x(x-a)\,p(0) + ap(a)\,\{x(x-a) + x^2\}$

(c) $p(x) = \left[1 - \dfrac{x(x-a)}{1-a}\right] p(0) + \dfrac{x(1-x)}{1-a}\dfrac{p(a) - p(0)}{a} + \dfrac{x(x-a)}{1-a}p(1)$

(d) none of these.

Example 68 In Example 67 if $a \to 0$, then $p(x)$ is given by

(a) $(1 - x^2)\,p(0) + x(1-x)\,p'(0) + x^2\,p(1)$

(b) $x^2\,p(0) + xp'(0) + p(1)$

(c) $x^2\,p(0) + xp'(0) + xp(1) + p'(1)$

(d) none of these

* It is called as Lagrange's interpolation polynomial.

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Example 69 A polynomial of degree 2 which takes values – 1, 0 and 1 at – 1, 0 and 1 respectively is

(a) $x^2 - x + 1$ (b) $x^2 - x + 3$

(c) $x^2 + x - 1$ (d) none of these.

Example 70 Condition under which $p(x)$ will be a polynomial of degree 2 is

(a) $\begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \neq 0$ (b) $\begin{vmatrix} x_0 & y_0^2 & 1 \\ x_1 & y_1^2 & 1 \\ x_2 & y_2^2 & 1 \end{vmatrix} \neq 0$

(c) $\begin{vmatrix} x_0^2 & y_1 & 1 \\ x_1^2 & y_1 & 1 \\ x_2^2 & y_2 & 1 \end{vmatrix} = 0$ (d) $\begin{vmatrix} x_0^2 & y_0^2 & 1 \\ x_1^2 & y_1^2 & 1 \\ x_2^2 & y_2^2 & 1 \end{vmatrix} = 0$

Ans. 66. (d), 67. (c), 68. (a), 69. (d), 70. (a)

Solution 66.

$$p(x) = \frac{(x-x_0-t)(x-x_1)}{(-t)(x_0-x_1)}\, y_0 + \frac{(x-x_0)(x-x_1)}{(t)(x_0+t-x_1)}\, y_0 + \frac{(x-x_0)(x-x_0-t)}{(x_1-x_0)(x_1-x_0-t)}\, y_1$$

$$= \frac{(x-x_1)(2x_0-x_1-x+t)}{(x_0-x_1)(x_0-x_1+t)}\, y_0 + \frac{(x-x_0)(x-x_0-t)}{(x_1-x_0)(x_1-x_0-t)}\, y_1$$

67. $$p(x) = \frac{x(x-1)}{a(a-1)}\, p(a) + \frac{(x-a)(x-1)}{a}\, p(0) + \frac{x(x-a)}{1-a} p(1)$$

$$= \frac{x(1-x)}{1-a}\left[\frac{p(a)-p(0)}{a}\right] + \frac{x(x-a)}{1-a}\, p(1) + \alpha(x, a)\, p(0)$$

where $$\alpha(x, a) = \frac{(x-1)x}{a(a-1)} + \frac{(x-1)(x-a)}{a}$$

$$= \frac{x-1}{a}\left[\frac{x}{a-1} + x - a\right] = 1 - \frac{x(x-a)}{1-a}$$

68. Take limit as $a \to 0$ in (c) of Example 67.

69. $p(x)$ is a polynomial of degree 1. In fact $p(x) \equiv x$.

70. If point (x_0, y_0), (x_1, y_1), (x_2, y_2) are not collinear, then a parabola can pass through these points.

Paragraph for Question Nos. 71 to 75

Let x_0, x_1 and x_2 be three distinct real numbers. Define three polynomials $l_0(x)$, $l_1(x)$, $l_2(x)$ and $l(x)$ as follows:

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

and $$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$l(x) = (x-x_0)(x-x_1)(x-x_2)$$

Example 71 $p(x) = l_0(x) + l_1(x) + l_2(x)$ equals

(a) 1 (b) x

(c) x^2 (d) $1 + x + x^2$

Example 72 $p(x) = (x_1 + x_2)\, l_0(x) + (x_2 + x_0)\, l_1(x) + (x_0 + x_1)\, l_2(x)$

then $p(x)$ equals

(a) x (b) $x - \frac{1}{2}(x_0 + x_1 + x_2)$

(c) $x_0 + x_1 + x_2 - x$ (d) none of these

Example 73 The Polynomial

$$\frac{l(x)}{(x-x_0)l'(x_0)} + \frac{l(x)}{(x-x_1)l'(x_1)} + \frac{l(x)}{(x-x_2)l'(x_2)}$$

equals

(a) 1 (b) x

(c) $x^2 + x$ (d) none of these

Example 74 $\sum_{k=0}^{2} l_k(x)\, x^2$ equals

(a) 0 (b) x

(c) x^2 (d) none of these

Example 75 If $\frac{1}{l(x)} = \frac{A_0}{x-x_0} + \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2}$ then A_i(for $i = 0, 1, 2$) equals

(a) $\frac{1}{l'(x_i)}$ (b) $\frac{l'(x_i)}{x_i}$

(c) $\frac{x_i}{l'(x_i)}$ (d) $l'(x_i)$

Ans. 71. (a), 72. (c), 73. (a), 74. (c), 75. (a)

Solution 71. $p(x) - 1 = 0$ for $x = x_0, x_1, x_2$.

Thus, $p(x) = 1 \ \forall \ x$.

72. $p(x) - (x_0 + x_1 + x_2 - x) = 0$

for $x = x_0, x_1, x_2$.

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$\therefore \quad p(x) = x_0 + x_1 + x_2 - x \ \forall \ x$

73. $l'(x) = (x - x_1)(x - x_2) + (x - x_0)(x - x_2) + (x - x_0)(x - x_1)$

This gives

$$\frac{l(x)}{(x-x_0)l'(x_0)} = l_0(x) \text{ etc.}$$

Now, use Example 71.

74. Use Example 71.

75. A_i can be obtained by putting $x = x_i$ in $\frac{1}{l(x)}$ after deleting the factor $(x - x_i)$ of $l(x)$.

Paragraph for Question Nos. 76 to 80

A polynomial
$p(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n, a_0 \neq 0$ is said to be a reciprocal equation if $a_i = a_{n-i}$ for $0 \leq i \leq [n/2]$ where $[x]$ denote the greatest integer $\leq x$.

Example 76 If $p(x)$ is a reciprocal polynomial of odd degree, then one of the roots of $p(x) = 0$ is

(a) -1 (b) 1
(c) 0 (d) $(n + 1)/2$

Example 77 If $p(x)$ is a reciprocal equation and $a \neq 0$ is a root of $p(x) = 0$, then which one of the following is also a root of $p(x) = 0$?

(a) $-a$ (b) $1/a$
(c) $a - 1$ (d) none of these

Example 78 Sum of the rational roots of

$$x^5 = \frac{133x - 78}{133 - 78x} \quad (1)$$

is

(a) 2/9 (b) 9/2
(c) 13/6 (d) 6/13

Example 79 Remainder when $p(x^5)$ is divided by $p(x) = x^4 + x^3 + x^2 + x + 1$ is

(a) $x + 1$ (b) $x - 1$
(c) $x^3 + 1$ (d) 5

Example 80 Let $m, n \in \mathbf{N}$ and

$$p(x) = 1 + x + \ldots + x^m.$$

$p(x)$ will divide $p(x^n)$ if

(a) $hcf\ (m, n) = 1$
(b) $hcf\ (m + 1, n) = 1$
(c) $hcf\ (m, n + 1) = 1$
(d) $hcf\ (m + 1, n + 1) = 1$

Ans. 76. (a), 77. (b), 78. (c), 79. (d), 80. (b)

Solution As n is odd $(-1)^i$ and $(-1)^{n-i}$ are of opposite sign. Now use the fact that $a_i = a_{n-i}$.

77. As $p(x)$ is a reciprocal polynomial

$$p(x) = x^n\ p\left(\frac{1}{x}\right)$$

$\therefore \quad p(a) = 0 \Leftrightarrow p(1/a) = 0$

78. We can write (1) as

$78x^6 - 133x^5 + 133x - 78 = 0 \quad (2)$

Clearly 1 and -1 are root of (2)
Dividing (2) by $x^2 - 1$, we get quotient as

$78x^4 - 133x^3 + 78x^2 - 133x + 78 = 0 \quad (3)$

Dividing (3) by x^2 and putting $x + 1/x = y$ we get

$78y^2 - 133y - 78 = 0$
$\Rightarrow \quad y = 13/6, -6/13$
Now, $x + 1/x = 13/6 \Rightarrow x = 2/3, 3/2$
and $x + 1/x = -6/13$ gives imaginary roots.

$\therefore$ sum of rational roots is

$1 + (-1) + 2/3 + 3/2 = 13/6.$

79. $(x - 1)\,p(x) = x^5 - 1$

Now
$$p(x^5) = (x^{20} - 1) + (x^{15} - 1) + (x^{10} - 1) + (x^5 - 1) + 5$$
$$= (x^5 - 1)\,q(x) + 5$$
where
$$q(x) = (x^{15} + x^{10} + x^5 + 1) + (x^{10} + x^5 + 1) + (x^5 + 1) + 1$$

$\therefore$ the required remainder is 5.

80. $p(x) = \frac{1 - x^{m+1}}{1 - x}$.

We wish to find m, n so that

$$\frac{p(x^n)}{p(x)} = \frac{\left(1 - x^{(m+1)n}\right)(1 - x)}{(1 - x^n)(1 - x^{m+1})}$$

is a polynomial.
Since both $1 - x^n$ and $1 - x^{m+1}$ divide $1 - x^{(m+1)n}$ and since factors of $1 - x^{(m+1)n}$ are distinct, it is necessary and sufficient that $x^{m+1} - 1$ and $x^n - 1$ have no factor in common except $x - 1$, that is, $hcf\ (m + 1, n) = 1$.

Paragraph for Question Nos. 81 to 85

A polynomial

$$p(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

where $a_0, a_1, \ldots a_n \in \mathbf{I}, a_0 \neq 0$
is said to be irreducible over $\mathbf{I}$ if $p(x)$ cannot be written as product of two non-constant polynomials of degrees $< n$.
For example, $p(x) = x^3 + x^2 + x + 1$ is not irreducible over $\mathbf{I}$ as

$$p(x) = (x^2 + 1)(x + 1)$$

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where as $q(x) = x^2 + x + 1$ is irreducible over I.

Eisenstein's Irreducibility Criterion

Let p be a prime and $a_0, a_1, \ldots a_n \in \mathbf{I}$ be such that $p \mid a_i$ for $1 \le i \le n$, $p \nmid a_0$ and $p^2 \nmid a_n$, then the polynomial

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

is irreducible over **I**.

Example 81 Let $a, b \in \mathbf{I}$. The polynomial

$$p(x) = (x-a)^2 (x-b)^2 + 1.$$

(a) $p(x)$ is irreducible over **I**

(b) $p(x)$ has a rational zero

(c) $p(x)$ has two integral

(d) none of these

Example 82 Let $p(x) = x^5 - 5x^4 + 25x - 5$, then

(a) $p(x)$ is irreducible over **I**

(b) $p(x)$ has at least one rational zeros

(c) $p(x)$ has three integral zeros

(d) none of these.

Example 83 Let p be a prime and

$$p(x) = x^{p-1} + x^{p-2} + \ldots + x + 1$$

then

(a) $p(x) = 0$ has no real roots

(b) $p(x)$ has at least one rational root

(c) $p(x)$ is irreducible over **I**

(d) none of these

Example 84 Least degree of the polynomial with integral coefficients of which

$\alpha = 3^{1/2007}$ is a zero is

(a) 223 (b) 669

(c) 2007 (d) none of these

Example 85 If a, b, c and d are different integers then

$$p(x) = (x-a)(x-b)(x-c)(x-d) - 1$$

(a) is irreducible over **I**

(b) has at least two rational roots

(c) has four real roots

(d) none of these.

Ans. 81. (a), 82. (a), 83. (c), 84. (c), 85. (a)

Solution 81. Suppose $p(x) = q(x)\, r(x)$ where $q(x)$ and $r(x)$ are non-constant polynomials of degrees < 4. We have

$$q(a) = r(a) = q(b) = r(b) = \pm 1$$

$\Rightarrow$ both $p(x) \pm 1$ and $q(x) \pm 1$ are divisible by $(x-a)(x-b)$. As deg $q(x)$ + deg $r(x) = 4$, we have

$$q(x) \pm 1 = r(x) \pm 1 = (x-a)(x-b)$$

$$\therefore \quad q(x)\, r(x) = [(x-a)(x-b) \pm 1]^2$$

$$\Rightarrow \quad p(x) = p(x) \pm 2(x-a)(x-b)$$

$\Rightarrow \quad (x-a)(x-b) = 0$. A contradiction.

82. Use Eisenstein's Irreducibility criterion with $p = 5$.

83. $p(x) = \dfrac{x^p - 1}{x-1}$

$$\Rightarrow \quad p(x+1) = \frac{(x+1)^p - 1}{x}$$

$$= x^{p-1} + {}^pC_1 x^{p-2} + \ldots + {}^pC_{p-2}\, x + {}^pC_{p-1}$$

For $1 \le i \le p-1$,

$${}^pC_i = \frac{p!}{(p-1)!\, i!}$$

As p is prime, $p \nmid (p-i)!$ and $p \nmid i!$

for $1 \le i \le p-1$. Also, $p \mid p!$

Thus, $p \mid {}^pC_i$ for $1 \le i \le p-1$.

As $p \nmid 1$ and $p^2 \nmid {}^pC_{p-1}$, $p(x+1)$ and hence $p(x)$ is irreducible over **I** by the Eisenstein's irreducible over **I**.

84. Clearly α is zero of $f(x) = x^{2007} - 3$. Also by the Eisenstein's criterion $f(x)$ is irreducible over **I**. If $g(x)$ is the desired polynomial of the least degree then by the division algorithm there exist $q(x)$ and $r(x)$ such that

$$f(x) = g(x)\, q(x) + r(x)$$

where $r(x) = 0$ or deg $r(x) <$ deg $g(x)$

Let $r(x) \ne 0$, then since

$$r(\alpha) = f(\alpha) - g(\alpha)\, q(\alpha) = 0$$

and deg $r(x) <$ deg $g(x)$, we reach a contradiction.

Thus, $r(x) = 0$ and $f(x) = g(x)\, q(x)$

and $f(x)$ is irreducible over **I** and deg $g(x) \ge 1$, $q(x)$ must be a constant. Therefore, $f(x)$ is the desired polynomial.

85. Let $p(x) = q(x)\, r(x)$ where $q(x)$, $r(x)$ are non-constant polynomials with integral coefficients. We have

$$q(\alpha) = -r(\alpha) = \pm 1 \text{ for } \alpha = a, b, c, d$$

Also degree of $q(x)$, $r(x)$ and hence of $q(x) + r(x) < 4$.

Since $q(\alpha) + r(\alpha) = 0$ for $\alpha = a, b, c, d$,

we get $q(x) + r(x) \equiv 0$.

$$\therefore \quad p(x) = -(q(x))^2$$

A contradiction as coefficient of x^4 on the L.H.S. is positive and on the R.H.S. it is negative.

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SECTION IV

INTEGER ANSWER TYPE

Example 86 If $x, y, z \in \mathbf{R}$, $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$, find the maximum possible value of z.

Ans. 2

Solution $x + y = 4 - z$, $x^2 + y^2 = 6 - z^2$,

$\therefore$ $2xy = (x + y)^2 - (x^2 + y^2)$

$= (4 - z)^2 - (6 - z^2) = 2z^2 - 8z + 10$

The quadratic equation whose roots are x and y is

$$t^2 - (x + y)t + xy = 0$$

or $$t^2 - (4 - z)t + z^2 - 4z + 5 = 0$$

Since x and y are real

$$(4 - z)^2 - 4(z^2 - 4z + 5) \geq 0$$

$\Leftrightarrow$ $3z^2 - 8z + 4 \leq 0 \Leftrightarrow (3z - 2)(z - 2) \leq 0$

$\Leftrightarrow$ $2/3 \leq z \leq 2$.

That z can take value 2 can be checked by taking $x = y = 1$.

Example 87 Find the number of rational roots of

$$p(x) = 2x^{98} + 3x^{97} + 2x^{96} + \ldots + 2x + 3 = 0.$$

Ans. 2

Solution $p(x) = (2x + 3)(x^{97} + x^{96} + \ldots + 1)$.

$= (2x + 3)(x + 1)(x^{96} + x^{94} + \ldots + x^2 + 1)$

Also, $x^{96} + x^{94} + \ldots + x^2 + 1 > 1 \ \forall \ x \in \mathbf{R}$.

Example 88 Find the sum of all the integeral roots of

$(\log_5 x)^2 + \log_{5x}(5/x) = 1$ (1)

Ans. 6

Solution Clearly $x > 0$ and $x \neq 1/5$

$$\log_{5x}(5/x) = \frac{\log_5 5 - \log_5 x}{\log_5 5 + \log_5 x}$$

Putting $\log_5 x = t$, then equation (1) becomes

$$t^2 + \frac{1 - t}{1 + t} = 1 \quad \Leftrightarrow \quad t^3 + t^2 - 2t = 0$$

$\Leftrightarrow$ $t(t - 1)(t + 2) = 0 \quad \Leftrightarrow \quad t = 0, 1, -2$

So integral roots of (1) are 1 and 5.

Example 89 Find the least integral value of m for which every solution of the inequality $1 \leq x \leq 2$ is a solution of the inequality $x^2 - mx + 1 < 0$.

Ans. 3

Solution Clearly $m > 0$. Now $x^2 - mx + 1 < 0$

$\Leftrightarrow$ $$\left(x - \frac{m}{2}\right)^2 < \frac{m^2}{4} - 1$$

Thus, $m^2 - 4 > 0$ or $m > 2$

and $$\frac{m}{2} - \sqrt{\frac{m^2}{4} - 1} < x < \frac{m}{2} + \sqrt{\frac{m^2}{4} - 1}$$

We must have

$$\frac{m}{2} - \sqrt{\frac{m^2 - 4}{4}} < 1 \text{ and } 2 < \frac{m}{2} + \sqrt{\frac{m^2}{4} - 1}$$

$\Rightarrow$ $m - 2 < \sqrt{m^2 - 4}$ and $4 - m < \sqrt{m^2 - 4}$

$\Rightarrow$ $m^2 - 4m + 4 < m^2 - 4$

and $16 - 8m + m^2 < m^2 - 4$

$\Rightarrow$ $2 < m$ and $m > 5/2$

$\therefore$ the least integral value of m is 3.

Example 90 Find the sum all the real numbers satisfying the equation

$x^2 + |x - 1| = 1$. (1)

Ans. 1

Solution For $x \geq 1$, (1) becomes

$$x^2 + x - 1 = 1 \Leftrightarrow (x + 2)(x - 1) = 0$$

$$\Rightarrow x = 1$$

For $x < 1$, (1) becomes

$$x^2 + 1 - x = 1 \Rightarrow x = 0.$$

Example 91 Find the greatest integral value of a for which the equation

$$(2 - x)(x + 1) = a \quad (1)$$

has non-negative roots.

Ans. 2

Solution (1) can be written as

$$x^2 - x + a - 2 = 0$$

Its roots will be real if $1 - 4(a - 2) > 0 \Rightarrow a < 9/4$

Also, $a - 2$ product of roots of (1) ≥ 0. Thus, $2 \leq a < 9/4$.

Example 92 Find the sum of all the real roots of the equation

$$(3 - x)^5 + (x - 5)^5 = -32 \quad (1)$$

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Ans. 8

Solution Put $4 - x = t$, so that (1) becomes

$$(t-1)^5 - (t+1)^5 = -32$$

$$\Rightarrow \quad 10t^4 + 20t^2 - 30 = 0$$

$$\Rightarrow \quad (t^2+3)(t^2-1) = 0$$

When x is real, $t^2 + 3 > 0$

$\therefore t = \pm 1 \Rightarrow x = 3, 5.$

Example 93 Find the number of polynomials $p(x)$ with integral coefficients satisfying the conditions

$$p(1) = 2, p(3) = 1.$$

Ans. 0

Solution Let $p(x)$ be one such polynomial.

Let $p_1(x)$, be such that

$$p(x) - 1 = (x-3)\, p_1(x)$$

$$\Rightarrow \quad p(1) - 1 = -2p_1(1)$$

$$\Rightarrow \quad 1 = -2p_1(1)$$

But this is not possible as both 2 and $p_1(1)$ are integers.

Example 94 Find the degree of the remainder when $x^{2007} - 1$ is divided by $(x^2+1)(x^2+x+1)$

Ans. 3.

Solution Write as

$$x^{2007} - 1 = q(x)(x^2+1)(x^2+x+1) + ax^3 + bx^2 + cx + d$$

and substitute $x = \omega, \omega^2, i, -i$ to show $a = 1$.

Example 95 Let $p(x)$ be a polynomial with integral coefficients. Let a, b, c be three distinct integers such that

$$p(a) = p(b) = p(c) = -1.$$

Find the number of integral roots of $p(x)$.

Ans. 0

Solution a, b, c are roots of $p(x) + 1 = 0$.

Let $p(x) + 1 = (x-a)(x-b)(x-c)\, q(x)$ (1)

where $q(x)$ is some polynomial with integral coefficients.

Let m be an integral root of $p(x)$. Then (1) gives

$$1 = (m-a)(m-b)(m-c)\, q(m)$$

As a, b, c are distinct at least one of $m-a, m-b, m-c$ is different from 1 and -1. This is a contradiction.

SECTION V

MATRIX MATCH TYPE

Example 96 Match the equations on left with the properties on right.

Column I	Column II
(a) $a < b < c < d$ and equation is $(x-a)(x-c) + \pi(x-b)(x-d) = 0$	(p) real roots
(b) $a > 0, a + b + c < 0$ and equation is $ax^2 + bx + c = 0$	(q) distinct real roots
(c) $b, c, \in \mathbf{I}$ and the equation $x^2 + bx + c = 0$ has rational roots	(r) integral roots
(d) $a, b, c, d \in \mathbf{R}$ are in $G.P.$ and equation is $(a^2+b^2+c^2)x^2 + 2(ab + bc + ca)x + b^2 + c^2 + d^2 = 0$	(s) discriminant ≥ 0

Ans.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

Solution (a) Let $P(x) = (x-a)(x-c) + \pi(x-b)(x-d)$

Note that P is continuous on $\mathbf{R}$ and

$P(a) > 0, P(b) < 0, P(c) < 0, P(d) > 0$

$\therefore$ $P(x) = 0$ has a root in (a, b) and a root in (c, d).

(b) As $a > 0$, the parabola $y = ax^2 + bx + c$ open upwords. Also, as $y(1) < 0$, $y = ax^2 + bx + c$ meets the x–axis twice.

$\therefore$ $ax^2 + bx + c = 0$ has distinct real roots.

(c) $x^2 + bx + c = 0$ has integral roots. See Theory page 2.1.

(d) If r is the common ratio of the $G.P.$, we can write the given equation as

$(x^2 + 2rx + r^2)(1 + r^2 + r^4) = 0$

$\Rightarrow$ $(x+r)^2 = 0 \Rightarrow x = -r, -r.$

Example 97 Match the equations on the left with the properties on the right.

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Column I	Column II
(a) $a, b, c \in \mathbf{R}$ are such that $a + b + c = 0$. Equation is $3ax^2 + 2bx + c = 0$	(p) real roots
(b) $a, b, c \in \mathbf{R}$ are such that $2a + 3b + 6c = 0$. Equation is $ax^2 + bx + c = 0$	(q) discriminant > 0
(c) $a, b, c \in \mathbf{R}$ are such that $(a + c)^2 < b^2$. Equation is $ax^2 + bx + c = 0$	(r) at least one root in (0, 1)
(d) $a, b, c \in \mathbf{R}$ are such that $1 + \frac{\lvert b\rvert}{a} + \frac{c}{a} < 0$. Equation $ax^2 + bx + c = 0$	(s) a root in $(-\infty, -1)$ or a root in $(1, \infty)$

		p	q	r	s
Ans.	a	(p)	(q)	(r)	(s)
	b	(p)	(q)	(r)	(s)
	c	(p)	(q)	(r)	(s)
	d	(p)	(q)	(r)	(s)

Solution (a) Consider the polynomial
$P(x) = ax^3 + bx^2 + cx$
and use Rolle's Theorem on [0, 1].

(b) Consider the polynomial
$P(x) = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + \frac{1}{6}cx$
and use Rolle's theorem on [0, 1].

(c) Let $P(x) = ax^2 + bx + c$
Note that $P(-1)\ P(1) < 0$
$\Rightarrow$ $P(x)$ vanishes at least once in $(-1, 1)$
$\Rightarrow$ $P(x) = 0$ has real roots.

(d) Let $P(x) = ax^2 + bx + c$
and $Q(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$

As $1 + \frac{\lvert b\rvert}{a} + \frac{c}{a} < 0$, we get
$Q(-1)$ or $Q(1) < 0$
Since $Q(x) \to \infty$ as $x \to \pm\infty$, we get $Q(x) = 0$ or $P(x) = 0$ has a root in $(-1, \infty)$ or $(1, \infty)$.

Example 98 If $x \in \mathbf{R}$, Match each expression with its range.

(a) $\frac{x^2 - 2x + 3}{x^2 - 2x - 8}$	(p) $(-\infty, 3] \cup [4, \infty)$
(b) $\frac{x^2 - 12}{2x - 7}$	(q) $\mathbf{R} - \{1\}$
(c) $\frac{x^2 - 3x + 2}{x^2 + x - 6}$	(r) $\mathbf{R}$
(d) $\frac{x^2 - 3x + 1}{x - 1}$	(s) $(-\infty, -2/9] \cup (1, \infty)$

		p	q	r	s
Ans.	a	(p)	(q)	(r)	(s)
	b	(p)	(q)	(r)	(s)
	c	(p)	(q)	(r)	(s)
	d	(p)	(q)	(r)	(s)

Solution

(a) $y = \frac{x^2 - 2x + 3}{x^2 - 2x - 8}$

$\Rightarrow (x^2 - 2x - 8)\ y = x^2 - 2x + 3$

$\Rightarrow (y - 1)\ x^2 - 2(y - 1)\ x - (8y + 3) = 0$

As $x \in \mathbf{R}$, $4(y - 1)^2 + 4\ (y - 1)\ (8y + 3) \geq 0$

$\Rightarrow (y - 1)\ (9y + 2) \geq 0 \Rightarrow y \leq -2/9$ or $y \geq 1$.

But $y = 1 \Rightarrow 3 = -8$. Not possible.

Thus, $y \in (-\infty, -2/9] \cup (1, \infty)$

(b) $y = \frac{x^2 - 12}{2x - 7}$

$\Rightarrow x^2 - 2yx + 7y - 12 = 0$

As $x \in \mathbf{R}$, $4y^2 - 4\ (7y - 12) \geq 0$

$\Rightarrow y^2 - 7y + 12 \geq 0 \Rightarrow (y - 3)\ (y - 4) \geq 0$

$\Rightarrow y \in (-\infty, 3] \cup [4, \infty)$

(c) $y = \frac{x^2 - 3x + 2}{x^2 + x - 6} \Rightarrow (y - 1)\ x^2 + (y + 3)\ x - (6y + 2) = 0$

As $x \in \mathbf{R}$, $(y + 3)^2 + 4(y - 1)\ (6y + 2) \geq 0$

$\Rightarrow 25y^2 - 10y + 1 = (5y - 1)^2 \geq 0$

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However, $y = 1 \Rightarrow x = 2$ for which expression becomes $\frac{0}{0}$. Therefore, $y \neq 1$. Thus, range of the given expression $\mathbf{R} - \{1\}$.

(d) $y = \dfrac{x^2-3x-1}{x-1} = x - 2 - \dfrac{3}{x-1}$

$$\frac{dy}{dx} = 1 + \frac{3}{(x-1)^2} > 0 \ \forall \ x \neq 1$$

$\therefore$ y increases as x increases.

Thus, range is **R**.

Example 99 Let α, β, γ be three numbers such that

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{1}{2}, \ \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{9}{4},$$

and $\alpha + \beta + \gamma = 2$.

(a) $\alpha\beta\gamma$	(p) 6
(b) $\beta\gamma + \gamma\alpha + \alpha\beta$	(q) 8
(c) $\alpha^2 + \beta^2 + \gamma^2$	(r) -2
(d) $\alpha^3 + \beta^3 + \gamma^3$	(s) -1

Ans.

	p	q	r	s
a	p	q	**r**	s
b	p	q	r	**s**
c	**p**	q	r	s
d	p	**q**	r	s

Solution

$$2\left(\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta}\right) = \left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^2 - \left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)$$

$$\Rightarrow \quad \frac{2(\alpha+\beta+\gamma)}{\alpha\beta\gamma} = -2 \ \Rightarrow \ \alpha\beta\gamma = -2$$

Also, $\dfrac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma} = \dfrac{1}{2} \ \Rightarrow \ \beta\gamma + \gamma\alpha + \alpha\beta = -1.$

$\therefore \quad \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta)$

$= 6$

Lastly $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$

$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta)$

Example 100 The set of value(s) of $k \in \mathbf{R}$ for which

(i) $kx^2 - (k+1)x + 2k - 1 = 0$ has no real roots	(a) $\{1, -2\}$
(ii) $x^2 - 2(4k-1)x + 15k^2 - 2k - 7 > 0$ for each x	(b) $(-\infty, -1/7) \cup (1, \infty)$
(iii) Sum of the roots of $x^2 + (2 - k - k^2)x - k^2 = 0$ is zero	(c) $\{-4\}$
(iv) The roots of $x^2 + (2k-1)x + k^2 + 2 = 0$ are in the ratio 1 : 2	(d) (2, 4)

Ans.

	p	q	r	s
a	p	**q**	r	s
b	p	q	r	**s**
c	**p**	q	r	s
d	p	q	**r**	s

Solution (a) $(k+1)^2 - 4k(2k-1) < 0, \ k \neq 0$

$\Rightarrow \quad 7k^2 - 6k - 1 > 0$

$\Rightarrow \quad (7k+1)(k-1) > 0$

$\Rightarrow \quad k < -1/7 \text{ or } k > 1$

(b) $(4k-1)^2 - (15k^2 - 2k - 7) < 0$

$\Rightarrow \quad k^2 - 6k + 8 < 0 \Rightarrow (k-2)(k-4) < 0$

$\Rightarrow \quad 2 < k < 4$

(c) $2 - k - k^2 = 0 \Rightarrow k = 1, -2$

(d) $\alpha + 2\alpha = -(2k-1),$

$\alpha(2\alpha) = k^2 + 2$

$\Rightarrow \quad \dfrac{2}{9}(2k-1)^2 = k^2 + 2 \ \Rightarrow k = -4$

Example 101 The number of rational roots of

(a) $x^3 - px^2 + 1 = 0, \ p > 2$	(p) 3
(b) $x^{10} - x^9 - 2 = 0$	(q) 2
(c) $(x+1)(x+2)(x+3)(x+4) = 24$	(r) 1
(d) $(\log_3 x)^2 + \log_{3x}(3/x) = 1$	(s) 0

Ans.

	p	q	r	s
a	p	q	r	**s**
b	p	q	**r**	s
c	p	**q**	r	s
d	**p**	q	r	s

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Solution

(a) Any rational root of $x^3 - px^2 + 1 = 0$ must be an integer. But for $a \in \mathbf{I}$, $a^2(p - a) = 1$ is not possible if $p > 2$.

(b) As in (i) any root of $x^{10} - x^9 - 2 = 0$ must be an integer. Clearly $x = -1$ satisfies the given equation. For, $x \neq -1$, $x \in \mathbf{I}$, $x^9(x - 1) = 2$ is not possible.

(c) The given equation can be written as
$(x^2 + 5x + 4)(x^2 + 5x + 6) = 24$

Put $x^2 + 5x = t$ to obtain

$t^2 + 10t = 0 \Rightarrow t = 0, t = -10.$

For $t = 0$, $x^2 + 5x = 0 \Rightarrow x = 0, x = -5$

For $t = -10$, $x^2 + 5x = -10$ does not have rational roots.

(d) Put $\log_3 x = t$ to obtain

$$t^2 + \frac{1-t}{1+t} = 1 \Rightarrow t^3 + t^2 - 2t = 0$$

$$\Rightarrow t(t + 2)(t - 1) = 0 \Rightarrow t = 0, 1, -2$$

This gives $x = 1, 3, 1/9$.

Example 102 The number of real roots of

(a) $\sqrt{x+9} + \sqrt{x} = 2.7$ (p) 2

(b) $\dfrac{\sqrt{13-x^2}}{|x+1|} = 1$ (q) 1

(c) $\sqrt{x+5} - \sqrt{x} = 1$ (r) infinite

(d) $\sqrt{x+2\sqrt{x-1}} - \sqrt{x-2\sqrt{x-1}} = 2$ (s) 0

Ans.

	p	q	r	s
a				●
b	●			
c		●		
d			●	

Solution

(a) $x \geq 0 \Rightarrow \sqrt{x+9} \geq 3$

(b) $13 - x^2 = (x + 1)^2 \Rightarrow x^2 + x - 6 = 0$
$\Rightarrow x = -3, 2$

(c) $x + 5 = 1 + x + 2\sqrt{x} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$

(d) Put $x - 1 = t^2$, to obtain
$|t + 1| - |t - 1| = 2$
which is satisfied for each $t \geq 1$.

Example 103 Let α, β be roots of the equation $ax^2 + bx + c = 0$, then equation whose roots are

(a) $-1/\alpha, -1/\beta$ (p) $c^2x^2 + (2ac - b^2)x + a^2 = 0$

(b) $-\alpha, -\beta$ (q) $a^2x^2 + (2ac - b^2)x + c^2 = 0$

(c) α^2, β^2 (r) $cx^2 - bx + a = 0$

(d) $1/\alpha^2, 1/\beta^2$ (s) $ax^2 - bx + c = 0$

Ans.

	p	q	r	s
a			●	
b				●
c		●		
d	●	●		

Solution

(a) Replace x by $-1/x$ to obtain $a - bx + cx^2 = 0$

(b) Replace x by $-x$ to obtain $ax^2 - bx + c = 0$

(c) Replace x by $\sqrt{x}$ to obtain

$$ax + b\sqrt{x} + c = 0$$
$$\Rightarrow a^2x^2 + 2acx + c^2 = b^2x$$
$$\Rightarrow a^2x^2 + (2ac - b^2)x + c^2 = 0$$

(d) Change x to $1/x$ in the above equation.

Example 104 Let α, β be roots of $ax^2 + bx + c = 0$. Match the equation on the left with its roots on the right.

(a) $(x - b)^2 + b(x - b) + ac = 0$ (p) $2\alpha, 2\beta$

(b) $ax^2 + 2bx + 4c = 0$ (q) $-\alpha/a, -\beta/a$

(c) $4a^2x^2 - b^2 + 4ac = 0$ (r) $a\alpha + b, a\beta + b$

(d) $a^3x^2 - abx + c = 0$ (s) $\alpha + b/2a, \beta + b/2a$

Ans.

	p	q	r	s
a			●	
b	●			
c				●
d		●		

Solution

(a) Write equation as

$$a\left(\frac{x-b}{a}\right)^2 + b\left(\frac{x-b}{a}\right) + c = 0$$

$$\Rightarrow \frac{x-b}{a} = \alpha, \beta$$

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(b) $a\left(\frac{x}{2}\right)^2 + b\left(\frac{x}{2}\right) + c = 0$

$\Rightarrow \quad x/2 = \alpha, \beta$

(c) $x = \frac{\pm\sqrt{b^2 - 4ac}}{2a} = \alpha + \frac{b}{2a}, \beta + \frac{b}{2a}$

(d) $a(-ax)^2 + b(-ax) + c = 0$

$\Rightarrow \quad -ax = \alpha, \beta \Rightarrow x = -\alpha/a, -\beta/a.$

Example 105 Let $y = \frac{x^2 - 12x + 100}{x^2 + 12x + 100}, x \in \boldsymbol{R}$.

(a) Greatest value of y (p) -10

(b) Least value of y (q) 10

(c) Greatest value of y is attained at (r) 1/4

(d) Least value of y is attained at (s) 4

Ans.

	p	q	r	s
a	p	q	r	**s**
b	p	q	**r**	s
c	**p**	q	r	s
d	p	**q**	r	s

Solution $(x^2 + 12x + 100)y = x^2 - 12x + 100$

$\Rightarrow \quad (y - 1)x^2 + 12(y + 1)x + 100(y - 1) = 0$

As x is real

$$36(y + 1)^2 - 100(y - 1)^2 \geq 0$$

$\Rightarrow \quad (y - 1/4)(y - 4) \leq 0$

$\Rightarrow \quad 1/4 \leq y \leq 4$

For $y = 4$, we get $x = -10$

For $y = -4$, we get $x = 10$

Example 106 Let α, β be roots of the equation $ax^2 + 2bx + c = 0$. The equation whose roots are

(a) $\pm 2\alpha, \pm 2\beta$ (p) $(a^2 + c^2x^2)^2 - 4(ac - 2b^2)x^2 = 0$

(b) $\pm 1/\alpha, \pm 1/\beta$ (q) $(a^2x^2 + c^2)^2 - 4(ac - 2b^2)x^2 = 0$

(c) $\pm \alpha^2, \pm \beta^2$ (r) $a^2x^4 + 8(ac - 2b^2)x^2 + 16c^2 = 0$

(d) $\pm 1/\alpha^2, \pm 1/\beta^2$ (s) $c^2x^4 + 2(ac - 2b^2)x^2 + a^2 = 0$

Ans.

	p	q	r	s
a	**p**	q	**r**	s
b	p	q	r	**s**
c	p	**q**	r	s
d	**p**	q	r	s

Solution

(a) $2\alpha, 2\beta$ are roots of $ax^2 + 4bx + 4c = 0$

and $-2\alpha, -2\beta$ are roots

$ax^2 - 4bx + 4c = 0.$

$\therefore$ Required equation is

$(ax^2 + 4c)^2 - 16b^2x^2 = 0$

or $a^2x^4 + 8(ac - 2b^2)x^2 + 16c^2 = 0$

(b) $1/\alpha, 1/\beta$ are roots of $a + 2bx + cx^2 = 0$

and $-1/\alpha, -1/\beta$ are roots of

$a - 2bx + cx^2 = 0$

$\therefore$ Required equation

$(cx^2 + a)^2 - 4b^2x^2 = 0$

or $c^2x^4 + 2(ac - 2b^2)x^2 + a^2 = 0$

(c) α^2, β^2 are roots of $ax + 2b\sqrt{x} + c = 0$

$(ax + c)^2 = 4b^2x$

$\Rightarrow \quad a^2x^2 + 2(ac - 2b^2)x + c^2 = 0.$

Also, $-\alpha^2, -\beta^2$ are roots of

$a^2x^2 - 2(ac - 2b^2)x + c^2 = 0$

$\therefore$ Required equation is

$(a^2x^2 + c^2)^2 - 4(ac - 2b^2)^2 x^2 = 0.$

(d) Replace x by $1/x$ in the above equation.

SECTION VI

REASONING TYPE

Example 107 **Statement-1:** If $a, b, c \in \mathbf{R}$ and $2a + 3b + 6c = 0$, then the equation $ax^2 + bx + c = 0$ has at least one root root in [0, 1].

Statement-2: If a continuous function f defined on $\mathbf{R}$ assumes both positive and negative values, then it vanishes at least once.

Ans. (b)

Solution Consider $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx$$

Then f is continuous and differentiable on $\mathbf{R}$ and

$$f(1) = \frac{1}{3}a + \frac{1}{2}b + c = \frac{1}{3}(2a + 3b + 6c) = 0, f(0) = 0.$$

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By the Rolle's theorem there exists at least one $\alpha \in (0, 1)$ such that $f'(\alpha) = 0$

But $$f'(x) = ax^2 + bx + c$$

Thus, $ax^2 + bx + c = 0$ has at least one root in $(0, 1)$.
Statement-2 is true but its not the correct explanation of statement-1.

Example 108 **Statement-1:** If $a \neq 0$ and the equation $ax^2 + bx + c = 0$ has two roots α and β such that $\alpha < -2$ and $\beta > 2$, then $a + |b| + c$ and a have the opposite signs.
Statement-2: If $a > 0$ and $\gamma, \delta (\gamma < \delta)$ are the roots of $ax^2 + bx + c = 0$, then graph of the parabola $y = ax^2 + bx + c$, for $\gamma < x < \delta$ lies below the x-axis.

Ans. (a)

Solution α, β are roots of $ax^2 + bx + c = 0 \Leftrightarrow \alpha, \beta$ are roots of $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

As coefficient of $x^2 > 0$, graph of the parabola $y = x^2 + \frac{b}{a}x + \frac{c}{a}$ for $\alpha < x < \beta$ lies below the x-axis. As $-1, 1 \in (\alpha, \beta)$, $(-1)^2 + \frac{b}{a}(-1) + \frac{c}{a} < 0$ and $1^2 + \frac{b}{a} + \frac{c}{a} < 0$

$$\Rightarrow \quad 1 - \frac{b}{a} + \frac{c}{a} < 0 \quad \text{and} \quad 1 + \frac{b}{a} + \frac{c}{a} < 0$$

$$\Rightarrow \quad 1 + \frac{|b|}{a} + \frac{c}{a} < 0 \quad \Rightarrow \quad \frac{1}{a}(a + |b| + c) < 0$$

Thus, $a + |b| + c$ and a have the opposite signs.

Example 109 **Statement-1:** If a, b and c are distinct real numbers, then

$$a\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\frac{(x-c)(x-a)}{(b-c)(b-a)} + c\frac{(x-a)(x-b)}{(c-a)(c-b)} = x$$

for each real x.
Statement-2: If $ax^2 + bx + c = 0$ for three distinct real values of x, then

$$a = b = c = 0, \quad i.e.\ ax^2 + bx + c = 0\ \forall\ x \in \mathbf{R}.$$

Ans. (a)

Solution To see that statement-2 is true, see theory Page 2.1.

Let $$g(x) = a\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\frac{(x-c)(x-a)}{(b-c)(b-a)} + c\frac{(x-a)(x-b)}{(c-a)(c-b)} - x$$

Note that $g(x)$ is a quadratic expression and $g(a) = g(b) = g(c) = 0$. Therefore

$$g(x) = 0\ \forall\ x \in \mathbf{R}$$

Example 110 **Statement-1:** Let $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbf{R}$. If $f(x)$ assumes real values for real values of x and non-real values for non-real values of x, then $a = 0$
Statement-2: If a, b, c are complex numbers, $a \neq 0$, then $a + i\beta, \beta \neq 0$ is a root of $ax^2 + bx + c = 0$ if and only if $\alpha - i\beta$ is a root of $ax^2 + bx + c = 0$

Ans. (c)

Solution If $a \neq 0$, we rewrite $f(x)$ as follows:

$$f(x) = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$f\left(-\frac{b}{2a} + i\right) = a\left\{(i)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$= a\left(-1 + \frac{4ac - b^2}{4a^2}\right)$$

which is real.
This contradicts our assumption that $f(x)$ is non-real for non-real x.
Therefore, $a = 0$.
Statement-2 is false since $-i$ is a root of $x^2 + x + 1 + i = 0$ but i is not a root of $x^2 + x + 1 + i = 0$.

Example 111 **Statement-1:** If a and b are integers and roots of $x^2 + ax + b = 0$ are rational then these roots must be integers.
Statement-2: If a and b are integers then roots of $x^2 + ax + b = 0$ are necessarily integers.

Ans. (c)

Solution Let $\frac{p}{q}$ where $p, q \in \mathbf{I}$ and $hcf(p, q) = 1$ be a root of $x^2 + ax + b = 0$.
Then

$$\frac{p^2}{q^2} + \frac{ap}{q} + b = 0$$

$$\Rightarrow \quad p^2 + apq + bq^2 = 0$$

$$\Rightarrow \quad p^2 = -q(ap + bq)$$

$$\Rightarrow \quad q \mid p^2 \quad \Rightarrow \quad q \mid p \qquad [\because hcf(p, q) = 1]$$

As $q \mid p$ and $hcf(p, q) = 1$ we get $q = 1$

Thus, $\frac{p}{q} = p$ is an integer.
Statement-2 is false as $x^2 + x + 1 = 0$ does not have integral roots.

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Example 112 **Statement-1:** $a+b+c>0$, $bc+ca+ab>0$, $abc>0 \Leftrightarrow a>0, b>0, c>0$.
Statement-2: If $p, q, r>0$, then the equation $x^3-px^2+qx-r=0$ cannot have negative roots.
Ans. (a)

Solution Statement-2 is true as $x<0$ implies

$$x^3-px^2+qx-r<0 \qquad [\because p, q, r>0]$$

Let $p=a+b+c$, $q=bc+ca+ab$ and $r=abc$, then $p, q, r>0$.
Equation whose roots are a, b, c is $(x-a)(x-b)(x-c)=0$

or $$x^3-px^2+qx-r=0$$

As this equation cannot have negative roots $a, b, c>0$.

Example 113 **Statement-1:** If $a, b, c \in \mathbf{R}-\{0\}$, then at least one $ax^2+bx+c=0$, $bx^2+cx+a=0$ and $cx^2+ax+b=0$ has imaginary roots.
Statement-2: If $a, b, c \in \mathbf{R}$, $a \neq 0$, then imaginary roots of the equation $ax^2+bx+c=0$ occur in conjugate pair.

Ans. (b)

Solution If each of the three equation has real roots, then

$$b^2-4ac \geq 0,\ c^2-4ab \geq 0 \text{ and } a^2-4ab \geq 0$$

$\Rightarrow$ $a^2b^2c^2 \geq 64a^2b^2c^2$. A contradiction.
For truth of statement-2, see Theory.

Example 114 **Statement-1:** Let $ac=bc$ and let x_1, x_2 be the roots of $x^2+ax+bc=0$ and x_2, x_3 be the roots of $x^2+bx+ca=0$, then x_1, x_3 are the roots of $x^2+cx+ab=0$.

Statement-2: If $a, b \in \mathbf{R}$, then $|ab| < \sqrt{a^2+b^2}$.

Ans. (c)

Solution As $ac \neq bc$, $c \neq 0$ and $a \neq b$.
We have

$$x_1^2+ax_1+bc=0$$
$$x_1^2+bx_1+ca=0$$

Subtracting, we get

$$(a-b)x_1+c(b-a)=0$$

$\Rightarrow$ $$x_1=c \text{ as } a \neq b.$$

Thus, $c^2+ac+bc=0 \Rightarrow a+b+c=0$
Also, $x_1=b$ and $x_3=a$.
Equation whose roots are x_1 and x_3 is

$$x^2-(a+b)x+ab=0 \text{ or } x^2+cx+ab=0.$$

Statement-2 is false can be checked by taking $a=b=0$.

Example 115 **Statement-1:** The equation

$$f(x)=1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}=0$$

has two pairs of repeated roots.

Statement-2: Polynomial equation $P(x)=0$ has a repeated root α if $P(\alpha)=0$ and $P'(\alpha)=0$.

Ans. (d)

Solution If α is a repeated root of $f(x)=0$, then $f(\alpha)=0$ and $f'(\alpha)=0$.

$\therefore$ $$1+\frac{\alpha}{1}+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\frac{\alpha^4}{4!}=0$$

and $$1+\frac{\alpha}{1}+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}=0$$

$\Rightarrow$ $$\frac{\alpha^4}{4!}=0 \Rightarrow \alpha=0.$$

But $\alpha=0$ does not satisfy $f(x)=0$.
For truth of statement-2, see Theory.

Example 116 **Statement-1:** If all the four roots of

$$x^4-4x^3+ax^2-bx+1=0$$

are positive, then $a=6$ and $b=4$.
Statement-2: If polynomial equation $P(x)=0$ has four positive roots, then the polynomial equation $P'(x)=0$ has 3 positive roots.

Ans. (b)

Solution Let x_1, x_2, x_3, x_4 be four positive roots of $x^4-4x^3+ax^2-bx+1=0$, then

$$x_1+x_2+x_3+x_4=4 \text{ and } x_1x_2x_3x_4=1.$$

$\Rightarrow$ $$\frac{1}{4}(x_1+x_2+x_3+x_4)=(x_1x_2x_3x_4)^{1/4}$$

That is, A.M. = G.M., but this is possible if and only if $x_1=x_2=x_3=x_4=1$.
Thus, the given equation becomes

$$(x-1)^4=0 \Rightarrow a=6, b=4$$

Statement-2 follows immediately from the Rolle's theorem.

Example 117 **Statement-1:** If $a, b, c \in Q$ and $2^{1/3}$ satisfies $a+bx+cx^2=0$, then $a=0, b=0, c=0$.
Statement-2: A polynomial equation with rational coefficients cannot have irrational roots.

Ans. (c)

Solution By multiplying a, b, c by an appropriate natural number, we may assume that a, b, c are integers. We may further assume that a, b, c have no factor in common. Now,

$$a+b2^{1/3}+c2^{2/3}=0$$

$\Rightarrow$ $$a^3=-2(b+2^{1/3}c)^3=-2(b^3+2c^3+3abc)$$

$\Rightarrow$ $2|a^3 \Rightarrow 2|a \Rightarrow 8a=2a_1$ for some $a_1 \in \mathbf{I}$.

Thus,

$$4a_1^3=-(b^3+2c^3-6a_1bc)$$

$\Rightarrow$ $$b^3=-2(2a_1^3+c^3-3a_1bc)$$

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$\Rightarrow \quad 2|b^3 \Rightarrow 2|b \Rightarrow b = 2b_1$ for some $b_1 \in \mathbf{I}$.

Therefore

$$4b_1^3 = -(2a_1^3 + c^3 - 6a_1b_1c)$$

$$\Rightarrow \quad c^3 = -2(a_1^3 + 2b_1^3 - 3a_1b_1c)$$

$\Rightarrow \quad 2|c^3 \Rightarrow 2|c$

This is a contradiction.

Statement-2 is false as $2^{1/3}$ is irrational but is a root of $x^3 - 2 = 0$.

Example 118 Let a, b, c, p, q be real numbers. Suppose a, b are the roots of equation $x^2 + 2px + q = 0$ and $\alpha, 1/\beta$ are roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \in \{-1, 0, 1\}$.

Statement-1: $(p^2 - q)(b^2 - ac) \geq 0$ and

Statement-2: $b \neq pa$ or $c \neq qa$

Ans. (a)

Solution Suppose $a \in \mathbf{R}$.

As $a, b, c, p, q \in \mathbf{R}$, the other root of $x^2 + 2px + q = 0$ and $ax^2 + 2bx + c = 0$ must be $\bar{\alpha}$.

Thus $\bar{\alpha} = \beta = 1/\beta \Rightarrow \beta^2 = 1$.

But $\beta^2 \notin \{-1, 0, 1\}$.

$\therefore \quad \alpha \in \mathbf{R} \Rightarrow \beta, 1/\beta \in \mathbf{R}$

$\Rightarrow \quad D_1 = 4(p^2 - q) \geq 0$ and $D_2 = 4(b^2 - ac) \geq 0$

Next, let $b = pa$ and $c = qa$, then

$ax^2 + 2bx + c = a(x^2 + 2px + q)$

$\Rightarrow$ both roots of the two equations are common.

$\Rightarrow \quad \beta = 1/\beta \Rightarrow \beta^2 = 1$. A contradiction.

Statement-2 is true but is not the correct explanation for the statement-1.

EXERCISE

LEVEL 1

SECTION I

SINGLE CORRECT CHOICE TYPE

1. If $a, b, c \in \mathbf{R}$ and $(a + b + c)c < 0$, then the quadratic equation $p(x) = ax^2 + bx + c = 0$ has
   (a) two negative roots
   (b) two real roots
   (c) two imaginary roots
   (d) none of these.
2. The number of rational roots of the equation
$$\frac{x^2-13x+15}{x^2-14x+15} - \frac{x^2-15x+15}{x^2-16x+15} = -\frac{1}{12}$$
is
   (a) 0 (b) 1
   (c) 2 (d) 3
3. The number of irrational roots of the equation
$$\frac{4x}{x^2+x+3} + \frac{5x}{x^2-5x+3} = -\frac{3}{2}$$
is
   (a) 0 (b) 1
   (c) 2 (d) 3
4. If $\alpha, \beta, \gamma, \delta$ are the roots of
$(x^2 + x + 4)^2 + 3x(x^2 + x + 4) + 2x^2 = 0$
then $|\alpha| + |\beta| + |\gamma| + |\delta|$ is equal
   (a) 6 (b) 8
   (c) 12 (d) 25
5. $x = 1$ is a root of
$(x^2 - x + 1)^4 - 6x^2(x^2 - x + 1)^2 + 5x^4 = 0$
of multiplicity
   (a) 2 (b) 3
   (c) 4 (d) 6
6. The number of irrational roots of the equation
$(x^2 - 3x + 1)(x^2 + 3x + 2)(x^2 - 9x + 20) = -30$
is
   (a) 0 (b) 2
   (c) 4 (d) 6
7. The number of roots of the equation
$$\sqrt{3x^2+6x+7} + \sqrt{5x^2+10x+14} = 4 - 2x - x^2$$
is
   (a) 4 (b) 3
   (c) 2 (d) 1
8. The number of real values of x which satisfy the equation
$$\left|\frac{x^2 - 6\sqrt{x} + 7}{x^2 + 6\sqrt{x} + 7}\right| = 1.$$
   (a) 0 (b) 1
   (c) 2 (d) infinite
9. The number of real values of x which satisfy the equation
$$\left|\frac{x}{x-1}\right| + |x| = \frac{x^2}{|x-1|}$$
   (a) 1 (b) 2
   (c) 5 (d) infinite
10. Sum of all the real values of x which satisfy the equation
$$\frac{\sqrt{2-x}}{\sqrt{2+x}} = \frac{2-x}{2+x}$$
   (a) 0 (b) 2
   (c) 7.5 (d) 11.5
11. The set of real values of a for which the equation
$$\frac{2a^2 + x^2}{a^3 - x^3} - \frac{2x}{ax + a^2 + x^2} + \frac{1}{x - a} = 0$$

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has a unique solution is

(a) $(-\infty, 1)$ (b) $(-1, \infty)$

(c) $(-1, 1)$ (d) none of these

12. The set of real values of a for which sum of the roots of the equation

$$\frac{1}{x}+\frac{1}{a}=\frac{1}{a^2}=\frac{1}{x+a-a^2}$$

is less than $a^3/4$ is

(a) $(0, 2) \cup (2, \infty)$ (b) $(3, \infty)$

(c) $(-1, 0) \cup (3, \infty)$ (d) $(2, \infty)$

13. The real values of a for which $\frac{x^2-ax-2}{x^2-3x+4} > -1$ for each $x \in \mathbf{R}$, is

(a) $(-1, 2)$ (b) $(0, 7)$

(c) $(-7, 1)$ (d) $(2, 7)$

14. If $a, b, c \in \mathbf{R}$ are distinct, then the condition(s) on a, b, c for which the equation

$$\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\frac{1}{x-b-c+a}=0$$

has real roots is

(a) $a + b + c \neq 0, a \neq 0$

(b) $a - b - c \neq 0, a \neq 0$

(c) $2a = b + c$

(d) none of these

15. If $f(x) = ax^2 + bx + c, f(-1) < 1, f(1) > -1$ and $f(-3) < -4$

then

(a) $a = 0$ (b) $a < 0$

(c) $a > 0$ (d) sign of a cannot be determined

16. The set of all values of k for which the equation $x^2 + 2(k - 1)x + (k - 5) = 0$ has at least one non-negative root is

(a) $[1, \infty)$ (b) $[-1, 1]$

(c) $(-\infty, -5]$ (d) $(-\infty, 5]$

17. If $3\pi/4 < \alpha < \pi$, then the set of values of α for which

$$(\sin \alpha)x^2 + (2 \cos \alpha)x + \frac{1}{2}(\sin \alpha + \cos \alpha)$$

is square of a linear polynomial is

(a) $\left\{\frac{5\pi}{6}, \frac{11\pi}{12}\right\}$ (b) $\left\{\frac{11\pi}{12}\right\}$

(c) $\left\{\frac{5\pi}{6}\right\}$ (d) ϕ

18. If $0 \leq \phi \leq 3\pi$, then the set of all values of ϕ for which sum of the squares of the roots of the equation

$$x^2 + (\sin \phi - 1)x - \frac{1}{2}\cos^2 \phi = 0$$

is greatest is

(a) $\left\{\pi, \frac{3\pi}{2}, 2\pi, 3\pi\right\}$ (b) $\{\pi, 3\pi\}$

(c) $\left\{2\pi, \frac{5\pi}{4}\right\}$ (d) $\left\{\frac{3\pi}{2}\right\}$

19. The set of values of b for which

$2\log_{1/36}(bx + 28) = -\log_6(12 - 4x - x^2)$

has exactly one solution is

(a) $(-\infty, -14) \cup [14/3, \infty)$

(b) $[14/9, \infty)$

(c) $(-\infty, -14] \cup \{4\} \cup [14/3, \infty]$

(d) $\{4\}$

20. If $2x^2 + 3x + 4 = 0$ and $ax^2 + bx + c = 0$, where $a, b, c \in \mathbf{N}$ have a common root, then the least value of $a + b + c$ is

(a) 10 (b) 9

(c) 8 (d) 6

21. Let p, q, r, s be real numbers such that $pr = 2(q + s)$. Consider quadratic equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$. Then

(a) none of these has real roots

(b) both have real roots

(c) at least one has real roots

(d) at most one has real roots

22. If $\cot \theta$, $\operatorname{cosec} \theta$ are the roots of the equation $ax^2 + bx + c = 0$ and $\Delta = b^2 - 4ac$, then a^4 equals

(a) $b\Delta$ (b) $b^2\Delta$

(c) $bc\Delta$ (d) $c\Delta$

23. Let $a < 0, a \neq -2$. The equation $x^2 + a|x| + 1 = 0$ has

(a) no real roots

(b) at least two real roots

(c) exactly four real roots or no real roots

(d) none of these

24. Let $f(x) = x^3 + 3x^2 + 6x + 2009$ and

$$g(x) = \frac{1}{x - f(1)} + \frac{2}{x - f(2)} + \frac{3}{x - f(3)}.$$

The number of real solutions of $g(x) = 0$ is

(a) 0 (b) 1

(c) 2 (d) infinite

25. Let f be a continuous function defined on $[-2009, 2009]$ such that $f(x)$ is irrational for each $x \in [-2009, 2009]$ and $f(0) = 2 + \sqrt{3} + \sqrt{5}$. The equation $f(2009)x^2 + 2f(0)x + f(2009) = 0$ has

(a) only rational roots

(b) only irrational roots

(c) one rational and one irrational root

(d) imaginary roots

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26. Let $\beta, \alpha, \beta \in \mathbf{R}$, $\beta \neq 0$ and $\alpha + i\beta$ be a root of $x^3 + qx + r = 0$ where $q, r \in \mathbf{R}$. A cubic equation with real coefficients one of whose roots is α, is
    (a) $x^3 - qx + r = 0$
    (b) $x^3 + 2qx + r = 0$
    (c) $x^3 + 4qx - 8r = 0$
    (d) none of these
27. Suppose a, b are complex numbers and $x^3 + ax + b = 0$ has a pair of complex conjugate roots, then
    (a) a is real and b is imaginary
    (b) a is imaginary and b is real
    (c) both a and b are real
    (d) none of these
28. Suppose that the quadratic equations $3x^2 + px + 1 = 0$ and $2x^2 + qx + 1 = 0$ have a common root then the value of $5pq - 2p^2 - 3q^2$ equals
    (a) 0 (b) 1
    (c) -1 (d) 2
29. The number of solution of
$$10^{2/x} + 25^{1/x} = \left(\frac{65}{8}\right)(50^{1/x}) \text{ is}$$
    (a) 0 (b) 2
    (c) 4 (d) infinite
30. The equation $\left(\frac{10}{9}\right)^x = -3x^2 + 2x - \frac{9}{11}$ has
    (a) no solution
    (b) exactly one solution
    (c) exactly two solutions
    (d) none of these
31. Let $a, b, c \in \mathbf{R}$ and $a \neq 0$ be such that $(a + c)^2 < b^2$, then the quadratic equation $ax^2 + bx + c = 0$ has
    (a) imaginary roots
    (b) real roots
    (c) two real roots lying between $(-1, 1)$
    (d) none of these
32. The integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integeral roots are
    (a) $-1, 3$ (b) 2, 3
    (c) 12, 8 (d) $-8, -12$
33. The number of real solution of $4^{x+1.5} + 9^{x+0.5} = (10)(6^x)$ is
    (a) zero (b) one
    (c) two (d) infinite
34. The number of real solution of $2^{\sin^2 x} + 5(2^{\cos^2 x}) = 7$ is
    (a) zero (b) 1
    (c) finitely many (d) infinitely many
35. Let $a, b, c \in \mathbf{R}$ be such that $a + b + c < 0$, $a - b + c < 0$ and $c > 0$. If α and β are roots of the equation $ax^2 + bx + c = 0$, then value of $[\alpha] + [\beta]$ is
    (a) 2 (b) 1
    (c) -1 (d) 0
36. If roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ lie in the interval $(-2, 4)$, then
    (a) $-1 < m < 3$ (b) $1 < m < 5$
    (c) $1 < m < 3$ (d) $-1 < m < 5$
37. The value of $x = \sqrt{8 + 2\sqrt{8 + 2\sqrt{8 + 2\sqrt{8 + \cdots}}}}$ is
    (a) 10 (b) 6
    (c) 8 (d) 4
38. The number of solutions of the equation
$$\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}\,x + 4 \text{ is}$$
    (a) 1 (b) 2
    (c) 0 (d) infinite
39. If $0 < a < b < c < d$, then the quadratic equation $f(x) = ax^2 + \{1 - a(b + c)\}x + abc - d = 0$ has
    (a) real and distinct roots out of which one lies between c and d
    (b) real and distinct roots out of which one lies between a and b
    (c) real and distinct roots out of which one lies between b and c
    (d) non-real roots
40. If α, β are the roots of the equation $ax^2 + bx + c = 0$ such that $\alpha - \beta = \pi/2$ then the value of the determinant
$$\Delta = \begin{vmatrix} 1 & \cos(\beta - \alpha) & \cos\alpha \\ \cos(\alpha - \beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 2 \end{vmatrix}$$
is
    (a) $a^2 + b^2$ (b) $a^2 b^2$
    (c) $a^2 + b^2 + 1$ (d) 1
41. If x is real, then the least value of the expression
$$\frac{x^2 - 6x + 5}{x^2 + 2x + 2} \text{ is}$$
    (a) -1 (b) $-1/3$
    (c) $-1/2$ (d) none of these.
42. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then
    (a) $\alpha < 0 < \beta < |\alpha|$ (b) $\beta > |\alpha|$
    (c) $|\alpha| + |\beta| < 2$ (d) none of these

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43. If x is real, and $y = \dfrac{x^2+2x+c}{x^2+4x+3}$ takes all real values then
(a) $0 < c < 2$ (b) $0 \le c \le 1$
(c) $-1 < c < 1$ (d) $-3 \le c \le 1$

44. The smallest integer x for which the inequality $\dfrac{x-5}{x^2+5x-14} > 0$ is satisfied is given by
(a) –6 (b) –5
(c) –4 (d) –3

45. If $|a \pm b| > c$ and $a \neq 0$, then the quadratic equation
$a^2x^2 + (b^2 + a^2 - c^2)x + b^2 = 0$
(a) has two real roots
(b) both positive roots
(c) cannot have real roots
(d) none of these

SECTION II

MULTIPLE CORRECT CHOICE TYPE

46. If $abc < 0$, then the equation
$ax^2 + 2(b + c - a)x + bc = 0$, has
(a) real roots
(b) one positive and one negative root
(c) both positive roots
(d) both negative roots

47. The equation $1 + \sqrt{1-\sqrt{x^4-x^2}} = x$, $x \in \mathbf{R}$, has
(a) only positive solutions
(b) exactly one solution
(c) at least two solutions
(d) infinite number of solutions.

48. The equation
$x + \sqrt{x} + \sqrt{x+2} + \sqrt{x^2+2x} = 3$ has
(a) no solution
(b) at least one solution
(c) only positive solutions
(d) infinite number of solutions.

49. Let a, b, c be the sides of an obtuse angled triangle with $\angle C > \pi/2$. The equation
$a^2x^2 + (b^2 + a^2 - c^2)x + b^2 = 0$
has
(a) two positive roots
(b) one positive and one negative root
(c) two real roots
(d) two imaginary roots.

50. The equation
$\sqrt{x-1} + \sqrt{x+3} + 2\sqrt{(x-1)(x+3)} = 4 - 2x$
has
(a) exactly one integral solution
(b) all its solutions in [1, 2]
(c) sum of all the solutions is 1
(d) no solution

51. The equation $x^3 + 1 = 2(2x - 1)^{1/3}$ has
(a) one rational solution
(b) two irrational solutions
(c) sum of roots equal to zero
(d) product of the roots as – 1.

52. Let $P(x)$ be a polynomial such that $P(x+2) = x^2 + 3x - 2$. Which one of the followings are true statement?
(a) There are two polynomials satisfying the above relation.
(b) $P(x)$ must be of degree 2
(c) $P(x) = 0$ has two irrational roots
(d) Product of the roots of $P(x) = 0$ is – 4.

53. Let $P(x)$ be a polynomial such that
$P(x^2 + 2) = x^{17} - 3x^5 + x^3 - 3$, then
(a) $P(x) = 0$ has at least 34 roots.
(b) $P(x) = 0$ has exactly 17 roots
(c) Sum of the roots of $P(x) = 0$ is – 193.
(d) none of these.

54. Let $P(x) = a_0 + a_1x + \ldots + a_nx^n$, $a_n \neq 0$ be such that $P(x^2) = P(x)^2$, then
(a) $a_0 = 1$
(b) $a_0 = a_1 = \ldots = a_{n-1} = 0$
(c) $a_n = 1$
(d) $a_n = -1$

55. Let p, q be non-zero, real numbers and x_1, x_2, x_3 be the roots of $x^3 + px + q = 0$ such that
$x_3 = \dfrac{1}{x_1} + \dfrac{1}{x_2}$, then
(a) q is a root of $x^2 + px + q = 0$
(b) $q < 0$
(c) $p < 0$ (d) $x_1 + x_2 < 0$

56. Let $n \in \mathbf{N}$, $n > 1$ and $\alpha \in \mathbf{R}$ be such that $\sin \alpha \neq 0$, and let $p(x) = x^n \sin \alpha - x \sin n\alpha + \sin (n-1)\alpha$ then
(a) $\cos \alpha \pm i \sin \alpha$ are zeros of $p(x)$
(b) $p(x)$ is divisible by $x^2 - 2x \cos \alpha + 1$
(c) $p(x)$ is divisible by $x^2 + 2x \cos \alpha + 1$
(d) sum of all the roots of $p(x) = 0$ is 0.

57. Let $n \in \mathbf{N}$, $n > 1$ and $\alpha \in \mathbf{R}$ be such that $\sin \alpha \neq 0$ and let
$p(x) = (x \sin \alpha + \cos \alpha)^n - x \sin (n\alpha) - \cos (n\alpha)$, then

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(a) $\pm i$ are zeros of $p(x)$

(b) $p(x)$ is divisible by $x^2 + 1$

(c) ± 1 are zeros of $p(x)$

(d) $p(x)$ is divisible by $x^2 - 1$

58. Let $a, b, c > 0$ and for all $p, q \in \mathbf{R}$ with $p + q = 1$ $pa^2 + qb^2 > pqc^2$, then

(a) $a + b > c$ (b) $a + c > b$

(c) $b + c > a$ (d) $a + b = c$

59. Let a, b be two distinct roots of

$x^4 + x^3 - 1 = 0$, and $p(x) = x^6 + x^4 + x^3 - x^2 - 1$

(a) ab is a root of $p(x) = 0$

(b) $a + b$ is a root of $p(x) = 0$

(c) both $a + b$ and ab are roots of $p(x) = 0$

(d) none of ab, $a + b$ is a root of $p(x) = 0$

60. If $x^2 + px + 1$ is a factor of $ax^3 + bx + c$, then

(a) $ax + c$ is also a factor of $ax^3 + bx + c$

(b) $ap + c = 0$

(c) $a^2 - c^2 = ab$

(d) $(2a - b)^2 = b^2 + 4c^2$

61. Let a, b, c, d and p be integers such that $p \neq 0$. If $x + p$ is a factor of $ax^3 + bx^2 + cx + d$, then

(a) $p|d$ (b) $p^2|(pc - d)$

(c) $d = 0 \Rightarrow p$ is a root of $ax^2 - bx + c = 0$

(d) none of these.

62. The equation $\log_{x^2+6x+8}(\log_{2x^2+2x+3}(x^2 - 2x)) = 0$ has

(a) no solution

(b) exactly one solution

(c) at least one negative solution

(d) at least two solutions.

63. Let $-1 \leq p \leq 1$ and $p(x) = 4x^3 - 3x - p$

(a) $p(x) = 0$ has at least one root in [1/2, 1]

(b) $p(x)$ is strictly increasing on [1/2, 1]

(c) $\alpha = \cos\left\{\frac{1}{3}\cos^{-1}(p)\right\}$ is a root of $p(x) = 0$

(d) $p(x) = 0$ has real roots.

64. Let $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbf{R}$. Suppose $f(x)$ is an integer whenever x is an integer, then which of the following are integers?

(a) $2a$ (b) $a + b$

(c) $a - b$ (d) c

65. Let $ab < 0$, $a + b \neq 0$ and $p(x) = (a + b)x^2 - 4abx + ab(a + b)$.

(a) $p(x) = 0$ has real roots

(b) $p(x) = 0$ has roots of opposite signs

(c) $p(x) > 0 \ \forall \ x$ if $a + b > 0$

(d) $p(x) < 0 \ \forall \ x$ if $a + b < 0$

SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos. 66 to 70

Let α, β, γ be roots of the equation $ax^3 + bx^2 + cx + d = 0$. To obtain the equation whose roots $f(\alpha), f(\beta), f(\gamma)$ where f is a function, we put $y = f(\alpha)$ and obtain $\alpha = f^{-1}(y)$.

Now, we use the fact that α is root of the equation $ax^3 + bx^2 + cx + d = 0$ to obtain the desired equation.

Note Desired equation may require some manipulation after substitution.

For example if α, β, γ are roots of $ax^3 + bx^2 + cx + d = 0$ to find equation whose roots are $1/\alpha, 1/\beta, 1/\gamma$, we put $\frac{1}{\alpha} = y \Rightarrow \alpha = \frac{1}{y}$. As α is a root of $ax^3 + bx^2 + cx + d = 0$, we get

$$\frac{a}{y^3} + \frac{b}{y^2} + \frac{c}{y} + d = 0 \text{ or } a + by + cy^2 + dy^3 = 0$$

This is the desired equation.

The same result holds for all polynomial equations.

66. If α, β are roots of $ax^2 + bx + c = 0$, the roots of $a(x - 1)^2 + b(x - 1)(x - 2) + c(x - 2)^2 = 0$ are

(a) $\frac{2\alpha - 1}{\alpha - 1}, \frac{2\beta - 1}{\beta - 1}$ (b) $\frac{\alpha - 1}{\alpha - 2}, \frac{\beta - 1}{\beta - 2}$

(c) $\frac{\alpha + 1}{\alpha - 1}, \frac{\beta + 1}{\beta - 1}$ (d) $\frac{3\alpha - 1}{\alpha - 1}, \frac{3\beta - 1}{\beta - 1}$

67. If α, β are roots of $px^2 + qx + r = 0$, then the equation whose roots of $\alpha^2 + r/p$ and $\beta^2 + r/p$ is

(a) $p^3x^2 + pq^2x + r = 0$

(b) $px^2 - qx + r^2 - p = 0$

(c) $p^3x^2 - pq^2x + q^2r = 0$

(d) none of these

68. If α, β, γ are the roots of $x^3 - 13x + 11 = 0$ then equation whose roots are $\beta + \gamma, \gamma + \alpha, \alpha + \beta$ is

(a) $x^3 - 13x - 11 + 0$

(b) $x^3 + 13x - 11 = 0$

(c) $x^3 + 13x + 11 = 0$

(d) $x^3 - 13x + 11 = 0$

69. If α, β, γ are the roots of $x^3 + 3px + q = 0$, then the equation whose roots are $(\beta - \gamma)^2$, $(\gamma - \alpha)^2$ and $(\alpha - \beta)^2$ is

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(a) $y^3 - 3(p-q)y + pq = 0$

(b) $y^3 + 3(p-q)y - pq = 0$

(c) $y^3 + 3(p+q)y - pq = 0$

(d) none of these

70. If α, β, γ are roots of $x^3 + 3x + 2 = 0$, then an equation whose roots are $(\alpha-\beta)(\alpha-\gamma), (\beta-\gamma)(\beta-\alpha), (\gamma-\alpha)(\gamma-\beta)$ is

(a) $y^3 - 6y^2 + 216 = 0$

(b) $y^3 + 9y^2 - 216 = 0$

(c) $y^3 + 3y^2 - 128 = 0$

(d) $y^3 + 6y^2 + 184 = 0$

Paragraph for Question Nos. 71 to 75

Let $f(x) = ax^2 + bx + c$. If x_1, x_2, x_3 are three distinct real numbers such that $f(x_1) = f(x_2) = f(x_3) = 0$, then $f(x) \equiv 0$, i.e. $f(x) = 0 \ \forall \ x \in \mathbf{R}$.

71. If f and x_1, x_2, x_3 are as in the above paragraph, then

(a) $a \neq 0, b \neq 0, c = 0$

(b) $a = b \neq 0, c = 0$

(c) $a = b = c = 0$

(d) $a \neq 0, b = c = 0$.

72. If x_1, x_2, x_3 are three distinct real numbers and

$$g(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + \frac{(x-x_3)(x-x_1)}{(x_2-x_3)(x_2-x_3)} + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

then $g(x)$ is identically equal to

(a) 0

(b) 1

(c) $(x-x_1)(x-x_2)(x-x_3)$

(d) none of these.

73. If $g(x) = x_1 \dfrac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + x_2 \dfrac{(x-x_3)(x-x_1)}{(x_2-x_1)(x_2-x_3)} + x_3 \dfrac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$

then $g(x)$ is identically equal to

(a) 0 (b) 1

(c) x

(d) $(x - x_1 - x_2 - x_3)^2 + (x_1^2 + x_2^2 + x_3^2)x - x_1x_2x_3$

74. If $g(x) = x_1(x_1-1)\dfrac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + x_2(x_2-1)\dfrac{(x-x_3)(x-x_1)}{(x_2-x_1)(x_2-x_3)} + x_3(x_3-1)\dfrac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$

then $g(x)$ is identically equal to

(a) x (b) $x^2 - x$

(c) $x^3 - x^2$ (d) $x^3 - x^2 + x + 1$

75. If $g(x) = \dfrac{x_2x_3(x-x_1)^2}{(x_1-x_2)(x_1-x_3)} + \dfrac{x_3x_1(x-x_2)^2}{(x_2-x_3)(x_2-x_1)} + \dfrac{x_1x_2(x-x_3)^2}{(x_3-x_1)(x_3-x_2)}$

then $g(x)$ is identically equal to

(a) 1 (b) x

(c) x^2 (d) $x^2 - x + 1$

Paragraph for Question Nos. 76 to 80

By the Rolle's theorem between any two real zeros of a polynomial $p(x)$ lies a zero of $p'(x)$.

76. If $p, q, r \in \mathbf{R}$ and $x^3 + px^2 + qx + r = 0$ has three real roots, then

(a) $p^2 < 3q$ (b) $p^2 \leq 3q$

(c) $p^2 \geq 3q$ (d) none of these.

77. The number of real roots of $x^3 - 3x + 1 = 0$ lying in the interval $(0, 1)$ is

(a) 0 (b) 1

(c) 2 (d) 3

78. Let $f(x)$ be a polynomial such that α, β with $\alpha < \beta$ are two consecutive zeros of $f'(x)$. The number of zeros of $f(x)$ lying in the interval $[\alpha, \beta]$ is

(a) at most 1 (b) at most 2

(c) at least 1 (d) at least 2

79. If $a, b, c \in \mathbf{R}$ and $3b^2 - 8ac < 0$, then the equation $f(x) = ax^4 + bx^3 + cx^2 + 2009x + 2010 = 0$ has

(a) all real roots

(b) all imaginary roots

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(c) cannot have all real roots
(d) two real and two imaginary roots

80. If $a + b + c = 0$, then equation $3ax^2 + 2bx + c = 0$ has
(a) at least one root in [0, 1]
(b) no real roots
(c) one positive and one negative root
(d) none of these.

Paragraph for Question Nos. 81 to 85

Descrate's Rule of Signs

Let $p(x)$ be a polynomial.

(a) *Positive roots* The number of positive roots of the equation $p(x) = 0$ cannot exceed the number of changes of sign from positive to negative and negative to positive in terms of the coefficients of $p(x)$.

(b) *Negative roots* The number of negative roots $p(x) = 0$ cannot exceed the number of changes in the signs of $p(-x)$.

81. The least number of imaginary roots of the equation $P(x) = x^9 - x^5 + x^4 + x^2 + 1 = 0$ is
(a) 2 (b) 8
(c) 4 (d) 6

82. The equation $p(x) = x^5 + x^3 - 8x - 5 = 0$ has
(a) exactly 3 real roots
(b) exactly one real root
(c) at least one root in (0, 1)
(d) none of these.

83. The equation $p(x) = x^5 - x + 16 = 0$ has
(a) no imaginary roots
(b) exactly two imaginary roots
(c) exactly four imaginary roots
(d) at least one root in $(-\infty, -2)$

84. The number of imaginary roots of $p(x) = x^9 - x^5 + x^4 + x^2 + 1 = 0$ is
(a) 2 (b) 4
(c) 6 (d) 8

85. If $a, b, c > 0$, then the equation $p(x) = x^4 - ax^3 - bx - c = 0$ has
(a) two negative roots
(b) all the four imaginary roots
(c) one positive, one negative and two imaginary roots
(d) two positive and two negative roots.

SECTION IV

INTEGER ANSWER TYPE

86. If $a, b, c > 0$, $a^2 = bc$ and $a + b + c = abc$, find the least possible value of a^2.

87. If $x_1 + x_2 + x_3 = 1$, $x_2x_3 + x_3x_1 + x_1x_2 = 1$ and $x_1x_2x_3 = 1$, find value of $|x_1| + |x_2| + |x_3|$.

88. Find the number of real roots of
$P(x) = x^{100} - 2x^{99} + 3x^{98} - \ldots - 100x + 101 = 0$.

89. Find the number of positive integral solutions
$x^4 - y^4 = 3879108$

90. Find the number of solutions of
$$2\cos^2\left(\frac{x}{2}\right) = (0.2)^x + (0.2)^{-x}$$

91. Let $f(x) = ax^2 + 2bx + c$, $a, b, c \in \mathbf{R}$.
If $f(x)$ takes real values for real values of x and imaginary value for imaginary values of x, find the value of a.

92. Find the real value of $-a$ for which the roots x_1, x_2, x_3 of $x^3 - 6x^2 + ax - a = 0$ satisfy $(x_1 - 3)^3 + (x_2 - 3)^3 + (x_3 - 3)^3 = 0$

93. Sum of all the real values of x for which
$$\sqrt{2-x} + \frac{4}{\sqrt{2-x}+3} = 2$$

94. Let a and b be two integers such that $10a + b = 5$ and $p(x) = x^2 + ax + b$. Find the integer n such that $p(10)\,p(11) = p(n)$.

95. If x_1 and x_2 are roots of the equation $x^2 + x + c = 0$, $c \neq 0$, and
$$\frac{x_1^3}{2+x_2} + \frac{x_2^3}{2+x_1} = -\frac{1}{2},$$
find value of $-4c$.

96. If x_1, x_2, x_3 are such that
$x_1 + x_2 + x_3 = 2$
$x_1^2 + x_2^2 + x_3^2 = 6$
$x_1^3 + x_2^3 + x_3^3 = 8$
find value of $-(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)$.

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SECTION V

MATRIX MATCH TYPE

97. Let $a < b < c < d$. Match with the column in which equation has root in

(a) $(x-a)(x-c) + 2009(x-b)(x-d) = 0$	(p) (c, d)
(b) $2009(x-a)(x-c) - 2008(x-b)(x-d) = 0$	(q) (a, b)
(c) $(x-a)(x-b)(x-c) + (x-d) = 0$	(r) $(-\infty, a)$
(d) $(x-a)(x-b) + (x-c) = 0$	(s) (b, c)

98. Let α, β, γ be the roots of $x^3 + px + q = 0$, then value of

(a) $\frac{1}{\beta+\gamma}+\frac{1}{\gamma+\alpha}+\frac{1}{\alpha+\beta}$	(p) $\frac{p^2}{q^2}$
(b) $\alpha^4 + \beta^4 + \gamma^4$	(q) $-(p^3+3q^2)/q^3$
(c) $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$	(r) $2p^2$
(d) $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$	(s) p/q

99. Let $a, b, c \in \mathbf{R}$ be such $b^2 - ac \geq 0$. The equation $ax^2 + 2bx + c = 0$ has

(a) two positive roots	(p) $a, -c$ are of the same sign
(b) two negative roots	(q) $a, -b, c$ are of the same sign
(c) one positive and one negative root	(r) a, b, c are of the same sign
(d) equal roots	(r) a, b, c are in G.P.

100. Consider the quadratic equations

$$x^2 + 2(a+1)x + 9a - 5 = 0$$

(a) $a < 1$	(b) real roots
(b) $a > 6$	(q) one positive and one negative root
(c) $1 < a < 6$	(r) imaginary roots
(d) $2 < a < 5$	(s) both negative roots.

101. The number of real roots of

(a) $x^4 + x^2 + 2 = \log_{0.75}(2009)$	(p) 0
(b) $(14)^{2/x} + 49^{1/x} = (4.25)(98)^{1/x}$	(q) 2
(c) $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x-2\sqrt{x-1}} = 1$	(r) infinite
(d) $\frac{1}{5-4\log_9 x} + \frac{4}{1+\log_9 x} = 3$	(s) 1

102. α, β are roots of $ax^2 + bx + c = 0$. Match equation with its roots.

(a) $ax^2 + (2a+b)x + a + b + c = 0$	(p) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$
(b) $(a-b+c)x^2 + (b-2c)x + c = 0$	(q) $\frac{\alpha-1}{\alpha+1}, \frac{\beta-1}{\beta+1}$
(c) $(a-b+c)x^2 + 2(a-c)x + a + b + c = 0$	(r) $\alpha - 1, \beta - 1$
(d) $ax^2 + (b-2a)x + a - b + c = 0$	(r) $\alpha + 1, \beta + 1$

SECTION VI

REASONING TYPE

103. If $P(x)$, $Q(x)$ and $R(x)$ are polynomials in x such that

$$P(x^3) + xQ(x^3) = (x^2 + x + 1)R(x)$$

Statement-1: 1 is a root of $P(x) = 0$
Statement-2: 1 is a root of $Q(x) = 0$

104. Let p, q, r, s be four real numbers such that $pr = 2(q+s)$.
Statement-1: At least one of $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ has real roots.
Statement-2: $x^2 + px + q = 0$ has real roots if and only if $x^2 + rx + s = 0$ has imaginary roots.

105. **Statement-1:** If $a, b, c \in \mathbf{R}$ and $3c < 4a + 4b$ and the equation $ax^2 + 2bx - 3c = 0$ has no real roots, then $c < 0$.
Statement-2: If $a, b, c \in \mathbf{R}$, the equation $ax^2 + bx + c = 0$, has no real roots if $b^2 - 4ac < 0$.

106. **Statement-1:** Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. If f is onto, then $\alpha \in [2, 14]$.

Statement-2: If the expression $y = \frac{\alpha x^2 + 3x - 4}{\alpha + 3x - 4x^2}$, $x \in \mathbf{R}$ assumes all real values, then $\alpha \in [1, 7]$.

107. Let $a, b, p, q \in \mathbf{R}$, $p \neq q$, $b \neq 0$ and the ratio of the roots of equation $ax^2 + bx + b = 0$ is $p : q$.

Statement-1: $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{b}{a}} = 0$
Statement-2: $a(p+q) + bpq = 0$

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LEVEL 2

SECTION I

SINGLE CORRECT CHOICE TYPE

1. The number of cubic polynomials $p(x)$ with integral coefficients such that for three distinct integers a, b and c such that $p(a) = b$, $p(b) = c$ and $p(c) = a$ is
   (a) 0 (b) 1
   (c) dependent on a, b, c (d) infinite
2. Let $p(x)$ be a polynomial with integral coefficients, such that $p(0)$ and $p(1)$ are both odd integers, then the number of integral roots of $p(x) = 0$, is
   (a) 0 (b) 1
   (c) 2 (d) none of these
3. If $n \in \mathbf{N}$, then number of real roots of
   $$1 + x + \frac{1}{2!}x^2 + \ldots + \frac{1}{(2n)!}x^{2n} = 0$$
   is
   (a) n (b) 2
   (c) 0 (d) none of these
4. The number of real values of x satisfying
   $$64^{1/x} - 2^{(3x+3)/x} + 12 = 0$$
   is
   (a) 0 (b) 1
   (c) 2 (d) infinite
5. Let S denote the sum of all the roots of the roots of the equation $27(16^x) - 30(36^x) + 8(81^x) = 0$ then S equals
   (a) 2/3 (b) 1
   (c) 3/2 (d) 5/2
6. The number of real values of a for which
   $$4^x - a(2^x) - a + 3 = 0$$
   has at least one solution is
   (a) 0 (b) 1
   (c) 2 (d) infinite
7. The solution set of $\frac{1}{5-4\log_4 x} + \frac{4}{1+\log_4 x} = 3$ is
   (a) {4, 8} (b) {2, 8}
   (c) {2, 4} (d) none of these
8. The solution set of $3x^{\log_3 4} + 4^{\log_3 x} = 64$ is
   (a) {9} (b) {27}
   (c) {81} (d) {16}
9. Let S denote the set of all values of a for which
   $$4^{x^2} + 2(2a+1)2^{x^2} + 4a^2 - 3 > 0$$
   is satisfied for all real values of x, then S equals
   (a) $(-\infty, -1)$ (b) $\left(\sqrt{3}/2, \infty\right)$
   (c) $\left(-1, \sqrt{3}/2\right)$ (d) none of these
10. The number of integral solutions of
    $$x\left(\frac{5-x}{x+1}\right)\left(x+\frac{5-x}{x+1}\right) = 6$$
    is
    (a) 0 (b) 1
    (c) 2 (d) 3
11. The number of subsets of the real solutions of
    $$x^4 = \frac{11x-6}{6x-11}$$
    is
    (a) 2 (b) 8
    (c) 16 (d) 32
12. Let S denote the set of integral values of x for which
    $$x^2 + 1 < 3x - x^2 - 3$$
    then the number of elements in S contains is
    (a) 0 (b) 1
    (c) 3 (d) infinite
13. The equation $(2.4)^x = (2.6)^x - 1$ has
    (a) no solution
    (b) exactly one solution
    (c) at least two solutions
    (d) infinite number of solutions
14. Let S denote the set of all irrational solutions of
    $$(3^x)\,(8^{x/(x+1)}) = 36$$
    then the number of elements in S is
    (a) 0 (b) 1
    (c) 2 (d) infinite
15. If $a, b, c \in \mathbf{R}$ and $(b-1)^2 < ac$, then the system of equations
    $$ax_1^2 + 2bx_1 + c = 2x_2$$
    $$ax_2^2 + 2bx_2 + c = 2x_3$$
    .....
    .....
    .....
    $$ax_{n-1}^2 + 2bx_{n-1} + c = 2x_n$$
    $$ax_n^2 + 2bx_n + c = 2x_1$$
    has
    (a) no solution
    (b) exactly one solution
    (c) infinite number of solutions
    (d) exactly n solutions.

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16. Let $x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > 0$, and

$A = x_1 + x_2 + ... + x_6$

$B = x_1x_3 + x_1x_5 + x_3x_5 + x_2x_4 + x_2x_6 + x_4x_6$

$C = x_1x_3x_5 + x_2x_4x_6$,

then the equation $2x^3 - Ax^2 + Bx - C = 0$ has
(a) exactly one real root
(b) three positive roots
(c) three negative roots
(d) none of these

17. Let A, B, C be the angles of triangle and $\tan A$, $\tan B$, $\tan C$ be the three roots of $x^4 - ax^3 + bx^2 - cx + d = 0$, then the fourth root of this equation is also a root of

(a) $x^2 + bx + d = 0$ (b) $x^2 - ax - c = 0$
(c) $x^2 - bx + d = 0$ (d) none of these

18. The equation $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$ have a common root and $x^2 - cx + d = 0$ has equal roots, then $b + d$ equals

(a) ac (b) $ac/2$
(c) $2ac$ (d) $\sqrt{ac}$

19. If $4\,\text{cosec}^2\alpha\, x^2 + 2x + \left(\beta^2 - \beta + \frac{1}{2}\right) = 0$ has real roots, then $\sin^2\alpha + (\cos(\sin^{-1}\beta))^2$ equals

(a) 3/4 (b) 7/4
(c) 11/4 (d) 15/4

20. Suppose $a_1, a_2, ..., a_{n-1} \geq 0$ and the equation $P(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + 1$ has n real roots, then least value of $P(2)$ is

(a) 3^n (b) 2^n
(c) 1 (d) none of these.

SECTION II

MULTIPLE CORRECT ANSWERS TYPE

21. The equation $x^{(3\log_3 x + 7/2)(\log_3 x)^2} = \sqrt{3}$ has
(a) a rational root
(b) two irrational roots
(c) sum of all the roots of the equation 7/2
(d) product of the roots of the equation is less than 1.

22. If $a, b, c \in \mathbf{R}$ and $(a + c)^2 < b^2$ then the equation $ax^2 + bx + c = 0$ has
(a) real roots
(b) at least one root in (– 1, 1)
(c) one of the two roots lie in the region given by $|x| > 1$
(d) both the roots lie in (– 1, 1)

23. If S denotes the solution set of the equation

$$\left(x + \frac{1}{x}\right)^2 = 4\cos^2(\pi x)\sin^2\left(\frac{\pi x}{2}\right), \text{ then}$$

(a) S is finite
(b) $P(S)$, the power set of S has exactly four elements
(c) S is infinite
(d) $S \subseteq \mathbf{I}$, the set of integers.

24. If $\alpha, \beta, \gamma \in \mathbf{C}$ are such that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 6$, $\alpha^3 + \beta^3 + \gamma^3 = 8$, then

(a) $\alpha\beta\gamma = -2$
(b) $\alpha^4 + \beta^4 + \gamma^4 = 18$
(c) $\alpha^5 + \beta^5 + \gamma^5 = 32$
(d) $\alpha^5 + \beta^5 + \gamma^5 = 34$

25. Suppose $\pi/2 < \theta < \pi$ and $b > 0$. If α and β $(\alpha < \beta)$ are the roots of $(\sin\theta)x^2 + bx + \cos\theta = 0$, then
(a) $0 < \alpha < \beta$
(b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$
(d) $\alpha < 0 < \alpha < |\beta|$

26. Let $a_1, a_2, a_3, ...$ be a sequence of non-zero real numbers which are in A.P. For $k \in \mathbf{N}$, let

$f_k(x) = a_k x^2 + 2a_{k+1}x + a_{k+2}$,

(a) $f_k(x) = 0$ has real roots for each $k \in N$.
(b) Each of $f_k(x) = 0$ has one root in common.
(c) non-common roots of $f_1(x) = 0$, $f_2(x) = 0$, $f_3(x) = 0$, ... form an A.P.
(d) none of these.

27. Let $p(x)$ be a polynomial with real coefficients and a, b be two distinct real numbers. The remainder when $p(x)$ is divided by $(x - a)(x - b)$ is
(a) 0 if a and b are toots of $p(x) = 0$

(b) $\frac{p(a)}{a-b}(x - b)$ if b is a root of $p(x) = 0$

(c) $\frac{1}{a-b}[x(p(a) - p(b)) + ap(b) - bp(a)]$

(d) $\frac{1}{a-b}[x(p(b) - p(a)) + bp(a) - ap(b)]$

28. If p, q, r, s are positive real numbers and the equation $x^4 - px^3 + qx^2 - rx + s = 0$ has four positive real roots, then

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(a) $p^4 \geq 4s$ (b) $r^4 \geq 4s^3$

(c) $pr \geq 16s$ (d) $q^2 \geq 36s$

29. If α, β are the roots of $\lambda(x^2 + x) + x + 5 = 0$ and λ_1, λ_2 are the two values of λ for which α, β are connected by the relation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = 4$, then the value of $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}$ is

(a) 254 (b) 482

(c) 780 (d) 782

30. Suppose $a, b, c \in \mathbf{R}$, $a > 0$ be such that $a + b + c \geq 0, a - b + c \geq 0, a \geq c$, and let x_1, x_2 be the roots of $ax^2 + bx + c = 0$, then

(a) $|x_1|, |x_2| > 0$ (b) $|x_1|, |x_2| \leq 1$

(c) $|b| \leq 2a$ (d) $|x_1 x_2| \leq 1$

SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos. 31 to 35

A root α of the polynomial equation $p(x) = 0$ is said to be of multiplicity $n \in \mathbf{N}$ if

$$(x - a)^n \text{ divdies } p(x)$$

but $(x - a)^{n+1}$ does not divide $p(x)$.

For instance 2 is a root of multiplicity 3 of

$$(x - 2)^3 (x - 5)(x^2 + 1) = 0$$

If α is a root of multiplicity n of $p(x) = 0$, then α is a root of multiplicity $n - r$ of $p^r(x) = 0$.

31. Values of the parameter a for which the equation $p(x) = 3x^4 + 4x^3 - 6x^2 - 12x + a = 0$ has a multiple root.

(a) 11, – 5 (b) 11, – 7

(c) – 11, 5 (d) – 11, 7

32. If $b \neq 0$, the relation between a and b for which the equation $x^5 + px^4 + q = 0$ has a root of multiplicity two is

(a) $3125q = 256p^5$ (b) $3125q + 256p^5 = 0$

(c) $256p^5 = 1025q$ (d) none of these

33. If the equation $ax^3 + 3bx^2 + 3cx + d = 0$ has a root of multiplicity 2, then this root is given by

(a) $\frac{b}{ac + b^2}$ (b) $\frac{bc - ad}{2(ac - b^2)}$

(c) $\frac{bc}{ac - b^2}$ (d) $\frac{d}{ab - c^2}$

34. If $q \neq 0$ and the equation $x^n - npx + (n - 1)q = 0$ has a root of multiplicity 2, then

(a) $p^n = q^{n-1}$ (b) $p^n + q^{n-1} = 0$

(c) $np + q^n = 0$ (d) $p^n + nq^{n-1} = 0$

35. If α is a root of $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ of multiplicity 2, then α must be a root of

(a) $p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$

(b) $p_1 x^{n-1} + 2p_2 x^{n-2} + \ldots + np_n = 0$

(c) $np_1 x^{n-1} + (n - 1) p_2 x^{n-2} + \ldots + p_n = 0$

(d) none of these.

Paragraph for Question Nos. 36 to 40

Let $f(x) = ax^2 + 2bx + c, a \neq 0, b^2 - ac \geq 0$. Both the roots of $f(x) = 0$ will lie in the interval (x_1, x_2) if

$$x_1 < -\frac{b}{a} < x_2$$

and $$a f(x_1) > 0, a f(x_2) > 0$$

If $x_1 = -\infty$, we drop the condition $a f(x_1) > 0$ and in case $x_2 = \infty$, we drop the condition $a f(x_2) > 0$.

36. If the equation $(1 - a^2)x^2 + 2ax - 1 = 0$ has both the roots lying in (0, 1), then

(a) $a > 2$ (b) $-1 < a < 1$

(c) $a < -1$ (d) $1 < a < 2$

37. If $|p| < 1$ and both the roots of $x^2 + 2px + q = 0$ satisfying the inequality $|x| < 1$, then

(a) $1 + p + |q| > 0$ (b) $1 + |p| + q > 0$

(c) $p + q > 1$ (d) $|p + q| < 1$

38. If $a > 0, a + |b| + c > 0$ and $c < a$, then both the roots of the equation $ax^2 + bx + c = 0$ lie in the interval

(a) (0, 1) (b) (– 1, 0)

(c) (– 1, 1) (d) none of these.

39. If both the roots of the equation $x^2 - 2(a - 1)x + 2a + 1 = 0$ are positive, then

(a) $a > 2$ (b) $1 < a < 2$

(c) $a \geq 4$ (d) $1 \leq a \leq 4$

40. If $b^2 - ac > 0, ab < 0, ac > 0$, then the equation $ax^4 + 2bx^2 + c = 0$ has

(a) four real roots

(b) two negative and two imaginary roots

(c) two positive and two imaginary roots

(d) four imaginary roots

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SECTION IV

INTEGER THE ANSWER TYPE

41. Let $a, b \in \mathbf{R}$. If for $|x| \le 1$, $|ax^2 + bx + c| \le 1$, find the maximum possible value of $|2ax + b|$ for $x \in [-1, 1]$.

42. Let $a, b, c \in \mathbf{R}$. If $|ax^2 + bx + c| \le 1$ and $\frac{8}{3}a^2 + 2b^2$ is maximum, find $|a + c|$.

43. Let $f(x) = ax^2 + bx + c$. Given $f(x) = x$ has no real roots, find the number of number of real roots of $f(f(x)) = x$.

44. Let $p(x)$ be a monic polynomial with integral coefficients. Let a, b, c, d be four distinct integers such that $f(a) = f(b) = f(c) = f(d) = 5$. Find the number of integral values of k for which $f(k) = 8$.

45. If x, y and z are three real numbers such that $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$, then it is known that $x, y, z \in [2/3, k]$. Find k.

46. If $a, b\, c$ are distinct, find the sum of the roots of the equation

$$\frac{(x-a)(x+b)}{(x-a)(x-b)} + \frac{(x+a)(x+c)}{(x-b)(x-c)} + \frac{(x+c)(x+a)}{(x-c)(x-a)} = 3$$

47. If three distinct real numbers $a, b, c \in \mathbf{R}$ satisfy

$a(a^2 + p) = b(b^2 + p) + c(c^2 + p)$

for some $p \in \mathbf{R}$, find the value of $a + b + c$.

48. Find the numbers of values of $x \in \mathbf{R}$ for which $64^{1/x} + 48^{1/x} = 80^{1/x}$.

49. If $a, b, c \in \mathbf{R}$ satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$ find the value of

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^{2007} - \frac{2007}{a^{2007} + b^{2007} + c^{2007}} + \frac{2006}{(a+b+c)^{2007}}$$

50. Find the value of the parameter $a \in \mathbf{R}$ such that the roots α, β, γ of $x^3 + 6x^2 - ax + a = 0$ satisfy the relation $(\alpha+3)^3 + (\beta+3)^3 + (\gamma+3)^3 = 0$

SECTION V

MATRIX MATCH TYPE

51. The value of k for which the equation $x^3 - 3x + k = 0$ has

(a) all three real roots	(p) $\|k\| > 2$
(b) two equal roots	(q) $k = -2, 2$
(c) exactly one real root	(r) $\|k\| < 2$
(d) three equal roots	(s) no value of k

52. Range of the function

(a) $\frac{x^2 - x + 1}{x^2 + x + 1}$	(p) $[-18, 9]$
(b) $\frac{x^2 - 3x + 6}{x - 1}$	(q) $\left[0, \frac{1}{2}\right]$
(c) $\frac{x + \|x\|}{1 + x^2}$	(r) $\left[\frac{1}{3}, 3\right]$
(d) $(3 - x)\left[x + \sqrt{9 - x^2}\right]$	(s) $(-\infty, -5] \cup [3, \infty)$

53. Suppose $a, b, c \in \mathbf{R}$, $abc \ne 0$ and $b^2 - 4ac \ge 0$. Let α, β be the roots of $ax^2 + bx + c = 0$, then roots of

(a) $x^2 + \left(2 - \frac{b^2}{ac}\right)x + 1 = 0$	(p) $\frac{\alpha^2}{\beta^2}, \frac{\beta^2}{\alpha^2}$
(b) $x^2 + \left[2 - \frac{(a+c)^2}{ac}\right]x + 1 = 0$	(q) $\alpha\beta, \frac{1}{\alpha\beta}$
(c) $x^2 + \left[\frac{ab^2}{ac} - \frac{b^4}{a^2c^2} - 2\right]x + 1 = 0$	(r) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$
(d) $(2x + 1)^2 - \frac{2(a+c)^2}{ac}x = 0$	(s) $\frac{\alpha}{2\beta}, \frac{\beta}{2\alpha}$

54. The value of k for which the equation $kx^3 - 3x^2 + 1 = 0$ has

(a) all three real roots	(p) $\|k\| > 2$
(b) two equal roots	(q) $k = -2, 2$
(c) exactly one real roots	(r) $\|k\| < 2$
(d) all three equal roots	(s) no value of k

55. Polynomial $p(x)$ which satisfy

(a) $p(x + 3) = x^2 + 7x + 12$	(p) no such polynomial
(b) $p(x - 1) + 2p(x + 1) = 3x^2 - 7x$	(q) $x^2 + x$
(c) $p(x^2 + 1) = x^5 + x^3 - 1$	(r) $x^2 - 3x$
(d) $p(x^2 + 1) = x^7 - x^5 + 3$	(s) $x^3 - 7x^5$

SECTION VI

REASONING TYPE

56. Let $p, q \in \mathbf{R}$ be such that $p^2 - 4q \ge 0$.
Statement-1: If the quadratic equation $x^2 - px + q = 0$ has positive roots, then the equation can be written as $(ax + b)^2 = x$ for some real numbers a and b.
Statement-2: If the quadratic equation $x^2 - px + q = 0$ has positive roots, then $q > 0$ and $p \ge 2\sqrt{q}$.

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57. **Statement-1:** If m, n are positive integers, $x^m (b^n - c^n) + b^m (c^n - x^n) + c^m (x^n - b^n)$ is divisible by $x^2 - (b + c)x + bc$.
Statement-2: If a, b are different complex numbers and $r(x)$ is the remainder when polynomial $p(x)$ is divided by $(x - a)(x - b)$, then

$$r(x) = \frac{p(a) - p(b)}{a - b}x + \frac{ap(b) - ba(a)}{a - b}$$

58. **Statement-1:** Let $a, b, c\ p, q \in \mathbf{R}$ and suppose $ax^2 + bx + c = 0$ has no real roots. Let α, $\bar{\alpha}$ be the roots of $ax^2 + bx + c = 0$ and

$$\frac{px + q}{ax^2 + bx + c} = \frac{A}{x - \alpha} + \frac{B}{x - \bar{\alpha}}$$

then $B = \bar{A}$.
Statement-2: If $a, b, c \in \mathbf{R}$, and $ax^2 + bx + c = 0$ has no real roots, then $b^2 - 4ac < 0$.

PAST YEARS' IIT QUESTIONS

1. Find all the integral values of x for which $5x - 1 < (x + 1)^2 < 7x - 3$. [1978]
2. If a, b are the roots of $x^2 + px + 1 = 0$ and c, d are the roots of $x^2 + qx + 1 = 0$, show that $q^2 - p^2 = (a - c)(b - c)(a + d)(b + d)$ [1978]
3. Solve the equation

$$\sqrt{x+1} - \sqrt{x-1} = 1$$ [1978]

4. If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s.
Deduce the condition that the equations have a common root. [1979]
5. Prove that the minimum value of

$$\frac{(a+x)(b+x)}{c+x},\ x > -c \text{ is } \left(\sqrt{a-c} + \sqrt{b-c}\right)^2$$

[1979]

6. If l, m, n are real, $l \neq m$, then the roots of the equation
$(l - m)x^2 - 5(l + m)x - 2(l - m) = 0$
are
(a) real and equal (b) complex
(c) real and unequal (d) none of these.
[1979]
7. The entire graph of the equation $y = x^2 + kx - x + 9$ is strictly above the x-axis, if and only if,
(a) $k < 7$ (b) $-5 < k < 7$
(c) $k > -5$ (d) none of these
[1979]
8. If x, y and z are real and different and $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$, then u is always
(a) non negative (b) zero
(c) non positive (d) none of these.
[1979]
9. Let $a > 0$, $b > 0$ and $c > 0$. The both the roots of the equation $ax^2 + bx + c = 0$
(a) are real and negative
(b) have negative real parts
(c) none of these [1980]
10. The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2}$
$0 < x \leq \pi/2$ has
(a) no real solution
(b) one real solution
(c) more than one real solution [1980]
11. Both the roots of the equation
$(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0$
are always
(a) positive (b) negative
(c) real (d) none of these.
[1980]
12. If $x^2 + px + 1$ is a factor of $ax^3 + bx + c$, then
(a) $a^2 + c^2 = -ab$ (b) $a^2 - c^2 = -ab$
(c) $a^2 - c^2 = ab$ (d) none of these
[1980]
13. Let x and y be two real variables such that $x > 0$ and $xy = 1$. Find the minimum value of $x + y$. [1981]
14. Let a, b, c be non-zero real numbers such that

$$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\,dx = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)\,dx$$

Then the quadratic equation $ax^2 + bx + c = 0$ has
(a) no root in (0, 2)
(b) at least one root in (0, 2)
(c) a double root in (0, 2)
(d) two imaginary roots. [1981]

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15. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is
    (a) 4 (b) 1
    (c) 3 (d) 2
16. Show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real solution. [1982]
17. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where $p, q \in \mathbf{R}$, the $(p, q) = (-, -)$. [1982]
18. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is
    (i) $-4 < x \leq 0$ (ii) $0 < x < 1$
    (iii) $-100 < x < 100$ (iv) $-\infty < x < \infty$
    [1982]
19. True/False.
    The equation $2x^2 + 3x + 1 = 0$ has an irrational root. [1983]
20. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has
    (a) at least one root in [0, 1]
    (b) one root in [2, 3] and the other in [– 2, – 1]
    (c) imaginary roots
    (d) none of these [1983]
21. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the n-th power of the other, then show that
    $(ac^n)^{1/(n+1)} + (a^n c)^{1/(n+1)} + b = 0$ [1983]
22. Find all real values of x which satisfying
    $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$. [1983]
23. The equation $x - \dfrac{2}{x-1} = 1 - \dfrac{2}{x-1}$ has
    (a) no root (b) one root
    (c) two equal roots (d) infinitely many roots
    [1984]
24. For real x, the function $\dfrac{(x-a)(x-b)}{x-c}$ will assume all real values provided
    (a) $a > b > c$ (b) $a < b < c$
    (c) $a > c > b$ (d) $a < c < b$ [1984]
25. True/False
    If $a < b < c < d$, then the roots of the equation
    $(x - a)(x - c) + 2(x - b)(x - d) = 0$
    are real and distinct. [1984]
26. If the product of the roots of the equation $x^2 - 3kx + 2e^{\log k} - 1 = 0$ is 7, then the roots are real for $k =$ ——. [1984]
27. If a, b, c are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\dfrac{d}{a}, \dfrac{e}{b}, \dfrac{f}{c}$ are in
    (a) A.P. (b) G.P.
    (c) H.P. (d) none of these
    [1985]
28. True/False
    If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then $P(x)\,Q(x) = 0$ has at least two real roots. [1985]
29. Solve for x
    $\left(5+2\sqrt{6}\right)^{x^2-3} + \left(5-2\sqrt{6}\right)^{x^2-3} = 10$ [1985]
30. For $a \leq 0$, determine all the real roots of the equation
    $x^2 - 2a|x - a| - 3a^2 = 0$ [1986]
31. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$, $(a \neq b)$ have a common root, then the numerical value of $a + b$ is —— [1986]
32. If S is the set of all real x such that $\dfrac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains
    (a) $(-\infty, -3/2)$ (b) $(-3/2, -1/4)$
    (c) $(-1/4, 1/2)$ (d) $(1/2, 3)$
    (e) none of these. [1986]
33. Find the set of all x for which $\dfrac{2x}{2x^2+5x+2} > \dfrac{1}{x+1}$. [1987]
34. Solve $|x^2 + 4x + 3| + 2x + 5 = 0$ [1988]
35. If α and β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always
    (a) two real roots
    (b) two positive roots
    (c) two negative roots
    (d) one positive and one negative root. [1989]
36. Let a, b, c be real numbers, $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies
    (a) $\gamma = \dfrac{\alpha+\beta}{2}$ (b) $\gamma = \alpha + \dfrac{\beta}{2}$
    (c) $\gamma = \alpha$ (d) $\alpha < \gamma < \beta$ [1989]
37. The number of real solutions of $\sin(e^x) = 5^x + 5^{-x}$ is
    (a) 0 (b) 1
    (c) 2 (d) infinitely many
    [1990]

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38. Let $f(x)$ be a quadratic expression which is positive for all real x. If $g(x) = f(x) + f'(x) + f''(x)$, than for any real x,
(a) $g(x) < 0$ (b) $g(x) > 0$
(c) $g(x) = 0$ (d) $g(x) \geq 0$ [1990]

39. Show that value of $\frac{\tan x}{\tan 3x}$, whenever defined lies between 1/3 and 3. [1992]

40. Let $\{x\}$ and $[x]$ denote the fractional and integral part of a real number x respectively. Solve
$4\{x\} = x + [x]$ [1994]

41. If p, q, r are positive and are in A.P., the roots of quadratic equation $px^2 + qx + r = 0$ and all real for
(a) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (b) $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$
(c) all p and r (d) no p and r [1995]

42. If p and q are roots of $x^2 + px + q = 0$. Then
(a) $p = 1$ (b) $p = 1$ or 0
(c) $p = -2$ (d) $p = -2$ or 0 [1995]

43. Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$, $\beta > 1$, then show that
$1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$ [1995]

44. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + bx + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie. [1996]

45. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has
(a) no solution (b) one solution
(c) two solutions (d) more than two solutions [1997]

46. Find the set of all solutions of the equation
$2^{|y|} - |2^{y-1} - 1| = 2^{y-1} + 1$ [1997]

47. The sum of all the real roots of the equation
$|x-2|^2 + |x-2| - 2 = 0$ is ——. [1997]

48. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then $A =$ _____ and B = _____. [1997]

49. Prove that the values of the function $\frac{\sin x \cos 3x}{\sin 3x \cos x}$ do not lie between $\frac{1}{3}$ and 3 for any real x. [1997]

50. Let $f(x) = Ax^2 + Bx + C$, where A, B, C are real numbers. Prove that if $f(x)$ is an integer whenever x is an integer, then $2A$, $A + B$ and C are all integers. Conversely, prove that if $2A$, $A + B$ and C are all integers, then $f(x)$ is an integer whenever x is an integer. [1998]

51. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real less than 3, then
(a) $a < 2$ (b) $2 \leq a \leq 3$
(c) $3 < a \leq 4$ (d) $a > 4$ [1999]

52. The harmonic mean of the roots of the equation
$(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0$ is
(a) 2 (b) 4
(c) 6 (d) 8 [1999]

53. In a triangle PQR, $\angle R = \pi/2$. If $\tan(P/2)$ and $\tan(Q/2)$ are the roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$, then
(a) $a + b = c$ (b) $b + c = a$
(c) $a + c = b$ (d) $b = c$ [1999]

54. If α and β $(\alpha < \beta)$, are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then
(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$ [2000]

55. If $b > a$, then the equation $(x-a)(x-b) - 1 = 0$, has
(a) both roots in $[a, b]$
(b) both roots in $(-\infty, a)$
(c) both roots in $(b, +\infty)$
(d) one root in $(-\infty, a)$ and other in $(b, +\infty)$ [2000]

56. For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the roots is square of the other, then p is equal to
(a) 1/3 (b) 1
(c) 3 (d) 2/3 [2000]

57. If α, β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$ and $\alpha + \delta$, $\beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$, $(A \neq 0)$ for some constant δ, then prove that
$\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$ [2000]

58. Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If α, β, γ, δ are in G.P., then the integral values of p and q respectively, are
(a) $-2, -32$ (b) $-2, 3$
(c) $-6, 3$ (d) $-6, -32$ [2001]

59. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is
(a) $[0, 1]$ (b) $\left(0, \frac{1}{2}\right]$

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(c) $\left[\frac{1}{2}, 1\right]$ (d) (0, 1] [2001]

60. Let a, b, c be real numbers with $a \neq 0$, and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α and β. [2001]

61. Let $-1 \leq p \leq 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval [1/2, 1] and identify it. [2001]

62. The set of all real numbers x for which $x^2 - |x + 2| + x > 0$, is

(a) $(-\infty, -2) \cup (2, \infty)$

(b) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(c) $(-\infty, -1) \cup (1, \infty)$

(d) $(\sqrt{2}, \infty)$ [2002]

63. If $\alpha \in \left(0, \frac{\pi}{2}\right)$ then the expression $\sqrt{x^2 + x} + \frac{\tan^2 \alpha}{\sqrt{x^2 + x}}$ is always greater than or equal to

(a) $2 \tan \alpha$ (b) 2

(c) 1 (d) sec2 α [2003]

64. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that min $f(x) >$ max $g(x)$, then the relation between b and c, is

(a) no relation (b) $0 < c < b/2$

(c) $|c| < |b|\sqrt{2}$ (d) $|c| > |b|\sqrt{2}$ [2003]

65. If $a, b, \in \mathbf{R}$, then find all the values of a for which the quadratic equation $x^2 + (a - b)x - a - b + 1 = 0$ has unequal roots for all $b \in \mathbf{R}$. [2003]

66. If one root of the equation $x^2 + px + q = 0$ is square of the other, then p and q satisfy the relation

(a) $p^3 + q(3p - 1) + q^2 = 0$

(b) $p^3 - q(3p + 1) + q^2 = 0$

(c) $p^3 - q(3p - 1) + q^2 = 0$

(d) $p^3 + q(3p + 1) + q^2 = 0$ [2004]

67. If the quadratic expression

$x^2 + 2ax - 3a + 10 > 0 \ \forall \ x \in \mathbf{R}$,

then

(a) $a > 5$ (b) $|a| < 5$

(c) $-5 < a < 2$ (d) $2 < a < 3$ [2004]

68. Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$ and $\Delta = b^2 - 4ac$. If $\alpha + \beta, \alpha^2 + \beta^2, \alpha^3 + \beta^3$ are in G.P., then

(a) $\Delta = 0$ (b) $\Delta \neq 0$

(c) $b\Delta = 0$ (d) $c\Delta = 0$ [2005]

69. Let a, b be the roots of $x^2 - 10cx - 11d = 0$ and c, d be the roots of $x^2 - 10ax - 11b = 0$. Find the value of $a + b + c + d$. [2006]

70. Let a, b, c be the sides of a triangle where $a \neq c$ and $\lambda \in \mathbf{R}$. If the roots of th equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then

(a) $\lambda \in \left(-\infty, \frac{4}{3}\right)$ (b) $\lambda \in \left(\frac{5}{3}, \infty\right)$

(c) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (d) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$ [2006]

71. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is

(a) $\frac{2}{9}(p - q)(2q - p)$

(b) $\frac{2}{9}(q - p)(2p - q)$

(c) $\frac{2}{9}(q - 2p)(2q - p)$

(d) $\frac{2}{9}(2p - q)(2q - p)$. [2007]

MATRIX MATCH TYPE

72. Let $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6}$

Column I	Column II
(a) If $-1 < x < 1$, then $f(x)$ satisfies	(p) $0 < f(x) < 1$
(b) If $1 < x < 2$, then $f(x)$ satisfies	(q) $f(x) < 0$
(c) If $3 < x < 5$, then $f(x)$ satisfies	(r) $f(x) > 0$
(d) If $x > 5$, then $f(x)$ satisfies	(s) $f(x) < 1$ [2007]

REASONING TYPE

73. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, 1/\beta$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$.

Statement-1: $(p^2 - q)(b^2 - ac) \geq 0$

Statement-2: $b \neq pa$ or $c \neq qa$ [2008]

74. The smallest calue of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distianct and have values at least 4, is [2009]

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SECTION VI

103. (b) 104. (c) 105. (a)
106. (a) 107. (d)

LEVEL 2

SECTION I

1. (a) 2. (a) 3. (c)
4. (c) 5. (c) 6. (d)
7. (c) 8. (a) 9. (d)
10. (c) 11. (b) 12. (a)
13. (b) 14. (b) 15. (a)
16. (c) 17. (c) 18. (b)
19. (b) 20. (a)

SECTION II

21. (a), (b), (d) 22. (a), (b), (c)
23. (a), (b), (d) 24. (a), (b), (c)
25. (a), (b), (c) 26. (a), (b)
27. (a), (b), (c) 28. (a), (b), (c), (d)
29. (d) 30. (b), (c), (d)

SECTION III

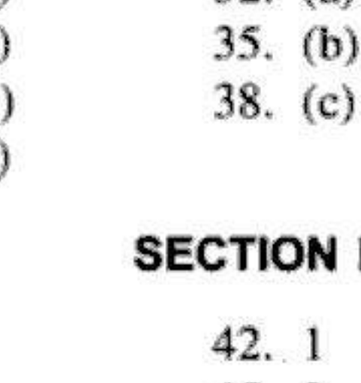
31. (a) 32. (a) 33. (b)
34. (a) 35. (b) 36. (a)
37. (b) 38. (c) 39. (c)
40. (a)

SECTION IV

41. 4 42. 1 43. 0
44. 0 45. 2 46. 5
47. 0 48. 4 49. 0
50. 9

SECTION V

51.



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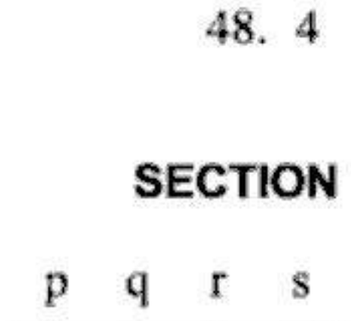

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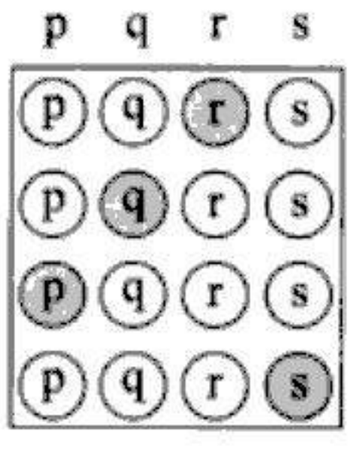

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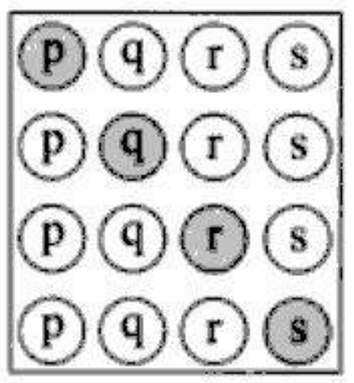

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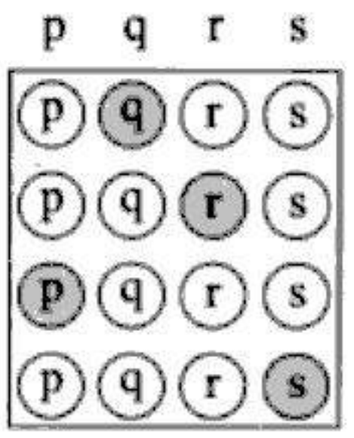

SECTION VI

56. (a) 57. (b) 58. (b)

PAST YEARS' IIT QUESTIONS

1. 3
3. 5/4
4. $(q - s)^2 + p(r - p)(q - s) + q(r - p)^2$;
   $(q - s)^2 + p(r - p)(q - s) + q(r - p)^2 = 0$
6. (c) 7. (b) 8. (a)
9. (b) 10. (a) 11. (c)
12. (c) 13. 2 14. (b)
15. (d) 17. (– 4, 7) 18. (iv)
19. False 20. (a)
22. $-1 \leq x < 1, 2 < x \leq 4$.
23. (a) 24. (c), (d) 25. True
26. 2 27. (a)
28. True 29. $\pm 2, \pm\sqrt{2}$
30. $(1 - \sqrt{2})a, (\sqrt{6} - 1)a$ 31. – 1
32. (a), (d) 33. (– 2, – 1) U (–2/3, – 1/2)
34. $-4, -1 - \sqrt{3}$ 35. (a), (d)
36. (d) 37. (a) 38. (b)
40. 0, 5/3 41. (b) 42. (b)
44. $\beta \in (-\infty, 1/3]$, $\gamma \in [-1/27, \infty)$
45. (a) 46. $y = -1$ or $y \geq 1$
47. 4 48. – 3, 77 51. (a)
52. (b) 53. (a) 54. (b)

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55. (d) 56. (c) 58. (a)
59. (d) 60. $\alpha^2\beta$, $\alpha\beta^2$
61. $\cos\left\{\frac{1}{3}\cos^{-1}(p)\right\}$ 62. (b)
63. (a) 64. (d) 65. $a > 1$
66. (c) 67. (c) 68. (d)
69. 1210 70. (a) 71. (d)
72. 73. (b)

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

74. 2

HINTS AND SOLUTIONS

LEVEL 1

1. $p(0)\,p(1) < 0$, therefore $p(x) = 0$ has at least one root in (0, 1).
2. $x = 0$ is not a root. Divide both the numerators and denominators by x and put $x + 15/x = y$ to obtain

$$\frac{y-13}{y-14} - \frac{y-15}{y-16} = -\frac{1}{12}$$

$$\Rightarrow \quad y = 20,\ 10.$$

But $x + 15/x = 20$ and $x + 15/x = 10$ do not have rational roots.

3. $x = 0$ is not a root. Divide both the numerators and denominators by x and put $x + 3/x = y$ to obtain

$$\frac{4}{y+1} + \frac{5}{y-5} = -\frac{3}{2} \Rightarrow y = -5,\ 3$$

$x + 3/x = -5$ has two irrational roots and $x + 3/x = 3$ has imaginary roots.

4. $(x^2 + x + 4 + x)(x^2 + x + 4 + 2x) = 0$

$$\Rightarrow x = -1 \pm \sqrt{3}\,i\ ,\ \left(-3 \pm \sqrt{7}\,i\right)/2$$

5. The given equation gives
$(x^2 - x + 1)^2 = x^2,\ 5x^2$
6. Use $x^2 + 3x + 2 = (x + 1)(x + 2)$,
and $x^2 - 9x + 20 = (x - 4)(x - 5)$
to obtain
$(x^2 + 3x + 2)(x^2 - 9x + 20)$
$= (x^2 - 3x - 4)(x^2 - 3x - 10)$
Now, put $x^2 - 3x = y$ and rewrite the equation as
$(y + 1)(y - 4)(y - 10) = -30$

Roots of this equation are $5,\ 4 \pm \sqrt{30}$.

7. Write the equation as

$$\sqrt{3(x+1)^2 + 4} + \sqrt{5(x+1)^2 + 9} = 5 - (x+1)^2$$

and observe that LHS > 5 and RHS < 5 for $x \neq -1$.

8. $\dfrac{x^2 - 6\sqrt{x} + 7}{x^2 + 6\sqrt{x} + 7} = \pm 1 \Rightarrow x = 0$ or $x^2 + 7 = 0$.

9. Since $\left|\dfrac{x}{x-1} + x\right| = \dfrac{x^2}{|x-1|}$, we must have x and $\dfrac{x}{x-1}$ are of the same sign i.e. $\dfrac{x^2}{x-1} \geq 0$

$\Rightarrow x = 0$ or $x - 1 > 0$

10. Clearly $-2 < x \leq 2$.
Also, $x = 2$ satisfies the given equation.
For $-2 < x < 2$, we get

$$\sqrt{\frac{2-x}{2+x}} = 1 \text{ or } x = 0.$$

11. For $a \neq 0$, $x = a/2$. For $a = 0$, no solution.
12. Roots are $-a$ and a^2. Now, $a^2 - a < a^3/4$.
Clearly a cannot be negative or 0. Thus,
$a^2 - 4a + 4 > 0 \Rightarrow (a - 2)^2 > 0$
13. As $x^2 - 3x + 4 > 0\ \forall\ x$,
$2x^2 - (a + 3)x + 2 > 0\ \forall\ x$
$\Leftrightarrow (a + 3)^2 < 16 \Leftrightarrow -4 < a + 3 < 4$
$\Leftrightarrow -7 < a < 1$.
14. $\left(\dfrac{1}{x-a} + \dfrac{1}{x-b-c+a}\right) + \left(\dfrac{1}{x-b} + \dfrac{1}{x-c}\right) = 0$

$$\Rightarrow \frac{2x-b-c}{(x-a)(x-b-c+a)} + \frac{2x-b-c}{(x-b)(x-c)} = 0$$

$\Rightarrow x = \dfrac{1}{2}(b + c)$ and $2x^2 - 2(b + c)x + ab + ac + bc - a^2 = 0$

The last equation has real roots.

15. That sign of a cannot be determined may be checked by taking examples. For instance,
$f(x) = x^2 - 4x - 5$ and $f(x) = 1 - x^2$.
16. Find values of k for which both the roots are negative.

$$(k-1)^2 - (k-5) \geq 0,$$

$$k - 1 > 0,\ k - 5 > 0 \quad \Rightarrow \quad k > 5.$$

17. $4\cos^2\alpha - 2\sin\alpha(\sin\alpha + \cos\alpha) = 0$

$$\Rightarrow \quad 1 + 3\cos 2\alpha = \sin 2\alpha$$

For $3\pi/4 < \alpha < \pi$, left side is positive and right side is negative.

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18. $\alpha^2 + \beta^2 = (\sin\phi - 1)^2 + 2\cos^2\phi$
$= 3 - 2\sin\phi - \sin^2\phi$
$= 4 - (\sin\phi + 1)^2$

19. $bx + 28 = 12 - 4x - x^2$
$\Rightarrow x^2 + (4 + b)x + 16 = 0$
For unique solution $(4 + b)^2 - 64 = 0$
$\Rightarrow b = 4, -12$, $b = -12$ is not possible.

20. $2x^2 + 3x + 4 = 0$ has imaginary roots. Thus, the given equations have identical roots.

$\therefore \frac{a}{2} = \frac{b}{3} = \frac{c}{4} = k$ (say)
$\Rightarrow a + b + c = 9k$. For least value take $k = 1$.

21. Let $D_1 = p^2 - 4q$, $D_2 = r^2 - 4s$
Now, $D_1 + D_2 = p^2 + r^2 - 4(q + s)$
$= p^2 + r^2 - 2pr = (p - r)^2 \geq 0$
$\Rightarrow$ At least one of $D_1, D_2 \geq 0$.

22. $\text{cosec}\,\theta + \cot\theta = -b/a$, $\text{cosec}\,\theta\cot\theta = c/a$
As $\text{cosec}^2\theta - \cot^2\theta = 1$, we get
$\text{cosec}\,\theta - \cot\theta = -a/b$.

$\therefore \text{cosec}\,\theta = -\frac{1}{2}\left(\frac{b}{a} + \frac{a}{b}\right)$ and $\cot\theta = \frac{1}{2}\left(\frac{a}{b} - \frac{b}{a}\right)$

Thus, $-\frac{1}{4}\left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right) = \frac{c}{a}$

$\Rightarrow a^4 - b^4 = -4acb^2$

$\Rightarrow a^4 = b^2(b^2 - 4ac) = b^2\Delta$

23. $D = a^2 - 4 > 0$ if $a < -2$ and $D < 0$ if $-2 < a < 0$. If $-2 < a < 0$, the equation has no real roots. If $a < -2$, the equation has exactly four real roots.

24. $f'(x) = 3(x + 1)^2 + 3 > 0\ \forall\ x$.
Thus, $f(x)$ increases on **R**. Let $a = f(1)$, $b = f(2)$ and $c = f(3)$, then $a < b < c$ and
$g(x) = (x - b)(x - c) + 2(x - a)(x - c) + 3(x - a)(x - b)$
As $g(a) > 0$, $g(b) < 0$ and $g(c) > 0$, $g(x) = 0$ has exactly two real solutions.

25. As f is continuous on $[-2009, 2009]$ and f assumes only irrational values, f must be constant. Thus, $f(x) = f(0)\ \forall\ x \in [-2009, 2009]$ and as such the given equation has only rational roots.

26. As $q, r \in \mathbf{R}$, $\alpha - i\beta$ must be a root of $x^3 + qx + r = 0$. Let the third root be γ, then
$\alpha + i\beta + \alpha - i\beta + \gamma = 0$
$\Rightarrow \gamma = -2\alpha$
As γ is a root of $x^3 + qx + r = 0$, $-8\alpha^3 - 2q\alpha + r = 0$
$\therefore$ required equation is $8x^3 + 2qx - r = 0$.

27. Let $\alpha + i\beta$, $\alpha - i\beta$, γ where $\beta \neq 0$ be roots of $x^3 + ax + b = 0$.
As $\alpha + i\beta + \alpha - i\beta + \gamma = 0$, we get γ is real.
$\therefore\ a = (\alpha + i\beta + \alpha - i\beta)(-2\alpha) + (\alpha + i\beta)(\alpha - i\beta)$
$= -3\alpha^2 + \beta^2$ is real.
Also, $b = 2\alpha(\alpha + i\beta)(\alpha - i\beta) = 2\alpha(\alpha^2 + \beta^2)$ is real.

28. Let common root be α, then
$3\alpha^2 + p\alpha + 1 = 0$ and $2\alpha^2 + q\alpha + 1 = 0$
Multiply the second expression by 3/2 and subtract from the first to obtain
$\left(p - \frac{3}{2}q\right)\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{1}{2p - 3q}$
Thus, $3 + p(2p - 3q) + (2p - 3q)^2 = 0$

29. Write the given equation as
$\left(\frac{100}{50}\right)^{1/x} + \left(\frac{25}{50}\right)^{1/x} = \frac{65}{8}$
$\Rightarrow 2^{1/x} = 8, 1/8 \Rightarrow x = -1/3, 1/3$

30. $-3x^2 + 2x - \frac{9}{11} = -3\left[\left(x - \frac{1}{3}\right)^2 + \frac{16}{99}\right] < 0\ \forall\ x$
and LHS $> 0\ \forall\ x$.

31. $(a + c)^2 < b^2 \Rightarrow (a - b + c)(a + b + c) < 0$
Thus, $ax^2 + bx + c = 0$ has at least one root in $(-1, 1)$.

32. $(x - a)(x - 1) = -1$
$\Rightarrow\ x - a = 1, x - 10 = -1$
or $\quad x - a = -1, x - 10 = 1$
$\Rightarrow\ a = 8, 12$.

33. Write the given equation as
$3t^2 - 10t + 8 = 0$ where $t = (3/2)^x$
$\Rightarrow t = 2, 4/3$.

34. Put $2^{\sin^2 x} = t$ to obtain
$t^2 - 7t + 10 = 0 \Rightarrow t = 2, 5$
But $1 \leq t \leq 2$.

35. One root lies between -1 and 0 and the other between 0 and 1.

36. $(x - m)^2 = 1 \Rightarrow x = m \pm 1$
$\therefore\ -2 < m - 1, m + 1 < 4$
$\Rightarrow -1 < m < 3$.

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37. $x = \sqrt{8+2x} \Rightarrow x^2 - 2x - 8 = 0$
$\Rightarrow x = 4, -2$. As $x > 0$, $x = 4$

38. $x^2 - 2\sqrt{3}x + 4 = \left(x - \sqrt{3}\right)^2 + 1$

$\Rightarrow \sin\left(\frac{\pi x}{2\sqrt{3}}\right) = 1$ and $x = \sqrt{3}$.

39. $f(x) = a(x-b)(x-c) + x - d$
$f(b) < 0, f(c) < 0, f(d) > 0$

$\therefore f(x) = 0$ has at least one root in (c, d).

40. $\Delta = \Delta_1^2$

where $\Delta_1 = \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ 1 & 0 & 1 \end{vmatrix}$
$= \sin(\beta - \alpha) = -1$

41. Let $y = \frac{x^2 - 6x + 5}{x^2 + 2x + 2}$

$\Rightarrow (y-1)x^2 + 2(y+3)x + (2y-5) = 0$
As x is real,
$(y+3)^2 - (y-1)(2y-5) \geq 0$
$\Rightarrow y^2 - 13y - 4 \leq 0$

42. $\alpha + \beta = -b < 0, \alpha\beta = c < 0$

As $\alpha < \beta$, we get $\alpha < 0, \beta > 0$ and $\beta < |\alpha|$

43. $x^2(y-1) + 2x(2y-1) + 3y - c = 0$
As x is real, $(2y-1)^2 - (y-1)(3y-c) \geq 0$
$\Rightarrow y^2 + (c-1)y - (c-1) \geq 0 \ \forall \ y \in \mathbf{R}$
$\Rightarrow (c-1)^2 + 4(c-1) \leq 0$
$\Rightarrow -3 \leq y \leq 1$

44. $\frac{x-5}{(x+7)(x-2)} > 0$

$\Rightarrow -7 < x < 2$ or $x > 5$
Thus, the smallest integer for which the inequality is satisfied is -6.

45. $D = (b^2 + a^2 - c^2)^2 - 4a^2b^2$
$= (a^2 + b^2 - 2ab - c^2)(a^2 + b^2 - c^2 + 2ab)$
$= [(a-b)^2 - c^2][(a+b)^2 - c^2] > 0$

46. $D = 4(b + c - a)^2 - 4abc > 0$

$\alpha, \beta = \frac{-(b+c-a) \pm \sqrt{D}}{2a}$, one is positive and other is negative.

47. As LHS ≥ 1, $x \geq 1$.
Write the given equation as
$1 - \sqrt{x^4 - x^2} = (x-1)^2$
$\Rightarrow x^2(2-x)^2 = x^2(x^2-1) \Rightarrow x = 5/4$.
It satisfies the given equation.

48. Put $y = \sqrt{x} + \sqrt{x+2} \Rightarrow y^2 = 2x + 2 + 2\sqrt{x^2+2x}$.
The given equation becomes $y^2/2 - 1 + y = 3$.
As $y > 0$, we get $y = 2$.

$\therefore x + 2 = 4 - 4\sqrt{x} + x \Rightarrow x = 1/4$
It satisfies the given equation.

49. See solution of Question number 45.

50. For equation to be meaningful, $x \geq 1$, $x \geq -3$, $x \leq 2$. Thus, $x \in [1, 2]$

Put $\sqrt{x-1} + \sqrt{x+3} = y$ to obtain $y^2 + y - 6 = 0$
$\Rightarrow y = 2$ as $y \geq 0$. This gives $x = 1$.

51. Put $(2x-1)^{1/3} = y$ or $y^3 = (2x-1)$.
The original equation becomes
$x^3 = 2y - 1$
$\therefore x^3 - y^3 = -2(x-y)$
$\Rightarrow x - y = 0$ or $x^2 + y^2 + xy = -2$
$\Rightarrow x = y$ as other equation is not possible
$\therefore 2x - 1 = x^3$ or $x^3 - 2x + 1 = 0$

$\Rightarrow x = 1, -(1+\sqrt{5})/2, -(1-\sqrt{5})/2$

52. Replace x by $x - 2$ to obtain
$P(x) = x^2 - x - 4$.

53. As $P(x^2 + 2)$ is of even degree, no such polynomial exists.

54. Clearly, $a_n^2 = a_n \Rightarrow a_n = 1$

Let k be the largest non-negative integer $< n$ such that $a_k \neq 0$, then
$P(x) = a_0 + a_1x + \ldots + a_k x^k + a_n x^n$.
Comparing coefficient of x^{n+k} on both sides of $P(x^2) = P(x)^2$, we obtain $0 = 2a_n a_k \Rightarrow a_k = 0$. A contradiction.

55. None of x_1, x_2, x_3, is 0. Also,
$x_1 + x_2 + x_3 = 0, x_1x_2x_3 = -q$.

As $x_3 = \frac{1}{x_1} + \frac{1}{x_2}$, we get $x_1x_2 = -1, x_3 = q$.

Thus $q^3 + pq + q = 0 \Rightarrow q^2 + p + 1 = 0 \Rightarrow p < 0$.

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56. $\sin\alpha(\cos\alpha \pm i\sin\alpha)^n - (\cos\alpha \pm i\sin\alpha)\sin(n\alpha) + \sin(n-1)\alpha = 0$

$\Leftrightarrow \sin\alpha\cos(n\alpha) - \cos\alpha\sin(n\alpha) + \sin(n-1)\alpha$

$\pm i[\sin\alpha\sin(n\alpha) - \sin\alpha\sin(n\alpha)] = 0$

which is true.

From here it also follows that $p(x)$ is divisible by $x^2 - 2x\cos\alpha + 1$.

57. Similarly to Question 56.

58. Putting $q = 1 - p$ in the given inequality we obtain $p^2c^2 + (a^2 - c^2 - b^2)p + b^2 > 0$.

It holds for each $p \in \mathbf{R}$ if and only if

$(a^2 - c^2 - b^2)^2 - 4b^2c^2 < 0$

$\Leftrightarrow (a^2 - c^2 - b^2 - 2bc)(a^2 - c^2 - b^2 + 2bc) < 0$

$\Leftrightarrow (a + b + c)(a - b - c)(a + b - c)(a - c + b) < 0$

59. Let $\alpha = a + b$, $\beta = ab$. Now, $a^3(a + 1) = 1$, and $b^3(b + 1) = 1$ gives

$1 = (ab)^3(a + b + ab + 1) \Rightarrow 1 = \beta^3(\alpha + \beta + 1)$

Also, $a^4 + a^3 = b^4 + b^3$, $a \neq b$

$(a^3 + a^2) + (b^3 + b^2) + ab(a + b) + ab = 0$

$\Rightarrow \quad \frac{1}{a} + \frac{1}{b} + \alpha\beta + \beta = 0$

$\Rightarrow \quad \alpha(1 + \beta^2) + \beta^2 = 0$

$\Rightarrow \quad \left(\frac{1}{\beta^3} - \beta - 1\right)(1 + \beta^2) + \beta^2 = 0$

$\Rightarrow \quad \beta^6 + \beta^4 + \beta^3 - \beta^2 - 1 = 0.$

60. Write

$ax^3 + bx + c = a(x^2 + px + 1)(x + \lambda)$

where λ is some constant.

$\Rightarrow \qquad 0 = a(\lambda + p),\ b = a(p\lambda + 1),\ c = a\lambda.$

Thus, $\quad 0 = c + ap \qquad (\because\ c = a\lambda)$

and $\quad b = a(1 - p^2) \Rightarrow ab = a^2 - c^2$

Also, $(2a - b)^2 = b^2 + 4c^2$

61. $-p$ is a root of $ax^3 + bx^2 + cx + d$

$\Rightarrow \qquad -ap^3 + bp^2 - cp + d = 0 \Rightarrow p|d$

Also, $\quad cp - d = p^2(b - ap) \Rightarrow p^2|(cp - d)$

If $d = 0$, $p(ap^2 - bp + c) = 0$.

62. $x^2 + 6x + 8 > 0,\ x^2 + 6x + 8 \neq 1$

$2x^2 + 2x + 3 > 0,\ 2x^2 + 2x + 3 \neq 1$

$\log_{2x^2+2x+3}(x^2 - 2x) = 1$

$\Rightarrow \quad x^2 + 6x + 8 > 0,\ x^2 + 6x + 7 \neq 0$

$2x^2 + 2x + 3 > 0,\ x^2 + x + 1 \neq 0$

and $x^2 - 2x = 2x^2 + 2x + 3$

$\Rightarrow \quad x = -1, -3$. But $x = -3$

does not satisfy $x^2 + 6x + 8 > 0$

63. $p'(x) = 12x^2 - 3 = 12(x^2 - 1/4) > 0$ if $1/2 < x < 1$.

Also $p\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{3}{2} - p \leq 0$

and $p(1) = 4 - 3 - p \geq 0$

Thus $p(x)$ has at least one root in $[1/2, 1]$

As $p(x)$ is strictly increasing on $[1/2, 1]$, $p(x)$ has exactly one root in $[1/2, 1]$. Let this root be α.

Put $\alpha = \cos\theta$. As $1/2 \leq \alpha \leq 1$, $0 \leq \theta \leq \pi/3$

Now, $0 = p(\alpha) = \cos(3\theta) - p$

$\Rightarrow \theta = \frac{1}{3}\cos^{-1}(p).$

64. $f(0)$, $f(1)$, $f(-1)$ are all integers

$\Rightarrow c,\ a + b + c,\ a - b + c$ are integers.

$\Rightarrow c,\ a + b,\ a - b$ are integers

$\Rightarrow c,\ a + b,\ a - b,\ 2a$ are integers.

65. $D = 16a^2b^2 - ab(a + b)^2 > 0$ as $ab < 0$ and product of roots < 0.

66. $a\left(\frac{x-1}{x-2}\right)^2 + b\left(\frac{x-1}{x-2}\right) + c = 0$

$\Rightarrow \frac{x-1}{x-2} = \alpha, \beta \Rightarrow x = \frac{2\alpha - 1}{\alpha - 1}, \frac{2\beta - 1}{\beta - 1}.$

67. Put $y = \alpha^2 + \frac{r}{p}$ or $\alpha = \sqrt{y - \frac{r}{p}}$.

$\therefore p\left(y - \frac{r}{p}\right) + q\sqrt{y - \frac{r}{p}} + r = 0$

$\Rightarrow p^3y^2 - pq^2y + q^2r = 0.$

68. As $\alpha + \beta + \gamma = 0$, $\beta + \gamma = -\alpha$.

The desired equation can be obtained by changing x to $-x$.

69. $y = (\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\beta\gamma = \alpha^2 + 4q/\alpha$

$= 3\left(\frac{q}{\alpha} - p\right)$

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89. Both x and y must be odd or both must be even. In both the cases LHS is divisible by 8 but the RHS is not divisible by 8.

90. For $x \neq 0$, $(0.2)^x + (0.2)^{-x} > 2$.
For $x = 0$, LHS = 2 = RHS.

91. If $a \neq 0, f(x) = a\left\{\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right\}$.

For $x = -\frac{b}{2a} + i$, $f(x)$ is real.

A contradiction. Thus, $a = 0$.

92. Let $S_k = x_1^k + x_2^k + x_3^k$.

$S_1 = 6$, $S_2 = S_1^2 - 2(x_2x_3 + x_3x_1 + x_1x_2) = 36 - 2a$

$S_3 = 6S_2 - aS_1 + 3a = 216 - 21a$.

Now, $0 = (x_1 - 3)^3 + (x_2 - 3)^2 + (x_3 - 3)^3$

$= S_3 - 9S_2 + 27S_1 - 243$

$= -27 - 3a$

$\Rightarrow -a = 9$.

93. Put $\sqrt{2-x} + 3 = t$ to obtain

$t - 3 + \frac{4}{t} = 2 \Rightarrow t^2 - 5t + 4 = 0$

$\Rightarrow t = 4, 1$ As $t \geq 3$, $\sqrt{2-x} = 1$

$\Rightarrow x = 1$.

94. Let $p(x) = (x - \alpha)(x - \beta)$

$p(10)\,p(11) = (10 - \alpha)(11 - \beta)(11 - \alpha)(10 - \beta)$

$= [110 - 10(\alpha + \beta) - \alpha + \alpha\beta]$

$[110 - 10(\alpha + \beta) - \beta + \alpha\beta]$

$= (110 + 10a + b - \alpha)(110 + 10a + b - \beta)$

$= (115 - \alpha)(115 - \beta) = p(115)$

95. Given relation implies

$4(x_1^3 + x_2^3) + 2(x_1^4 + x_2^4) + 4 + 2(x_1 + x_2) + x_1x_2 = 0$

But $x_1 + x_2 = -1$, $x_1x_2 = c$

gives $4c^2 + 5c = 0$. As $c \neq 0$, $-4c = 5$.

96. $2(x_2x_3 + x_3x_1 + x_1x_2) = (x_1 + x_2 + x_3)^2 -$

$(x_1^2 + x_2^2 + x_3^2) = -2$

$x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$

$= (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_2x_3 - x_3x_1 - x_1x_2)$

$= (2)(6 + 1) = 14$

$\Rightarrow x_1x_2x_3 = -2$.

Equation whose roots are x_1, x_2, x_3 is

$x^3 - 2x^2 - x + 2 = 0 \Rightarrow x = 2, 1, -1$.

Thus, $-(x_2 - x_3)(x_3 - x_1)(x_1 - x_2) = 6$.

97. (a) $f(a) > 0, f(b) < 0$

(b) $f(x) \to \infty$ as $x \to -\infty$ and $f(a) < 0$

(c) $f(c) < 0, f(d) > 0$

(d) $f(b) < 0, f(c) > 0$

98. (a) Use $\alpha + \beta + \gamma = 0$

(b) For large values of x.

$$\frac{1}{x-\alpha} = \frac{1}{x}\left(1-\frac{\alpha}{x}\right)$$

$$= \frac{1}{x}\left(1+\frac{\alpha}{x}+\frac{\alpha^2}{x^2}+\frac{\alpha^3}{x^3}+\frac{\alpha^4}{x^4}+----\right)$$

Thus, $\alpha^4 + \beta^4 + \gamma^4$

$= $ coefficient of $\frac{1}{x^5}$ in the expansion of $\frac{f'(x)}{f(x)}$.

In the present case

$f(x) = 3x^2 + p$

Therefore

$$\frac{f'(x)}{f(x)} = \frac{3x^2+p}{x^3+px+q} = \frac{3}{x}-\frac{2p}{x^3}-\frac{3q}{x^4}+\frac{2p^2}{x^5}+---$$

$\therefore \alpha^4 + \beta^4 + \gamma^4 = 2p^2$

(c) and (d) $\frac{1}{x-\alpha} = -\frac{1}{\alpha}\left(1-\frac{x}{\alpha}\right)^{-1}$

$$= -\frac{1}{\alpha}\left(1+\frac{x}{\alpha}+\frac{x^2}{\alpha^2}+----\right)$$

Thus, $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$

$= $ Coefficient of x^2 in the expansion of $-\frac{f'(x)}{f(x)}$ (1)

But

$$\frac{-f'(x)}{f(x)} = \frac{-p}{q}+\frac{p^2}{q^2}x-\left(\frac{p^3}{q^3}+\frac{3}{q}\right)x^2+----$$

Thus, $\alpha^{-3} + \beta^{-3} + \gamma^{-3} = -\left(\frac{p^3}{q^3}+\frac{3}{q}\right)$

and $\alpha^{-2} + \beta^{-2} + \gamma^{-2} = p^2/q^2$

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99. (a) $-\frac{2b}{a} > 0,\ \frac{c}{a} > 0 \Rightarrow a, -b, c$ are of the same sign

(b) $-\frac{2b}{a} < 0,\ \frac{c}{a} > 0 \Rightarrow a, b, c$ are of the same sign

(c) $\frac{c}{a} < 0 \Rightarrow a, -c$ are of the same sign.

100. Discriminant of the equation is

$D = 4(a+1)^2 - 4(9a-5) = 4\,(a-1)\,(a-6)$

Equation will have real roots if $a \le 1$ or $a \ge 6$ and imaginary roots if $1 < a < 6$.

No such value of a

Roots will positive if $D \ge 0$; $-(a+1) > 0$ and $9a - 5 > 0$.

Roots will be negative if $D \ge 0$; $-(a+1) < 0$ and $9a - 5 > 0$

$\Leftrightarrow \quad a > 6.$

101. (a) Use $\log_{0.75} x < 0$ if $x > 1$.

(b) $(196)^{1/x} + (49)^{1/x} = (4.25)\,(98)^{1/x}$

$$\Rightarrow \quad 2^{1/x} + \left(\frac{1}{2}\right)^{1/x} = \frac{9}{4} \Leftrightarrow 2^{1/x} = 3/2,\ 2/3.$$

Thus, there are two values of x.

(c) Put $\sqrt{x-1} = t$ or $x = 1 + t^2$ to obtain

$$\sqrt{t^2 - 4t + 4} + \sqrt{t^2 - 2t + 1} = 1$$

$$\Rightarrow \quad |t-2| + |t-1| = 1 \Rightarrow 1 \le t \le 2$$

$\therefore$ there are infinite roots.

(d) Put $\log_9 x = t$ to obtain

$$\frac{1}{5-4t} + \frac{4}{1+t} = 3 \quad \Rightarrow \quad t = 1/2,\ 1$$

$$\Rightarrow \quad x = 3,\ 9$$

102. (a) $a(x^2 + 2x + 1) + b(x+1) + c = 0$

$\Rightarrow x + 1 = \alpha, \beta \Rightarrow x = \alpha - 1\ \beta - 1$

(b) $ax^2 - bx\,(x-1) + c(x-1)^2 = 0$

$$\Rightarrow a\left(-\frac{x}{x-1}\right)^2 + b\left(-\frac{x}{x-1}\right) + c = 0$$

$$\Rightarrow \frac{-x}{x-1} = \alpha, \beta \Rightarrow x = \frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$$

(c) $a(x+1)^2 - b(x^2-1) + c(x-1)^2 = 0$

$$\Rightarrow a\left(\frac{x+1}{x-1}\right)^2 - b\left(\frac{x+1}{x-1}\right) + c = 0$$

$$\Rightarrow -\frac{x+1}{x-1} = \alpha, \beta \Rightarrow x = \frac{\alpha-1}{\alpha+1}, \frac{\beta-1}{\beta+1}.$$

(d) $a(x^2 - 2x + 1) + b(x-1) + c = 0$

$\Rightarrow x - 1 = \alpha, \beta \quad \Rightarrow \quad x = 1 + \alpha,\ 1 + \beta$

103. Let $\omega \ne 1$ be a cube root of unity. Putting $x = \omega, \omega^2$ we get

$$P(1) + \omega\, Q(1) = 0$$

and $\quad P(1) + \omega^2\, Q(1) = 0$

$\Rightarrow \quad P(1) = 0$ and $Q(1) = 0$

104. Let $D_1 = p^2 - 4q$ and $D_2 = r^2 - 4s$

Now, $D_1 + D_2 = p^2 + r^2 - 4(q+s)$

$= p^2 + r^2 - 2pr\ [\because\ 2(q+s) = pr]$

$= (p-r)^2 \ge 0$

$\Rightarrow$ at least one of D_1, D_2, is non-negative.

105. $ax^2 + 2bx - 3c = 0$ has no real roots

$\Rightarrow \quad 4b^2 + 12ac < 0 \quad \Rightarrow \quad ac < 0$

If $c > 0$, then $a < 0$

In this case

$$ax^2 + 2bx - 3c < 0\ \forall\ x \in \mathbf{R}$$

$\Rightarrow \quad a(2^2) + 2b(2) - 3c < 0$

$\Rightarrow \quad a + b < \frac{3c}{4}$. A contradiction.

Thus, $c < 0$.

For statement-2, see Theory.

106. $$y = \frac{\alpha x^2 + 3x - 4}{\alpha + 3x - 4x^2}$$

$\Rightarrow (\alpha + 3x - 4x^2)y = \alpha x^2 + 3x - 4$

$\Rightarrow (\alpha + 4y)\,x^2 + 3(1-y)x - (4 + \alpha y) = 0$

As $x \in \mathbf{R}$, $9(1-y)^2 + 4(\alpha + 4y)\,(4 + \alpha y) \ge 0$

$\Rightarrow (9 + 16\alpha)\,y^2 + 2\,(2\alpha^2 + 23)y + 16\alpha + 9 \ge 0$

This is true for each $y \in \mathbf{R}$.

$\Rightarrow 9 + 16\alpha > 0$ and $(2\alpha^2 + 23)^2 - (16\alpha + 9)^2 \le 0$

$\Rightarrow (2\alpha^2 + 23 + 16\alpha + 9)\,(2\alpha^2 + 23 - 16\alpha - 9) \le 0$

$\Rightarrow 9 + 16\alpha > 0$ and $4(\alpha+4)^2\,(\alpha-1)\,(\alpha-7) \le 0$

$\Rightarrow \alpha \in [1, 7]$

The function $f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ will be onto if the expression $\frac{(\alpha/2)x^2 + 3x - 4}{(\alpha/2) + 3x - 4x^2}$ takes all real values.

$\Rightarrow \alpha/2 \in [1, 7]$ or $\alpha \in [2, 14]$.

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107. Expression $\sqrt{\frac{p}{q}}, \sqrt{\frac{q}{p}}, \sqrt{\frac{b}{a}}$ have meanings if $p/q \geq 0$ and $b/a \geq 0$.

But $\sqrt{\frac{p}{q}}, \sqrt{\frac{q}{p}}, \sqrt{\frac{b}{a}} = 0 \Rightarrow \frac{p}{q} = 0, \frac{q}{p} = 0, \frac{b}{a} = 0$

This is not possible.

$$p\alpha + q\alpha = \frac{-b}{a} \Rightarrow \alpha = -\frac{b}{a(p+q)}$$

Now, use $p\alpha$ is a root of $ax^2 + bx + c = 0$.

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Note that if $r \neq 1$, then $S_n = \dfrac{a_{n+1} - a_1}{r-1}$

If $-1 < x < 1$, then

$$\lim_{n \to \infty} x^n = 0.$$

Therefore, the sum of the infinite G.P.

$$1 + x + x^2 + \cdots = \frac{1}{1-x}, \ (-1 < x < 1)$$

Thus $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$ (1)

Differentiating both the sides w.r.t. x, we get

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots (-1 < x < 1) \quad (2)$$

Differentiating (1) $(m-1)$ times w.r.t. x, we get for $-1 < x < 1$

$$(m-1)!\,(1-x)^{-m} = (m-1)! + m!x + \frac{(m+1)!}{2!}x^2 + \frac{(m+2)!}{3!}x^3 + \cdots$$

$$\Rightarrow \quad (1-x)^{-m} = 1 + {}^{m}C_1\, x + {}^{m+1}C_2\, x^2 + {}^{m+2}C_3\, x^3 + \cdots \quad (3)$$

for $-1 < x < 1$

Also, if n is any real number, then

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots (-1 < x < 1)$$

If $-1 < r < 1$, then the sum of the infinite G.P.

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$

Some Facts about G.P.

1. If each term of a G.P. is multiplied (divided) by a fixed non-zero constant, then the resulting sequence is also a G.P.
2. If $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ are two geometric progressions, then the sequence $a_1b_1, a_2b_2, a_3b_3, \cdots$ is also a G.P.
3. If $a_1, a_2, a_3, \ldots$ are in G.P., then $\dfrac{a_n}{a_m} = r^{n-m}$ where r is the common ratio of the G.P.
4. The *geometric mean* G of two positive numbers a and b is given by $\sqrt{ab}$. It is to be noted that a, G and b are in G.P. If $a_1, a_2, \cdots, a_n$ are n positive numbers, then their geometric mean is given by
$$G = (a_1\ a_2 \cdots a_n)^{1/n}$$
5. If we have to take three terms in G.P., we take them as a/r, a, ar and four terms as a/r^3, a/r, ar, ar^3.
6. If $a_1, a_2, a_3, \cdots$ is a G.P. $(a_i > 0\ \forall\ i)$, then $\log a_1$, $\log a_2$, $\log a_3$, $\cdots$ is an A.P. In this case the converse also holds.
Suppose $a_1, a_2, a_3, \cdots$ is a G.P. Let $a_i = AR^{i-1}$, where A is the first term and R is the common ratio of the G.P. Then
$$\log a_i = \log A + (i-1)\log R$$
This shows that $\log a_1$, $\log a_2$, $\log a_3$, $\cdots$ is an A.P. whose first term is $\log A$ and the common difference is $\log R$. Conversely, assume that $\log a_1, \log a_2, \log a_3, \cdots$ is an A.P. Suppose that $\log a_i = a + (i-1)d$, where a is the first term and d is the common difference of the A.P. Then
$$a_i = \alpha^{a+(i-1)d} = \alpha^a(\alpha^d)^{i-1}$$
where α is the base of the logarithm. This shows that $a_1, a_2, a_3, \cdots$ is a G.P., with the first term α^a and common ratio α^d.

3.4 ARITHMETICO-GEOMETRIC SEQUENCE

Suppose $a_1, a_2, \cdots, a_n, \cdots$ is an A.P., and $b_1, b_2, \cdots, b_n, \cdots$ is a G.P. Then the sequence $a_1b_1, a_2b_2, \cdots, a_nb_n, \cdots$ is said to be an *arithmetico-geometric sequence*. An arithmetico-geometric sequence is of the form

$$ab, (a+d)br, (a+2d)br^2, (a+3d)br^3, \cdots$$

Sum to *n* term of an Arithemtico-Geometric series.

Let $S_n = ab + (a+d)br + (a+2d)br^2 + \cdots + [a+(n-2)d]br^{n-2} + [a+(n-1)d]br^{n-1}$ (1)

We multiply each term by r and write the first term below the second, the second below the third and so on.

$$rS_n = abr + (a+d)br^2 + \cdots + [a+(n-3)d]br^{n-2} + [a+(n-2)d]br^{n-1} + [a+(n-1)d]br^n$$

Subtracting, we get

$$(1-r)S_n = ab + dbr + dbr^2 + \cdots + dbr^{n-2} + dbr^{n-1} - [a+(n-1)d]br^n$$

$$= ab + \frac{dbr(1-r^{n-1})}{1-r} - [a+(n-1)d]br^n \quad (r \neq 1)$$

$$\Rightarrow \quad S_n = \frac{ab}{1-r} + \frac{dbr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]br^n}{1-r} \quad (r \neq 1)$$

If $-1 < r < 1$, then

$$\lim_{n \to \infty} r^n = 0 \quad \text{and} \quad \lim_{n \to \infty} nr^n = 0$$

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Therefore, the sum of an infinite number of terms of the sequence in (1) is

$$S = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2} \quad (-1 < r < 1)$$

3.5 HARMONIC PROGRESSION

The sequence $a_1, a_2, \cdots, a_n, \cdots$ where $a_i \neq 0$ for each i is said to be in harmonic progression (H.P.) if the sequence $1/a_1, 1/a_2, \cdots, 1/a_n, \cdots$ is in A.P. Note that a_n, the nth term of the H.P., is given by

$$a_n = \frac{1}{a+(n-1)d} \text{ where } a = \frac{1}{a_1} \text{ and } d = \frac{1}{a_2} - \frac{1}{a_1}.$$

Some Facts About H.P.

1. If a and b are two non-zero numbers, then the *harmonic mean* of a and b is a number H such that the sequence a, H, b is an H.P. We have

$$\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \quad \text{or} \quad H = \frac{2ab}{a+b}$$

If $a_1, a_2, \cdots, a_n$ are n non-zero numbers, then the harmonic mean H of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

2. The n numbers $H_1, H_2, \cdots, H_n$ are said to be *harmonic means between* a and b if $a, H_1, H_2, \cdots, H_n, b$ are in H.P., that is, if $1/a, 1/H_1, 1/H_2, \cdots, 1/H_n, 1/b$ are in A.P. Let d be the common difference of this A.P. Then

$$\frac{1}{b} = \frac{1}{a} + (n+2-1)d$$

$$\Rightarrow \quad d = \frac{1}{n+1}\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{a-b}{(n+1)ab}$$

Thus

$$\frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{(n+1)ab},$$

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{(n+1)ab}, \cdots, \frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}$$

From here, we can get the values of $H_1, H_2, \cdots, H_n$.

3. If a and b are two positive real numbers and A, G, H are arithmetic mean, geometric mean and harmonic between a and b respectively, then $A \geq G \geq H$ and the equality holds if and only if $a = b$.

4. If $a_1, a_2, \ldots, a_n$ are n positive real numbers,

and $$A = \frac{1}{n}(a_1 + a_2 + \ldots + a_n),$$

$$C_n = (a_1\, a_2 \ldots a_n)^{1/n}$$

and $$H = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right),$$

then $A \geq G \geq H$.

3.6 SUMMATION OF SOME SERIES OF NATURAL NUMBERS

1. $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$

2. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

3. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\sum_{k=1}^{n} k\right)^2$

$$= (1 + 2 + 3 + \ldots + n)^2$$

$$= \left[\frac{1}{2}n(n+1)\right]^2 = \frac{1}{4}n^2(n+1)^2.$$

4. To find $\sum_{k=1}^{n} k^4$ and $\sum_{k=1}^{n} k^5$ we can use the identities:

$$k^5 - (k-1)^5 = 5k^4 - 10k^3 + 10k^2 - 5k + 1$$

and $k^6 - (k-1)^6 = 6k^5 - 15k^4 + 20k^3 - 15k^2 + 6k - 1$

The first one gives

$$\sum_{k=1}^{n}\{k^5 - (k-1)^5\} = 5\sum_{k=1}^{n} k^4 - 10\sum_{k=1}^{n} k^3 + 10\sum_{k=1}^{n} k^2 - 5\sum_{k=1}^{n} k + n$$

$$\Rightarrow \quad n^5 + 10\left\{\frac{1}{4}n^2(n+1)^2\right\} - 10\left\{\frac{1}{6}n(n+1)(2n+1)\right\} + 5\left\{\frac{1}{2}n(n+1)\right\} - n$$

$$= 5\sum_{k=1}^{n} k^4.$$

Similarly we can use 2nd identity to obtain sum of $\sum_{k=1}^{n} k^5$.

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3.7 SUM OF THE PRODUCTS OF TWO TERMS OF A SEQUENCE

To obtain the sum $\sum_{i<j} a_i a_j$, we use the identity

$$2\sum_{i<j} a_i a_j = (a_1 + a_2 + \ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2)$$

3.8 METHOD OF DIFFERENCE FOR SUMMATION OF SERIES

If possible express rth term as difference of two terms as follows:

$$t_r = f(r) - f(r \pm 1).$$

Now, it is easier to obtain

$$\sum_{r=1}^{n} t_r$$

as explained below.

For instance if the given series is $1.3.5 + 3.5.7 + 5.7.9 + \ldots$

then $t_r = (2r-1)(2r+1)(2r+3)$

and $t_{r+1} = (2r+1)(2r+3)(2r+5)$

$\Rightarrow (2r+5)t_r = (2r-1)t_{r+1}$

$\Rightarrow (2r-3)t_r + 8t_r = (2r-1)t_{r+1}$

$\Rightarrow 8t_r = f(r+1) - f(r)$

where $f(r) = (2r-3)t_r$

$= (2r-3)(2r-1)(2r+1)(2r+3)$

$= (4r^2-9)(4r^2-1)$

This gives

$$8\sum_{r=1}^{n} t_r = \sum_{r=1}^{n} [f(r+1) - f(r)] = f(n+1) - f(1)$$

$$\Rightarrow 8\sum_{r=1}^{n} t_r = [4(n+1)^2 - 9][4(n+1)^2 - 1] - (4-9)(4-1)$$

$$= 16(n+1)^4 - 40(n+1)^2 + 9 + 15$$

$$\Rightarrow \sum_{r=1}^{n} t_r = 2(n+1)^4 - 5(n+1)^2 + 3.$$

SOLVED EXAMPLES

SECTION I

SINGLE CORRECT CHOICE

Example 1 Let S denote the set of all real values of x such that

$(x^{2010} + 1)(1 + x^2 + x^4 + \ldots + x^{2008}) = 2010x^{2009}$,

then the number of elements in S is

(a) 0 (b) 1

(c) 2 (d) infinite

Ans. (b)

Solution We can write

$$\left(x + \frac{1}{x^{2009}}\right)(1 + x^2 + x^4 + \ldots + x^{2008}) = 2010$$

$$\Leftrightarrow 2010 = x + x^3 + x^5 + \ldots + x^{2009} + \frac{1}{x^{2009}} + \frac{1}{x^{2007}} + \ldots + \frac{1}{x}$$

$$= \left(x + \frac{1}{x}\right) + \left(x^3 + \frac{1}{x^3}\right) + \ldots + \left(x^{2009} + \frac{1}{x^{2009}}\right)$$

$$\geq \underbrace{2 + 2 + \ldots + 2}_{1005 \text{ times}} = 2 \times 1005$$

$$\Leftrightarrow x + \frac{1}{x} = 2, x^3 + \frac{1}{x^3} = 2, \ldots x^{2009} + \frac{1}{x^{2009}} = 2$$

$$\Leftrightarrow x = 1$$

$\therefore$ S consists of just one element.

Example 2 Let $a_n = \sum_{k=1}^{n} \frac{1}{k(n+1-k)}$, then for $n \geq 2$

(a) $a_{n+1} > a_n$ (b) $a_{n+1} < a_n$

(c) $a_{n+1} = a_n$ (d) $a_{n+1} - a_n = 1/n$

Ans. (b)

Solution We have

$$a_n = \frac{1}{n+1}\sum_{k=1}^{n}\left(\frac{1}{k} + \frac{1}{n+1-k}\right)$$

$$= \frac{2}{n+1}\sum_{k=1}^{n}\frac{1}{k}$$

For $n \geq 2$,

$$\frac{1}{2}(a_n - a_{n+1}) = \frac{1}{n+1}\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{n+2}\sum_{k=1}^{n+1}\frac{1}{k}$$

$$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}\sum_{k=2}^{n}\frac{1}{k} > 0$$

$\Rightarrow a_n > a_{n+1}$

Example 3 Sum of the series

$S = 1^2 - 2^2 + 3^2 - 4^2 + \ldots - 2008^2 + 2009^2$ is

(a) 2019045 (b) 1005004

(c) 2000506 (d) none of these

Ans. (a)

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Solution We can write S as

$$S = (1-2)(1+2) + (3-4)(3+4) + \ldots + (2007-2008)(2007+2008) + 2009^2$$

$$= -[1+2+3+4+\ldots+2008] + 2009^2$$

$$= -\frac{1}{2}(2008)(2009) + 2009^2 = 2019045.$$

Example 4 The value of n for which $704 + \frac{1}{2}(704) + \frac{1}{4}(704) + \ldots$ upto n terms $= 1984 - \frac{1}{2}(1984) + \frac{1}{4}(1984) - \ldots$ upto n terms is

(a) 5 (b) 3 (c) 4 (d) 10

Ans. (a)

Solution According to the given condition

$$(704)(2)\left\{1-\frac{1}{2^n}\right\} = \frac{1984(2)}{3}\left\{1-\frac{(-1)^n}{2^n}\right\}$$

$$\Rightarrow \quad 128 = \frac{2112}{2^n} - 1984\frac{(-1)^n}{2^n}$$

If n is odd, we get $2^n = 32$ or $n = 5$

If n is even, we get $128 = 128/2^n \Rightarrow n = 0$.

Example 5 If a, b, c are three unequal numbers such that a, b, c are in A.P. and $b-a, c-b, a$ are in G.P., then ratio $a : b : c$ is equal

(a) 1 : 2 : 3 (b) 1 : 3 : 4
(c) 2 : 3 : 4 (d) 1 : 2 : 4

Ans. (a)

Solution By the hypothesis, $b - a = c - b$ and $(c-b)^2 = a(b-a)$

$\Rightarrow \quad (b-a)^2 = a(b-a) \Rightarrow b - a = a \quad (\because b \neq a)$

$\Rightarrow \quad b = 2a$ and $c = 3a$

$\therefore \quad a : b : c = 1 : 2 : 3.$

Example 6 If $1, \log_y x, \log_z y, -15\log_x z$ are in A.P., then

(a) $z^3 = x$ (b) $x = y^{-2}$
(c) $z^{-2} = y$ (d) none of these

Ans. (a)

Solution Let d be the common difference. Then

$$\log_y x = 1 + d \Rightarrow x = y^{1+d}$$

$$\log_z y = 1 + 2d \Rightarrow y = z^{1+2d}$$

and $-15\log_x z = 1 + 3d \Rightarrow z = x^{-(1+3d)/15}$

$\therefore \quad x = y^{1+d} = z^{(1+2d)(1+d)}$

$$= x^{-(1+d)(1+2d)(1+3d)/15}$$

$\Rightarrow \quad (1+d)(1+2d)(1+3d) = -15$

$\Rightarrow \quad 6d^3 + 11d^2 + 6d + 16 = 0$

$\Rightarrow \quad (d+2)(6d^2 - d + 8) = 0 \Rightarrow d = -2$

[Note that $6d^2 - d + 8 = 0$ has complex roots]

$\therefore \quad x = y^{1+d} = y^{-1}, \; y = z^{1-4} = z^{-3}$

and $\quad x = (z^{-3})^{-1} = z^3.$

Example 7 If $\log 2$, $\log(2^x - 1)$ and $\log(2^x + 3)$ are in A.P., then x is equal to

(a) $\frac{5}{2}$ (b) $\log_2 5$ (c) $\log_3 2$ (d) $\frac{3}{2}$.

Ans. (b)

Solution $2, 2^x - 1, 2^x + 3$ are in G.P.

$\Rightarrow \quad (2^x - 1)^2 = 2(2^x + 3)$

$\Rightarrow \quad 2^{2x} - 4 \times 2^x - 5 = 0 \Rightarrow (2^x - 5)(2^x + 1) = 0$

As 2^x cannot be negative, we get

$2^x - 5 = 0 \Rightarrow 2^x = 5$ or $x = \log_2 5$.

Example 8 The sum S_n to n terms of the series

$$\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \cdots$$

is equal to

(a) $2^n - n - 1$ (b) $1 - 2^{-n}$
(c) $2^{-n} + n - 1$ (d) $2^n - 1$.

Ans. (c)

Solution Let t_r denote the rth term of the series. Then $t_r = 1 - (1/2^r)$. We have

$$S_n = \left(1-\frac{1}{2}\right) + \left(1-\frac{1}{4}\right) + \left(1-\frac{1}{2^3}\right) + \cdots + \left(1-\frac{1}{2^n}\right)$$

$$= n - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right)$$

$$= n - \frac{(1/2)[1-(1/2^n)]}{1-(1/2)} = n - 1 + 2^{-n}.$$

Example 9 If $1^2 + 2^2 + 3^2 + \ldots + 2003^2 = (2003)(4007)(334)$

and $(1)(2003) + (2)(2002) + (3)(2001) + \ldots + (2003)(1) = (2003)(334)(x)$,

then x equals

(a) 2005 (b) 2004 (c) 2003 (d) 2001

Ans. (a)

Solution Let

$$S = (1)(2003) + (2)(2002) + (3)(2001) + \ldots + (2003)(1)$$

and $\quad K = 1^2 + 2^2 + 3^2 + \ldots + 2003^2$

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Length of a side of S_n = Length of a diagonal of S_{n+1}

$$\Rightarrow \quad a_n = \sqrt{2}\, a_{n+1} \quad \Rightarrow \quad \frac{a_{n+1}}{a_n} = \frac{1}{\sqrt{2}}.$$

Thus, $a_1, a_2, a_2, \ldots$ is a G.P. with first term 10 and common ratio $1/\sqrt{2}$. Therefore,

$$a_n = 10\left(1/\sqrt{2}\right)^{n-1}$$

Also, Area $(S_n) = a_n^2 < 1$

$$\Rightarrow \quad [10\ (1/\sqrt{2})^{n-1}]^2 < 1 \quad \Rightarrow \quad 100 < 2^{n-1}$$

$\Rightarrow$ smallest possible value of n is 8.

Example 15 If a, b, c are in G.P., and $\log a - \log 2b$, $\log 2b - \log 3c$ and $\log 3c - \log a$ are in A.P., then a, b, c are the lengths of the sides of a triangle which is

(a) acute-angled (b) obtuse-angled

(c) right-angled (d) equilateral.

Ans. (b)

Solution We have $b^2 = ac$ and

$$2(\log 2b - \log 3c) = \log a - \log 2b + \log 3c - \log a$$

$$\Rightarrow \quad b^2 = ac \text{ and } 2b = 3c \quad \Rightarrow \quad b = \frac{2a}{3} \text{ and } c = \frac{4a}{9}$$

Since $a + b = 5a/3 > c$, $b + c = 10a/9 > a$ and $c + a = 13a/9 > b$, therefore a, b, c are the sides of a triangle. Also, as a is the greatest side, let us find angle A of ΔABC.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = -\frac{29}{48} < 0.$$

Hence, ΔABC is an obtuse-angled triangle.

Example 16 If $x_1, x_2, \cdots x_n$ are n non-zero real numbers such that $(x_1^2 + x_2^2 + \cdots + x_{n-1}^2)(x_2^2 + x_3^2 + \cdots + x_n^2) \le (x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n)^2$ then $x_1, x_2, \cdots, x_n$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these.

Ans. (b)

Solution We shall make use of the identity

$$(a_1^2 + a_2^2 + \cdots + a_m^2)(b_1^2 + b_2^2 + \cdots + b_m^2) - (a_1 b_1 + a_2 b_2 + \cdots + a_m b_m)^2$$
$$= (a_1 b_2 - a_2 b_1)^2 + (a_1 b_3 - a_3 b_1)^2 + \cdots + (a_{m-1} b_m - a_m b_{m-1})^2$$

Thus, $(x_1^2 + x_2^2 + \cdots + x_{n-1}^2)(x_2^2 + x_3^2 + \cdots + x_n^2) - (x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n)^2 \le 0$

$$\Rightarrow \quad (x_1 x_3 - x_2 x_2)^2 + (x_1 x_4 - x_3 x_3)^2 + \cdots + (x_{n-2} x_n - x_{n-1} x_{n-1})^2 \le 0$$

As $x_1, x_2, \cdots, x_n$ are real, this is possible if and only if

$$x_1 x_3 - x_2^2 = x_2 x_4 - x_3^2 = \cdots = x_{n-2} x_n - x_{n-1}^2 = 0$$

$$\Rightarrow \quad \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \cdots = \frac{x_n}{x_{n-1}}$$

$\Rightarrow$ $x_1, x_2, \cdots, x_n$ are in G.P.

Example 17 If three positive real numbers a, b, c $(c > a)$ are in H.P., then $\log (a + c) + \log (a - 2b + c)$ is equal to

(a) $2 \log (c - b)$ (b) $2 \log (a + c)$

(c) $2 \log (c - a)$ (d) $\log a + \log b + \log c$.

Ans. (c)

Solution a, b, c are in H.P., $b = (2ac)/(a + c)$. We have

$$\log (a + c) + \log (a - 2b + c)$$
$$= \log (a + c) + \log \left[(a + c) - \frac{4ac}{a + c}\right]$$
$$= \log [(a + c)^2 - 4ac] = \log (c - a)^2 = 2 \log (c - a)$$

Example 18 If $a_1, a_2, \cdots, a_n$ are in A.P. with common difference $d \ne 0$, then the sum of the series

$$\sin d\, [\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \cdots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n]$$

is

(a) $\sec a_1 - \sec a_n$ (b) $\operatorname{cosec} a_1 - \operatorname{cosec} a_n$

(c) $\cot a_1 - \cot a_n$ (d) $\tan a_1 - \tan a_n$.

Ans. (c)

Solution We have

$$d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}.$$

Thus,

$$\sin d\, [\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \cdots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n]$$
$$= \frac{\sin (a_2 - a_1)}{\sin a_1 \sin a_2} + \frac{\sin (a_3 - a_2)}{\sin a_2 \sin a_3} + \cdots + \frac{\sin (a_n - a_{n-1})}{\sin a_{n-1} \sin a_n}$$
$$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + \cdots + (\cot a_{n-1} - \cot a_n)$$
$$= \cot a_1 - \cot a_n.$$

Example 19 Sum to n terms of the series

$$1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \cdots$$

is

(a) $8n^3 + 12n^2 - 2n - 3$

(b) $n(8n^3 + 11n^2 - n - 3)$

(c) $n(2n^3 + 8n^2 + 7n - 2)$

(d) none of these.

Ans. (c)

Solution The rth term of the series is given by

$$t_r = (2r - 1)(2r + 1)(2r + 3) = 8r^3 + 12r^2 - 2r - 3$$

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Therefore, S_n, the sum to n terms of the series is given by

$$S_n = 8\sum_{r=1}^{n} r^3 + 12\sum_{r=1}^{n} r^2 - 2\sum_{r=1}^{n} r - 3n$$

$$= 8\left[\frac{n(n+1)}{2}\right]^2 + 12\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} - 3n$$

$$= 2n^2(n+1)^2 + 2n(n+1)(2n+1) - n(n+1) - 3n$$

$$= n(2n^3 + 4n^2 + 2n + 4n^2 + 6n + 2 - n - 1 - 3)$$

$$= n(2n^3 + 8n^2 + 7n - 2).$$

Example 20 Sum to n terms of the series

$$\frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{2\cdot3\cdot4\cdot5} + \frac{1}{3\cdot4\cdot5\cdot6} + \cdots$$

is

(a) $\dfrac{n^3}{3(n+1)(n+2)(n+3)}$

(b) $\dfrac{n^2+6n^2-3n}{6(n+2)(n+3)(n+4)}$

(c) $\dfrac{15n^2+7n}{4n(n+1)(n+5)}$

(d) $\dfrac{n^3+6n^2+11n}{18(n+1)(n+2)(n+3)}$.

Ans. (d)

Solution The rth term of the series is given by

$$t_r = \frac{1}{r(r+1)(r+2)(r+3)}$$

Splitting t_r into partial fraction, we get

$$t_r = \frac{1}{6r} - \frac{1}{2(r+1)} + \frac{1}{2(r+2)} - \frac{1}{6(r+3)}$$

$$= \frac{1}{6}\left(\frac{1}{r} - \frac{1}{r+1}\right) - \frac{1}{3}\left(\frac{1}{r+1} - \frac{1}{r+2}\right) + \frac{1}{6}\left(\frac{1}{r+2} - \frac{1}{r+3}\right)$$

$$\Rightarrow \sum_{r=1}^{n} t_r = \frac{1}{6}\left(1 - \frac{1}{n+1}\right) - \frac{1}{3}\left(\frac{1}{2} - \frac{1}{n+2}\right) + \frac{1}{6}\left(\frac{1}{3} - \frac{1}{n+3}\right)$$

$$= \frac{n^3+6n^2+11n}{18(n+1)(n+2)(n+3)}.$$

Example 21 Sum of all the products of the first n positive integers taken two at a time is

(a) $\dfrac{1}{24}(n-1)n(n+1)(3n+2)$

(b) $\dfrac{1}{48}(n-2)(n-1)n^2$

(c) $\dfrac{1}{6}n(n+1)(n+2)(n+5)$

(d) none of these.

Ans. (a)

Solution We have $(1 + 2 + \cdots + n)^2 = \sum k^2 + 2\sum_{r<s} rs$

$$\Rightarrow \left[\frac{n(n+1)}{2}\right]^2 = \frac{n(n+1)(2n+1)}{6} + 2\sum_{r<s} rs$$

$$\Rightarrow \sum_{r<s} rs = \frac{1}{2}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{24}n(n+1)[3n^2 + 3n - 2(2n+1)]$$

$$= \frac{1}{24}n(n+1)(3n^2 - n - 2)$$

$$= \frac{1}{24}(n-1)n(n+1)(3n+2).$$

Example 22 For a positive integer n, let

$a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{(2^n)-1}$. Then

(a) $a(100) < 100$ (b) $a(100) > 100$

(c) $a(200) < 100$ (d) none of these

Ans. (a)

Solution We have

$$a(n) = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \cdots + \frac{1}{7}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{15}\right) + \cdots + \left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}+1} + \cdots + \frac{1}{(2^n)-1}\right)$$

$$< 1 + \frac{2}{2} + \frac{4}{4} + \frac{8}{8} + \cdots + \frac{2^{n-1}}{2^{n-1}} = n$$

Thus, $a(100) < 100$.

Example 23 S_n, the sum to n terms of the series

$(n^2 - 1^2) + 2(n^2 - 2^2) + 3(n^2 - 3^2) + \ldots$

is

(a) $\frac{1}{4}n^2(n^2 - 1)$ (b) $\frac{1}{4}n(n+1)^2$

(c) 0 (d) $2n(n^2 - 1)$

Ans. (a)

Solution $S_n = n^2\left(\sum_{K=1}^{n} k\right) - \sum_{K=1}^{n} k^3$

$$= \frac{1}{2}n^3(n+1) - \frac{1}{4}n^2(n+1)^2$$

$$= \frac{1}{4}n^2(n+1)(2n - n - 1)$$

$$= \frac{1}{4}n^2(n^2 - 1).$$

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Example 24 Let x be the arithmetic mean and y, z be the two geometric means between any two positive numbers. Then value of $\frac{y^3+z^3}{xyz}$ is

(a) 2 (b) 3 (c) 1/2 (d) 3/2

Ans. (a)

Solution Let two positive numbers be a and b. Then $x = (a + b)/2$.

Also, a, y, z, b are in G.P. If r is the common ratio of this G.P., then $b = ar^3 \Rightarrow r = (b/a)^{1/3}$.

We have

$$\frac{y^3+z^3}{xyz} = \frac{a^3r^3+a^3r^6}{x(ar)(ar^2)} = \frac{a(1+r^3)}{x}$$

$$= \frac{a+b}{(a+b)/2} = 2$$

Example 25 Let $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$, then the sum to n terms of the series

$$\frac{1^2}{1^3}+\frac{1^2+2^2}{1^3+2^3}+\frac{1^2+2^2+3^2}{1^3+2^3+3^3}+\cdots$$

is

(a) $\frac{4}{3}H_n - 1$ (b) $\frac{4}{3}H_n + \frac{1}{n}$

(c) $\frac{4}{3}H_n$ (d) $\frac{4}{3}H_n - \frac{2}{3}\frac{n}{n+1}$

Ans. (d)

Solution We have

$$t_n = \frac{1^2+2^2+3^2+\cdots+n^2}{1^3+2^3+3^3+\cdots+n^3} = \frac{\frac{1}{6}n(n+1)(2n+1)}{\frac{1}{4}n^2(n+1)^2}$$

$$= \frac{2(2n+1)}{3n(n+1)} = \frac{2}{3}\left[\frac{1}{n}+\frac{1}{n+1}\right]$$

$$\Rightarrow \sum_{k=1}^{n} t_k = \frac{2}{3}\left\{\left(1+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}+\frac{1}{n+1}\right)\right\}$$

$$= \frac{4}{3}H_n - \frac{2}{3}\frac{n}{n+1}.$$

Example 26 If the sum of first n terms of an A.P. is cn^2, then sum of squares of these n terms is

(a) $\frac{1}{6}n(4n^2-1)c^2$ (b) $\frac{1}{3}n(4n^2+1)c^2$

(c) $\frac{1}{3}n(4n^2-1)c^2$ (d) $\frac{1}{6}n(4n^2+1)c^2$

Ans. (c)

Solution Let t_n denote the rth term of the the A.P., then

$$t_n = S_n - S_{n-1} = cn^2 - c(n-1)^2$$
$$= c[n^2 - (n^2 - 2n + 1)] = c(2n - 1)$$

Now, $t_1^2 + t_2^2 + \ldots + t_n^2$

$$= \sum_{k=1}^{n} t_k^2 = c^2 \sum_{k=1}^{n} (2k-1)^2$$

$$= c^2 \sum_{k=1}^{n} (4k^2 - 4k + 1)$$

$$= c^2\left[\frac{4}{6}n(n+1)(2n+1) - \frac{4}{2}n(n+1) + n\right]$$

$$= \frac{1}{3}c^2 n\,[2(n+1)(2n+1) - 6(n+1) + 3]$$

$$= \frac{1}{3}c^2 n\,(4n^2 - 1)$$

Example 27 The sum of first n terms of the series

$1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \ldots$

is $n(n + 1)^2/2$ when n is even. When n is odd the sum of the series is

(a) $n^2(3n + 1)/4$ (b) $n^2(n + 1)/2$

(c) $n^3(n - 1)/2$ (d) none of these

Ans. (b)

Solution Let $n = 2m$, then

$S_{2m} = 1^2 + 2\times2^2 + 3^2 + 2\times4^2 + \ldots + (2m - 1)^2 + 2(2m)^2$

$= 2m(2m + 1)^2/2 = m(2m + 1)^2$

When $n = 2m - 1$.

$1^2 + 2\times2^2 + 3^2 + 2\times4^2 + \ldots + (2m - 1)^2$

$= S_{2m} - 2(2m)^2 = m(2m + 1)^2 - 2(2m)^2$

$= m\,[4m^2 + 4m + 1 - 8m] = m(2m - 1)^2$

$= n^2\,(n + 1)/2.$

Example 28 The interior angles of a polygon are in A.P. If the smallest angle is 120° and the common difference is 5°. then number of sides in the polygon is.

(a) 7 (b) 8 (c) 9 (d) 16

Ans. (c)

Solution We know that the sum of the interior angles of a polygon of n sides is $(n \quad 2)\pi$. We are given that $a = 120$ and $d = 5$.

$\therefore \quad \frac{n}{2}[2(120) + (n - 1)\,(5)] = (n - 2)\,(180)$

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$\Rightarrow \quad n[48 + (n - 1)] = (n - 2)\ (72)$

$\Rightarrow \quad n^2 - 25n + 144 = 0$

$\Rightarrow \quad (n - 9)\ (n - 16) = 0 \quad \Rightarrow \quad n = 9,\ 16.$

But when $n = 16$, the greatest angle is equal to $a + (16 - 1)d = 120° + 15 \times 5° = 195°$ and other interior angles are 190°, 185° and 180°. But no interior angle of a polygon can be equal to 180°. Therefore $n = 16$ is not possible. Thus, $n = 9$.

Example 29 Let $A_n = \dfrac{1.2.3 + 2.3.4 + 3.4.5 + \ldots + \text{upto } n \text{ terms}}{n\,(1.2 + 2.3 + 3.4 + \ldots + \text{upto } n \text{ term})}$

then $\lim\limits_{n\to\infty} A_n$ is

(a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{5}{4}$

Ans. (a)

Solution The numerator of A_n

$$= \sum_{r=2}^{n+1} (r - 1)r\ (r + 1)$$

$$= \sum_{r=2}^{n+1} (r^3 - r) = \sum_{r=1}^{n+1} (r^3 - r)$$

$$= \frac{1}{4}\ (n + 1)^2\ (n + 2)^2 - \frac{1}{2}\ (n + 1)\ (n + 2)$$

Also, $\sum\limits_{r=1}^{n} r(r + 1) = \sum\limits_{r=1}^{n} [(r + 1)^2 - (r + 1)]$

$$= \sum_{s=1}^{n+1} (s^2 - s)$$

$$= \frac{1}{6}\ (n + 1)\ (n + 2)\ (2n + 3) - \frac{1}{2}\ (n + 1)\ (n + 2)$$

Thus,

$$L = \lim_{n\to\infty} \frac{\frac{1}{4}\left(1+\frac{1}{n}\right)^2\left(1+\frac{2}{n}\right)^2 - \frac{1}{2n^2}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{\frac{1}{6}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(2+\frac{3}{n}\right) - \frac{1}{2n}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \frac{1}{4} \times \frac{6}{2} = \frac{3}{4}.$$

Example 30 If the ratio of sums to n terms of two A.P's is $(5n + 3) : (3n + 4)$, then the ratio of their 17th terms is

(a) 172 : 99 (b) 168 : 103

(c) 175 : 99 (d) 171 : 103

Ans. (b)

Solution Let two A.P.'s be

$a, a + d, a + 2d, a + 3d, \ldots$

and $A, A + D, A + 2D, A + 3D, \ldots$

According to the given condition,

$$\frac{\frac{n}{2}[2a + (n-1)d]}{\frac{n}{2}[2A + (n-1)D]} = \frac{5n+3}{3n+4}$$

$$\Rightarrow \quad \frac{a + \frac{n-1}{2}d}{A + \frac{n-1}{2}D} = \frac{5n+3}{3n+4}$$

Now, but $\dfrac{n-1}{2} = 16$ or $n = 33$.

Example 31 If $H_1, H_2, \ldots, H_n$ are n harmonic means between a and $b (\neq a)$, then value of $\dfrac{H_1 + a}{H_1 - a} + \dfrac{H_n + b}{H_n - b}$ is equal to

(a) $n + 1$ (b) $n - 1$

(c) $2n$ (d) $2n + 3$

Ans. (c)

Solution As $a, H_1, H_2, \ldots, H_n, b$ are in H.P., $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2} \cdots \frac{1}{H_n}, \frac{1}{b}$ are in A.P.

Let d be the common difference of this A.P., then

$$\frac{1}{b} = \frac{1}{a} + (n + 1)d$$

and $$\frac{1}{H_n} - \frac{1}{H_1} = (n - 1)d$$

Now, $$\frac{H_1 + a}{H_1 - a} = \frac{1/a + 1/H_1}{1/a - 1/H_1} = \frac{1/a + 1/H_1}{-d}$$

and $$\frac{H_n + b}{H_n - b} = \frac{1/b + 1/H_1}{1/b - 1/H_n} = \frac{1/b + 1/H_n}{d}$$

$$\therefore \quad \frac{H_1 + a}{H_1 - a} + \frac{H_n + b}{H_n - b} = \frac{1}{d}\left[\frac{1}{b} - \frac{1}{a} + \frac{1}{H_n} - \frac{1}{H_1}\right]$$

$$= \frac{1}{d}[(n + 1)d + (n - 1)d] = 2n$$

Example 32 If a, b, c are three positive real numbers, then least value of the expression $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ is

(a) 3 (b) $3\sqrt{3}$

(c) 6 (d) 9

Ans. (d)

Solution Since A.M. ≥ G.M., we have

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$$\frac{1}{3}(a+b+c) \geq (abc)^{1/3} \quad (1)$$

and $$\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \left(\frac{1}{abc}\right)^{1/3} \quad (2)$$

Multiplying the above inequalities, we get

$$\frac{1}{9}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 1$$

$$\Rightarrow \quad (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

The least value is attained when $a = b = c = 1$.

Example 33 Let $a_1, a_2, \ldots a_n$ be n positive real numbers such that $a_1\, a_2 \ldots a_n = c$, where c is a fixed real number. The expression $a_1 + 2a_2 + 3a_3 + \ldots + na_n$ cannot be less than

(a) $n!\, c$ (b) $n\,(n!\, c)^{1/n}$

(c) $c^{1/n}$ (d) none of these

Ans. (b)

Solution We have

$$\frac{1}{n}(a_1 + 2a_2 + 3a_3 + \ldots + na_n) \geq [(a_1)\,(2a_2)\,(3a_3) \ldots (na_n)]^{1/n} = (n!\, c)^{1/n}$$

Example 34 The sum upto $(2n + 1)$ terms of the series $a^2 - (a + d)^2 + (a + 2d)^2 - (a + 3d)^2 + \ldots$ is

(a) $a^2 + 3nd^2$

(b) $a^2 + 2nad + n(n - 1)d^2$

(c) $a^2 + 3nad + n(n - 1)d^2$

(d) $a^2 + 2nad + n\,(2n + 1)\,d^2$

Ans. (d)

Solution We can write the sum upto $(2n + 1)$ terms as

$[a + (a + d)]\,(-d) + [(a + 2d) + (a + 3d)]\,(-d) + \ldots$

$[(a + (2n - 2)\,d) + (a + (2n - 1)d]\,(-d) + (a + 2nd)^2$

$= (-d)\,[a + (a + d) + (a + 2d) + \ldots + a + (2n - 1)d] + (a + 2nd)^2$

$= (-d)\,\frac{2n}{2}\,\{a + a + (2n - 1)\,d\} + (a + 2nd)^2$

$= -2nad - n(2n - 1)d^2 + a^2 + 4\,n(ad) + 4n^2d^2$

$= a^2 + 2nad + n(2n + 1)d^2.$

Example 35 Value of

$$y = (0.64)^{\log_{0.25}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots \text{ upto } \infty\right)} \text{ is}$$

(a) 0.9 (b) 0.8 (c) 0.6 (d) 0.25

Ans. (b)

Solution We have

$$S = \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots = \frac{1/3}{1-1/3} = \frac{1}{2}$$

$$\therefore \quad y = (0.64)^{\log_{0.25}(1/2)}$$

$$\Rightarrow \quad \log y = \frac{\log(1/2)}{\log(0.25)} \log (0.64)$$

$$= \frac{-\log 2}{-\log 4} \log (0.64) = \frac{1}{2} \log (0.64) = 0.8$$

$$\Rightarrow \quad y = 0.8.$$

Example 36 Let $ax^2 + \frac{b}{x} \geq c$ for all positive x, where $a > 0$ and $b > 0$. The value of the expression $27ab^2$ cannot be less than

(a) $4c^3$ (b) $4c^2$

(c) $8c^3$ (d) c^3

Ans. (a)

Solution As A.M. $\geq$ G.M., we get

$$\frac{1}{3}\left(ax^2 + \frac{b}{2x} + \frac{b}{2x}\right) \geq \left(ax^2 \cdot \frac{b}{2x} \cdot \frac{b}{2x}\right)^{1/3}$$

$$\Rightarrow \quad \frac{1}{3}\left(ax^2 + \frac{b}{x}\right) \geq \left(\frac{ab^2}{4}\right)^{1/3}$$

But the least value of $ax^2 + \frac{b}{x}$ is c, therefore

$$3\left(\frac{ab^2}{4}\right)^{1/3} \geq c$$

$$\Rightarrow \quad 27ab^2 \geq 4c^3$$

Example 37 If a, b and c are three positive real numbers, then the minimum value of the expression $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$ is

(a) 1 (b) 2

(c) 3 (d) 6

Ans. (d)

Solution We have

$$\frac{1}{2}\left(\frac{b}{a}+\frac{a}{b}\right) \geq \sqrt{\frac{b}{a}\cdot\frac{a}{b}} = 1$$

$$\frac{1}{2}\left(\frac{c}{a}+\frac{a}{c}\right) \geq \sqrt{\frac{c}{a}\cdot\frac{a}{c}} = 1 \text{ and } \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right) \geq \sqrt{\frac{b}{c}\cdot\frac{c}{b}} = 1$$

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Adding the above inequalities, we get

$$\frac{1}{2}\left(\frac{b}{a}+\frac{a}{b}+\frac{c}{a}+\frac{a}{c}+\frac{b}{c}+\frac{c}{b}\right)\geq 3 \Rightarrow \frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 6$$

Thus, the minimum value of the given expression cannot be less than 6. Also, for $a = b = c = 1$,

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}=6$$

Hence, the minimum value of the given expression is 6.

Example 38 If $S_n=\sum_{r=1}^{n} t_r=\frac{1}{6}\,n(2n^2+9n+13)$, then $\sum_{r=1}^{n}\sqrt{t_r}$ equals

(a) $\frac{1}{2}\,n(n+1)$ (b) $\frac{1}{2}\,n(n+2)$

(c) $\frac{1}{2}\,n(n+3)$ (d) $\frac{1}{2}\,n(n+5)$

Ans. (c)

Solution We have $t_n = S_n - S_{n-1}\ \forall\ n \geq 2$

$$\therefore \quad t_n = \frac{1}{6}\,[2(n^3-(n-1)^3)+9(n^2-(n-1)^2)+13(n-n+1)]$$

$$=\frac{1}{6}\,[6n^2-6n+2+9(2n-1)+13]$$

$$=\frac{1}{6}\,(6n^2+12n+6)=(n+1)^2$$

Also, $t_1 = S_1 = 4 = (1+1)^2$

$$\therefore \quad \sum_{r=1}^{n}\sqrt{t_r}=\sum_{r=1}^{n}(r+1)=\frac{1}{2}\,(n+1)(n+2)-1$$

$$=\frac{1}{2}\,n(n+3).$$

Example 39 If $0 < \theta, \phi < \pi/2$ and

$x=\sum_{n=0}^{\infty}\sin^{2n}\theta,\ y=\sum_{n=0}^{\infty}\cos^{2n}\phi$ and $z=\sum_{n=0}^{\infty}\cos^{n}(\theta+\phi)\cos^{n}(\theta-\phi)$, then

(a) $xyz + 1 = yz - zx$ (b) $xyz - 1 = yz + zx$

(c) $xyz - xy = yz - zx$ (d) $xyz + 1 = yz + zx$

Ans. (c)

Solution We have

$$x=\frac{1}{1-\sin^2\theta}=\frac{1}{\cos^2\theta},\ y=\frac{1}{1-\cos^2\phi}=\frac{1}{\sin^2\phi}$$

Also, $\cos(\theta+\phi)\cos(\theta-\phi)$

$$=\cos^2\theta-\sin^2\phi=\frac{1}{x}-\frac{1}{y}$$

As, $0 < \cos^2\theta < 1,\ 0 < \sin^2\phi < 1,$

$-1 < \cos^2\theta - \sin^2\phi < 1$

$$\Rightarrow \quad -1<\frac{1}{x}-\frac{1}{y}<1$$

Thus, $$z=\sum_{n=0}^{\infty}\left(\frac{1}{x}-\frac{1}{y}\right)^n=\frac{1}{1-\left(\frac{1}{x}-\frac{1}{y}\right)}$$

$$=\frac{xy}{xy-y+x}$$

$\Rightarrow \quad z(xy - y + x) = xy$ or $xyz - xy = yz - xz.$

Example 40 If $a_1, a_2, \ldots, a_n$ are in A.P. with common difference $d \neq 0$, then sum of the series $\sin d\,[\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} \sec a_n]$ is

(a) $\tan a_n - \tan a_1$ (b) $\cot a_n - \cot a,$

(c) $\sec a_n - \sec a_1$ (d) $\operatorname{cosec} a_n - \operatorname{cosec} a$

Ans. (a)

Solution As $a_1, a_2, a_3, \ldots a_{n-1}, a_n$ are in A.P.,

$$d = a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1}$$

$\sin d\,[\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} \sec a_n]$

$$=\frac{\sin(a_2-a_1)}{\cos a_1\cos a_2}+\frac{\sin(a_3-a_2)}{\cos a_2\cos a_3}+\ldots+\frac{\sin(a_n-a_{n-1})}{\cos a_{n-1}\cos a_n}$$

$= (\tan a_2 - \tan a_1) + (\tan a_3 - \tan a_2) + \ldots + (\tan a_n - \tan a_{n-1})$

$= \tan a_n - \tan a_1.$

Example 41 Value of $\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)$ is

(a) $\pi/2$ (b) $\pi/4$

(c) 1 (d) none of these

Ans. (b)

Solution We have

$$\frac{1}{2r^2}=\frac{2}{4r^2}=\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}$$

$$\tan^{-1}\left(\frac{1}{2r^2}\right)=\tan^{-1}(2r+1)-\tan^{-1}(2r-1)$$

$$\Rightarrow \quad \sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{n}\left\{\tan^{-1}(2r+1)-\tan^{-1}(2r-1)\right\}$$

$$=\tan^{-1}(2n+1)-\pi/4$$

$$\Rightarrow \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$$

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Example 42 Let $I_n = \int_0^{\pi/4} \tan^n x\, dx$. Then $I_2 + I_4$, $I_3 + I_5$, $I_4 + I_6$, $I_5 + I_7$, ...are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Ans. (c)

Solution We have for $r \geq 1$,

$$I_r + I_{r+2} = \int_0^{\pi/4} \tan^r x\,(1+\tan^2 x)\, dx$$

$$= \int_0^{\pi/4} \tan^r x \sec^2 x\, dx = \frac{1}{r+1}$$

Thus, the given sequence becomes,

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

which is clearly an H.P.

Example 43 If $\sum_{r=1}^{n} t_r = \frac{1}{6} n\,(n+1)\,(n+2)\ \forall\, n \geq 1$,

Then $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{t_r}$ is

(a) 2 (b) 3 (c) 3/2 (d) 6

Ans. (a)

Solution We have, for $n \geq 1$,

$$t_n = \sum_{r=1}^{n} t_r - \sum_{r=1}^{n-1} t_r$$

$$= \frac{1}{6} n\,(n+1)\,(n+2-n+1) = \frac{1}{2} n\,(n+1).$$

Thus, $$\sum_{r=1}^{n} \frac{1}{t_r} = 2 \sum_{r=1}^{n} \left[\frac{1}{r} - \frac{1}{r+1}\right] = 2\left(1 - \frac{1}{n+1}\right)$$

$$\Rightarrow \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{t_r} = \lim_{n\to\infty} 2\left(1 - \frac{1}{n+1}\right) = 2\,(1-0) = 2.$$

Example 44 If $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, then value of $S_n = 1 + \frac{3}{2} + \frac{5}{3} + \ldots + \frac{2n-1}{n}$ is

(a) $H_n + n$ (b) $2n - H_n$
(c) $(n-1) + H_n$ (d) $H_n + 2n$

Ans. (b)

Solution $S_n = (2-1) + \left(2 - \frac{1}{2}\right) + \ldots + \left(2 - \frac{1}{n}\right)$

$= 2n - H_n$.

Example 45 Sum to n terms of $S_n = 1 + \frac{3}{2} + \frac{7}{4} + \frac{15}{8} + \frac{31}{16} + \ldots$ is

(a) $2(n-1) + \frac{1}{2n-1}$ (b) $2n - \frac{1}{2^n}$
(c) $2 + \frac{1}{2^n}$ (d) $2n - 1 + \frac{1}{2^n}$

Ans. (a)

Solution

$$S_n = (2-1) + \left(2 - \frac{1}{2}\right) + \left(2 - \frac{1}{4}\right) + \ldots + \left(2 - \frac{1}{2^{n-1}}\right)$$

$$= 2n - \frac{1 - 1/2^n}{1 - 1/2} = 2(n-1) + \frac{1}{2^{n-1}}$$

SECTION II

MULTIPLE CORRECT CHOICE TYPE

Example 46 If a, b, c are in H.P.. then the expression $E = \left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{c} + \frac{1}{a} - \frac{1}{b}\right)$ equals

(a) $\frac{2}{bc} - \frac{1}{b^2}$ (b) $\frac{1}{4}\left(\frac{3}{c^2} + \frac{2}{ca} - \frac{1}{a^2}\right)$
(c) $\frac{3}{b^2} - \frac{2}{ab}$ (d) none of these.

Ans. (a), (b), (c)

Solution As a, b, c are in H.P., $1/a$, $1/b$, $1/c$ are in A.P. Let the common difference of this A.P. be d Now,

$$E = \left(\frac{1}{c} + d\right)\left(\frac{1}{c} - d\right) = \frac{1}{c^2} - d^2$$

$$= \frac{1}{c^2} - \left(\frac{1}{c} - \frac{1}{b}\right)^2 = \frac{2}{bc} - \frac{1}{b^2}$$

Next, $$E = \frac{1}{c^2} - \frac{1}{4}\left(\frac{1}{c} - \frac{1}{a}\right)^2 = \frac{1}{4}\left(\frac{3}{c^2} + \frac{2}{ca} - \frac{1}{a^2}\right)$$

and $$E = \left(\frac{2}{b} - \frac{1}{a}\right)^2 - \left(\frac{1}{b} - \frac{1}{a}\right)^2 = \frac{3}{b^2} - \frac{2}{ab}$$

Example 47 If b_1, b_2, b_3 $(b_1 > 0)$ are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by

(a) $r > 3$ (b) $r < 1$
(c) $r = 3.5$ (d) $r = 5.2$

Ans. (a), (b), (c), (d)

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$$\geq \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ times}} = \frac{n}{2}$$

Thus, $a(n) > \frac{n}{2} \Rightarrow a(2n) > n.$

Example 51 Sum to n terms of the series $S = 1^2 + 2(2)^2 + 3^2 + 2(4^2) + 5^2 + 2(6^2) + \ldots$ is

(a) $\frac{1}{2} n (n+1)^2$ when n is even

(b) $\frac{1}{2} n^2 (n+1)$ when n is odd

(c) $\frac{1}{4} n^2 (n+2)$ when n is odd

(d) $\frac{1}{4} n (n+2)^2$ when n is even.

Ans. (a), (b)

Solution Let $n = 2m$, then

$$S = [1^2 + 2^2 + 3^2 + \ldots + (2m)^2] + [2^2 + 4^2 + \ldots + (2m)^2]$$

$$= \frac{1}{6} (2m)(2m+1)(4m+1) + \frac{4}{6} m (m+1)(2m+1)$$

$$= \frac{1}{3} m(2m+1)[4m+1+2m+2]$$

$$= \frac{1}{2} n(n+1)^2$$

When $n = 2m - 1$, then

$$S = m(2m+1)^2 - 2(2m)^2$$

$$= m(2m-1)^2 = \frac{1}{2} n^2(n+1).$$

Example 52 If a, b, c are in A.P., and a^2, b^2, c^2 are in H.P., then

(a) $a = b = c$

(b) $a, b, -\frac{1}{2} c$ are in G.P.

(c) a, b, c are in H.P.

(d) $-\frac{1}{2} a, b, c$ are in G.P.

Ans. (a), (b), (d)

Solution From the given condition,

$$2b = a + c \quad \text{and} \quad b^2 = \frac{2a^2c^2}{a^2 + c^2}$$

Eliminating b, we get

$$\left(\frac{a+c}{2}\right)^2 = \frac{2a^2c^2}{a^2 + c^2}$$

$$\Rightarrow (a^2 + c^2)^2 + 2ac(a^2 + c^2) - 8a^2c^2 = 0$$

$$\Rightarrow [(a+c)^2 + 2ac](a-c)^2 = 0$$

$$\Rightarrow 4\left(b^2 + \frac{1}{2} ac\right)(a-c)^2 = 0$$

$$\Rightarrow a - c = 0 \quad \text{or} \quad b^2 = -\frac{1}{2} ac$$

If $a = c$, we get $a = b = c$.

If $b^2 = -\frac{1}{2} ac$, then either $a, b, -\frac{1}{2} c$ are in G.P.

or $-\frac{1}{2} a, b, c$ are in G.P.

Example 53 In the nth row of the triangle

1

2 3

4 5 6

7 8 9 10

.....

.....

(a) Last term $= \frac{1}{2} n (n+1)$

(b) First term $= \frac{1}{2} (n^2 - n + 2)$

(c) Sum $= \frac{1}{2} n (n^2 + 1)$

(d) Sum $= \frac{1}{2} n^2 (n+1)$

Ans. (a), (b), (c)

Solution Last term of nth row

$$= 1 + 2 + 3 + \ldots + n = \frac{1}{2} n(n+1)$$

As terms in the nth row forms an A.P. with common difference 1,

first term $= \text{last term} - (n-1)(1)$

$$= \frac{1}{2} n (n+1) - n + 1$$

$$= \frac{1}{2} (n^2 - n + 2)$$

Sum $= \frac{1}{2} n \left[\frac{1}{2}(n^2 - n + 2) + \frac{1}{2}(n^2 + n)\right]$

$$= \frac{1}{2} n (n^2 + 1)$$

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Example 54 If $S = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$ upto ∞, then

(a) $(0.25)^{\log_2 (S)} = 4$ (b) $(0.008)^{\log_5 (S)} = 8$

(c) $(0.008)^{\log_5 (S)} = 4$ (d) $(0.25)^{\log_2 (S)} = 8$

Ans. (a), (b)

Solution We have $S = \frac{1/3}{1-1/3} = \frac{1}{2}$

Thus, $(0.25)^{\log_2 (S)} = (0.5)^{2 \log_2 (1/2)}$

$= (0.5)^{-2} = 4.$

$(0.008)^{\log_5 (S)} = (0.2)^{3\log_5 (1/2)}$

$= \frac{1}{5^{-3 \log_5 2}} = 8.$

Example 55 Let $a_n = \underbrace{(111 \ldots 1)}_{n \text{ times}}$, then

(a) a_{912} is not prime. (b) a_{951} is not prime.

(c) a_{480} is not prime (d) a_{91} is not prime.

Ans. (a), (b), (c), (d)

Solution As a_{912}, a_{951} and a_{480} are divisible by 3, none of them is prime. For a_{91}, we have

$$a_{91} = \frac{1}{9} \underbrace{(99 \ldots 9)}_{91 \text{ times}} = \frac{1}{9}(10^{91} - 1)$$

$$= \frac{1}{9}[(10^7)^{13} - 1] = \left[\frac{(10^7)^{13} - 1}{10^7 - 1}\right]\left[\frac{10^7 - 1}{10 - 1}\right]$$

$$= [(10^7)^{12} + (10^7)^{11} + \ldots + 10^7 + 1] \times [10^6 + 10^5 + \ldots + 10 + 1]$$

$\Rightarrow$ a_{91} is not prime.

Example 56 If 2 and 31 appear as two terms in an A.P., then

(a) common difference of the A.P. is a rational number

(b) all the terms of the A.P. must be rational.

(c) all the terms of the A.P. must be integers.

(d) sum to any finite number of terms of the A.P. must be rational.

Ans. (a), (b), (d)

Solution Let d be the common difference and $a_m = 2$, $a_n = 31$, then $(n - m)d = 2a \Rightarrow d = \frac{2a}{n-m}$ which is rational.

Also $a_r - a_m = (r - m)d$

$\Rightarrow$ a_r is rational.

As each term is rational, sum to any finite number of terms must be rational.

Example 57 Let a and b be two positive real numbers. Suppose A_1, A_2 are two arithmetic means; G_1, G_2 are two geometric means and H_1, H_2 are two harmonic means between a and b; then

(a) $\frac{G_1 G_2}{H_1 H_2} = \frac{A_1 + A_2}{H_1 + H_2}$

(b) $\frac{G_1 G_2}{H_1 H_2} - \frac{5}{9} = \frac{2}{9}\left(\frac{a}{b} + \frac{b}{a}\right)$

(c) $\frac{H_1 + H_2}{A_1 + A_2} = \frac{9ab}{(2a+b)(a+2b)}$

(d) $\frac{G_1 G_2}{H_1 H_2} = \frac{H_1 + H_2}{A_1 + A_2}$

Ans. (a), (b), (c)

Solution $A_1 = a + \frac{1}{3}(b - a)$, $A_2 = b + \frac{2}{3}(b - a)$

$\Rightarrow$ $A_1 + A_2 = a + b$

Similarly, $G_1 = a\left(\frac{b}{a}\right)^{1/3}$, $G_2 = a\left(\frac{b}{a}\right)^{2/3}$

$\Rightarrow$ $G_1 G_2 = ab$

and $\frac{1}{H_1} = \frac{1}{a} + \frac{1}{3}\left(\frac{1}{b} - \frac{1}{a}\right)$, $H_2 = \frac{1}{a} + \frac{2}{3}\left(\frac{1}{b} - \frac{1}{a}\right)$

Now, $\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}$

$\Rightarrow$ $\frac{H_1 + H_2}{H_1 H_2} = \frac{a+b}{ab} = \frac{A_1 + A_2}{G_1 G_2}$

$\Rightarrow$ $\frac{G_1 G_2}{H_1 H_2} = \frac{A_1 + A_2}{H_1 + H_2}$

Now, $H_1 + H_2 = \frac{3ab}{a+2b} + \frac{3ab}{2a+b}$

$= \frac{9ab(a+b)}{(a+2b)(2a+b)}$

$\Rightarrow$ $\frac{A_1 + A_2}{H_1 + H_2} = \frac{2(a^2 + b^2) + 5ab}{9ab}$

Thus, $\frac{G_1 G_2}{A_1 A_2} - \frac{5}{9} = \frac{2}{9}\left(\frac{a}{b} + \frac{b}{a}\right)$.

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Example 58 Let three terms of a non-constant positive G.P. be the sides of a triangle. If r is the common ratio of the G.P. then

(a) $1 < r < 2\cos 36°$ (b) $2\sin 18° < r < 1$

(c) $r^2 + r^{-2} = 4$ (d) none of these

Ans. (a), (b)

Solution If $r > 1$ then ar^2 is the greatest term.

$\therefore \quad ar^2 < a + ar \Rightarrow r^2 - r - 1 < 0$

$$\Rightarrow \quad \frac{1}{2}(1-\sqrt{5}) < r < \frac{1}{2}(1+\sqrt{5})$$

As $r > 1$, $1 < r < \frac{1}{2}(\sqrt{5}+1) = 2\cos 36°$

In case $0 < r < 1$, $a < ar + ar^2$

$\Rightarrow \quad r^2 + r - 1 > 0$

$$\Rightarrow \quad r < -\frac{1}{2}(\sqrt{5}+1) \text{ or } r > \frac{1}{2}(\sqrt{5}-1)$$

Thus, $2\sin 18° < r < 1$

Example 59 Let $n > 2$ and $a = 2^n + 1$, $b = 1 + 2 + ... + 2^{n-1}$, then

(a) hcf $(a, b) = 1$

(b) a is prime implies b is composite

(c) a and b cannot be simultaneously prime

(d) none of these

Ans. (a), (b), (c)

Solution If p is a prime that divides both a and b, then p is odd and $p|(a-b)$ or $p|2$.

Example 60 Let a and k be two natural numbers greater than 1. If $a^k - 1$ is prime then

(a) $a = 2$ (b) k is prime

(c) $k = 2^m$ for some $m \in \mathbf{N}$ (d) none of these

Ans. (a), (b)

Solution $a^k - 1 = (a-1)m$

where $m = a^{k-1} + a^{k-2} + ... + a + 1$.

As $a, k > 1, m > 1$.

Since $a^k - 1$ is prime, $a - 1 = 1$ or $a = 2$.

If k is not prime, the $k = rs$ where $r, s > 1$.

In this case

$$2^k - 1 = \frac{b^s - 1}{b-1}\frac{b-1}{2-1}$$

where $b = 2^r > 2$.

As $$\frac{b^s - 1}{b-1} = b^{s-1} + b^{s-2} + ... + b + 1 > 1$$

and $$\frac{b-1}{2-1} = 2^{r-1} + 2^{r-2} + ... + 1 > 1.$$

$2^k - 1$ is not prime. A contradiction.

Example 61 Let $F_n = 2^{2^n} + 1$, where $n \in \mathbf{N}$.

(a) Last digit of F_n is 7 $\forall$ $n \geq 2$.

(b) hcf $(F_n, F_m) = 1$ $\forall$ $n \neq m$.

(c) $F_n - 2$ is divisible by F_m if $n > m$.

(d) none of these

Ans. (a), (b), (c)

Solution For $n > 1$, last digit of 2^{2^n} is 6.

Let $n > m$, $a = 2^{2^m}$ and $k = 2^{n-m}$, then

$F_n - 2 = a^k - 1 = \alpha F_m$

where $\alpha = a^{k-1} + a^{k-2} + ... + a - 1 > 1$.

Any prime p dividing both F_n and F_m must be odd, but then p must divide 2.

Example 62 Let $\alpha(n) = 1 - \frac{1}{2} + \frac{1}{3} - ..... + (-1)^{n-1}\frac{1}{n}$

(a) $\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} = \alpha(2n)$

(b) $\alpha(2n) < 1$ $\forall$ n

(c) $\alpha(2n) \geq 0.5$ $\forall$ n

(d) $0.5 < \alpha(n) < 1$ $\forall$ n.

Ans. (a), (b), (c)

Solution $\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + ... + \frac{1}{n}\right)$$

$$= \left(1 + \frac{1}{3} + ... + \frac{1}{2n-1}\right) + \frac{1}{2}\left(1 + \frac{1}{2} + ... + \frac{1}{n}\right) - \left(1 + \frac{1}{2} + ... + \frac{1}{n}\right)$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - ..... + \frac{1}{2n-1} - \frac{1}{2n}$$

Also,

$$\alpha(2n) < \underbrace{\frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n}}_{n \text{ times}} = 1$$

and $\alpha(2n) \geq \frac{1}{2n} \geq \frac{1}{2} \forall n$.

(d) is not correct as $\alpha(1) = 1$.

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Example 63 Let $\alpha(n)$ be as in Example 62, $\frac{p_n}{q_n} = \alpha(n)$ where $p_n, q_n \in \mathbf{N}$ and hcf $(p_n, q_n) = 1$.

(a) $1979 | p_{1319}$
(b) q_n is even for each $n > 1$
(c) p_n is odd
(d) none of these

Ans. (a), (b), (c)

Solution $\alpha(1319) = \alpha(1318) + \frac{1}{1319}$

$$= \frac{1}{660} + \frac{1}{661} + \ldots\ldots + \frac{1}{1318} + \frac{1}{1319}$$

[See Example 62]

$$= \left(\frac{1}{660} + \frac{1}{1319}\right) + \left(\frac{1}{661} + \frac{1}{1318}\right) + \ldots\ldots + \left(\frac{1}{989} + \frac{1}{990}\right)$$

$$= 1979 \frac{p}{q} \text{ where } p, q \in \mathbf{N}.$$

$\Rightarrow \quad qp_{1319} = 1979\, pq_{1319}$.

As 1979 is a prime and all the factors of q are smaller than 1979, 1979 must divide p_{1319}.

You may check yourself that for $n > 1$, p_n is odd and q_n is even.

Example 64 Let $I_n = \int_0^1 x^n \tan^{-1} x\, dx$. If

$a_n I_{n+2} + b_n I_n = c_n \ \forall\ n \geq 1$, then

(a) $a_1, a_2, a_3, \ldots$ are in A.P.
(b) $b_1, b_2, b_3, \ldots$ are in A.P.
(c) $c_1, c_2, c_3, \ldots$ are in H.P.
(d) $\frac{\pi}{2} - c_1, \frac{\pi}{2} - c_2, \frac{\pi}{2} - c_3 \ldots$ are in G.P.

Ans. (a), (b), (d)

Solution $I_{n+2} = \frac{1}{n+3} x^{n+3} \tan^{-1} x \Big]_0^1 - \frac{1}{n+3} \int_0^1 \frac{x^{n+3}}{1+x^2}\, dx$

$\Rightarrow \quad (n+3) I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}\, dx$

Similarly,

$$(n+1) I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\, dx$$

$\therefore \quad (n+3) I_{n+2} + (n+1) I_n$

$$= \frac{\pi}{2} - \int_0^1 x^{n+1}\, dx = \frac{\pi}{2} - \frac{1}{n+2}$$

$\therefore \quad a_n = n + 3,\ b_n = n + 1,\ c_n = \frac{\pi}{2} - \frac{1}{n+1}$.

Example 65 The numbers 1, 5, 25 can be three terms (not necessarily consecutive) of

(a) at least one A.P.
(b) at least one G.P.
(c) infinite number of A.P.'s
(d) infinite number of G.P.'s

Ans. (a), (b), (c), (d)

Solution Let 1, 5, 25 be the pth, qth, rth terms of an A.P. with common difference d, then

$(q-p)d = 5 - 1$ and $(r-p)d = 25 - 1$

$\Rightarrow \quad \frac{q-p}{1} = \frac{r-p}{6} = k(\text{say})$

$\Rightarrow \quad q = p + k,\ r = p + 6k$ where k is any natural number.

Let 1, 5, 25 be the pth, qth, rth terms of a G.P., with common ratio R, then

$R^{q-p} = 5,\ R^{r-p} = 25$

$\therefore \quad r - p = 2q - 2p \quad \Rightarrow \quad r + p - 2q = 0.$

There exist infinitely many triplets of natural numbers satisfying this relation.

SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos. 66 to 70

Sum of the following three series is given

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots = \log 2 \qquad (1)$$

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \ldots\ldots = \frac{\pi}{2\sqrt{2}} \qquad (2)$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \ldots\ldots = \frac{\pi}{4} \qquad (3)$$

Example 66 Sum of the series

$$1 - 2\left(\frac{1}{(3)(5)} + \frac{1}{(7)(9)} + \frac{1}{(11)(13)} + \ldots\ldots\right)$$

is

(a) $\pi/2$ (b) $\pi/2 - 1$ (c) $\pi/4$ (d) $\pi/4 + 1$

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Example 67 Sum of the series

$$1-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}\ldots\ldots \text{ upto } \infty$$

is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{8}(1+\sqrt{2})$

(c) $\frac{\pi}{4\sqrt{2}}-1$ (d) $\frac{\pi}{4\sqrt{2}}$

Example 68 Sum of the series

$$\frac{1}{1.2.3}+\frac{1}{5.6.7}+\frac{1}{9.10.11}+\ldots\ldots \text{ upto } \infty$$

is

(a) $\frac{1}{4}\log 2$ (b) $\frac{1}{6}\log 2$

(c) $\frac{1}{3}\log 2$ (d) $\log 4$

Example 69 Sum of the series

$$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\ldots. \text{ upto } \infty$$

is

(a) $\frac{\pi}{4}-\log\sqrt{2}$ (b) $\frac{\pi}{4}+\log\sqrt{2}$

(c) $\pi-\log 2$ (d) $\pi+\log 2$

Example 70 Sum of the series

$$\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right)+\ldots \text{ upto } \infty$$

is

(a) $\frac{1}{2}\log 2$ (b) $\frac{1}{3}\log 2$

(c) $\log 4$ (d) $\pi-\log 2$

Ans. 66. (c), 67. (b), 68. (a), 69. (a), 70. (a)

Solution 66. The given series can be written as

$$1-\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\ldots..\right)=\frac{\pi}{4}$$

67. Add series (2) and (3).

68. $$t_r=\frac{1}{(4r-3)(4r-2)(4r-1)}$$

$$=\frac{1}{2}\left(\frac{1}{4r-3}-\frac{2}{4r-2}+\frac{1}{4r-1}\right)$$

$$=\frac{1}{2}\left(\frac{1}{4r-3}-\frac{1}{4r-2}+\frac{1}{4r-1}-\frac{1}{4r}\right)$$

$$-\frac{1}{4}\left(\frac{1}{2r-1}-\frac{1}{2r}\right)$$

$$\therefore \quad \sum_{r=1}^{\infty}t_r=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{4r-3}-\frac{1}{4r-2}+\frac{1}{4r-1}-\frac{1}{4r}\right)$$

$$-\frac{1}{4}\sum_{r=1}^{n}\left(\frac{1}{2r-1}-\frac{1}{2r}\right)$$

Taking limit as $n\to\infty$, we get

$$\sum_{r=1}^{\infty}t_r=\frac{1}{2}\log 2-\frac{1}{4}\log 2=\frac{1}{4}\log 2.$$

69. Multiply (1) by $\frac{1}{2}$ and subtract from (3).

70. $$t_r=\frac{1}{2r-1}-\frac{1}{4r-2}-\frac{1}{4r}=\frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r}\right)$$

$$\therefore \quad \sum_{r=1}^{\infty}t_r=\frac{1}{2}\log 2$$

Paragraph for Question Nos. 71 to 75

Given a sequence $t_1, t_2, \ldots$ if its possible to find a function $f(r)$ such that

$$t_r=f(r+1)-f(r)$$

then

$$\sum_{r=1}^{n}t_r=f(n+1)-f(1)$$

Example 71 Sum of the series

$\sum_{r=1}^{\infty}\frac{1}{4r^2-1}$ is

(a) 2 (b) 1 (c) 1/2 (d) 1/4

Example 72 If $u_1=1$, $u_{n+1}=2u_n+1$, then u_{n+1} equals

(a) 2^n+1 (b) $2^{n+1}-1$

(c) 2^n-2 (d) $2^{n+1}-2$

Example 73. Sum of the $\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}$ is

(a) 1 (b) 1/2 (c) 1/4 (d) 1/8

Example 74 Sum of the series $\sum_{r=1}^{n}r(r+1)(r+2)$ is

(a) 4 $({}^{n+3}C_2)$ (b) 6 $({}^{n+3}C_4)$

(c) n^2-n (d) n^2+n

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Example 75 Sum of the series $\sum_{r=1}^{n} r(r+3)(r+6)$ is

(a) $\frac{1}{3}n(n+3)(n+9)$

(b) $n^4 + 7n^2 + 20n$

(c) $\frac{1}{4}n(n+3)(n+5)(n+9)$

(d) none of these

Ans. 71. (c), 72. (b), 73. (c), 74 (b), 75. (d)

Solution 71. $\frac{1}{4r^2-1} = \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$

$$\Rightarrow \quad \sum_{r=1}^{n}\frac{1}{4r^2-1} = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

Now take limit as $n \to \infty$

72. $u_{n+1} - u_n = 2(u_n - u_{n-1})$

$= 2^2 (u_{n-1} - u_{n-2})$

$= 2^3 (u_{n-2} - u_{n-3})$

$= \ldots\ldots\ldots\ldots$

$= 2^{n-1} (u_2 - u_1) = 2^n$

$\Rightarrow \quad u_{n+1} = u_n + 2^n = u_{n-1} + 2^{n-1} + 2^n$

$= u_{n-2} + 2^{n-2} + 2^{n-1} + 2^n$

$= \ldots\ldots\ldots\ldots$

$= u_1 + 2 + \ldots\ldots + 2^{n-1} + 2^n$

$= 2^{n+1} - 1.$

73. Let $t_r = \frac{1}{r(r+1)(r+2)}$

$$\Rightarrow \quad rt_r = \frac{1}{(r+1)(r+2)} = (r+3)t_{r+1}$$

$\Rightarrow rt_r - (r+1)t_{r+1} = 2t_{r+1}$

$$\Rightarrow \quad 2\sum_{r=1}^{n-1} t_{r+1} = t_1 - nt_n$$

$$\Rightarrow \quad 2\sum_{r=1}^{n} t_r = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$$

$$\Rightarrow \quad 2\sum_{r=1}^{\infty} t_r = \frac{1}{2} - 0 = \frac{1}{2}.$$

74. Let $t_r = r(r+1)(r+2)$

$\Rightarrow (r+3)t_r - r\,t_{r+1} = 0$

$\Rightarrow \quad 4t_r = rt_{r+1} - (r-1)t_r$

$$\Rightarrow \quad 4\sum_{r=1}^{n} t_r = nt_{n+1} = 4!\ ({}^{n+3}C_4).$$

> ***Tip***: In general $\sum_{r=1}^{n} r(r+1)\ldots\ldots(r+k-1)$
>
> $$= \frac{1}{k}n(n+1)\ldots\ldots(n+k)$$

75. $t_r = r(r+3)(r+6)$

$= r[r^2 + 9r + 18]$

$= r[(r+1)(r+2) + 6(r+1) + 10]$

$= r(r+1)(r+2) + 6r(r+1) + 10r$

Now, use the above tip.

Paragraph for Question Nos. 76 to 80

For $k, n \in \mathbf{N}$, we define

$B(k, n) = 1.2.3 \ldots k + 2.3.4 \ldots (k+1)$

$+ \ldots + n(n+1) \ldots (n+k-1)$,

$S_0(n) = n$

and $S_k(n) = 1^k + 2^k + \ldots + n^k$

To obtain value of $B(k, n)$, we rewrite $B(k, n)$ as follows:

$$B(k, n) = k!\left[\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \ldots\ldots + \binom{n+k-1}{k}\right]$$

$$= k!\binom{n+k}{k+1} = \frac{n(n+1)\ldots\ldots(n+k)}{k+1}.$$

Example 76 $S_2(n) + S_1(n)$ equals

(a) $B(2, n)$ (b) $\frac{1}{2}B(2, n)$

(c) $\frac{1}{6}B(2, n)$ (d) none of these

Example 77 $S_3(n) + 3S_2(n)$ equals

(a) $B(3, n)$ (b) $B(3, n) - 2B(2, n)$

(c) $B(3, n) - 2B(1, n)$ (d) $B(3, n) + 2B(1, n)$

Example 78 $\binom{k+1}{1}S_k(n) + \binom{k+1}{2}S_{k-1}(n) + \ldots +$

$\binom{k+1}{k}S_1(n) + \binom{k+1}{k+1}S_0(n)$

equals

(a) $(n+1)^k$ (b) $(n+1)^k - 1$

(c) $n^k - (n-1)^k$ (d) $(n+1)^k - (n-1)^k$

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Example 79 $\sum_{k=1}^{n} k^4$ equals

(a) $\frac{1}{6} n(n+1)(2n+1)(3n^2+2n-1)$

(b) $\frac{1}{6} n(n+1)(2n+1)(3n^2+3n-1)$

(c) $\frac{1}{6} n(n+1)(2n+1)(3n^2+1)$

(d) none of these.

Example 80 $\sum_{k=1}^{n} k(k+1)(k+2)^2$ equals

(a) $\frac{1}{20} n(n+1)(n+2)(n+3)(4n+15)$

(b) $\frac{1}{20} n(n+1)(n+2)(n+3)(3n+12)$

(c) $\frac{1}{20} n(n+1)(n+2)(n+3)(2n+13)$

(d) $\frac{1}{20} n(n+1)(n+2)(n+3)(n+14)$

Ans. 76. (a), 77. (c), 78. (b), 79. (d), 80. (a)

Solution

76. $S_2(n) + S_1(n) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \sum_{k=1}^{n} k(k+1)$

$= B(2, n).$

77. $S_3(n) + 3S_2(n)$

$= \sum_{k=1}^{n} k^2(k+3) = \sum_{k=1}^{n} k(k+1)(k+2) - 2\sum_{k=1}^{n} k$

$= B(3, n) - 2B(1, n)$

78. $\binom{k+1}{1} S_k(n) + \binom{k+1}{2} S_{k-1}(n) + \ldots + \binom{k+1}{k} S_1(n) + \binom{k+1}{k+1} S_0(n)$

$= \sum_{r=1}^{n} \left[\binom{k+1}{1} r^k + \binom{k+1}{2} r^{k-1} + \ldots + \binom{k+1}{k} r + \binom{k+1}{k+1} \right]$

$= \sum_{r=1}^{n} [(r+1)^{k+1} - r^{k+1}] = (n+1)^{k+1} - 1$

79. $\sum_{k=1}^{n} k^4 = S_4(n)$

But by Example 78

$5S_4(n) = (n+1)^5 - 1 - \binom{5}{2} S_3(n) - \binom{5}{3} S_2(n) - \binom{5}{4} S_1(n) - n$

$= (n+1)^5 - 1 - \frac{10}{4} n^2(n+1)^2 - \frac{10}{6} n(n+1)(2n+1) - \frac{5}{2} n(n+1) - n$

$= n\left[n^4 + \frac{5}{2}n^3 + \frac{5}{3}n^2 - \frac{1}{6}\right]$

$= \frac{n}{6}(6n^4 + 15n^3 + 10n^2 - 1)$

$= \frac{n}{6}(n+1)(2n+1)(3n^2+3n-1).$

Alternative Solution

Write

$k^4 = k(k+1)(k+2)(k+3) + ak(k+1)(k+2) + bk(k+1) + ck$

Put $k = -1, -2, -3$ to obtain

$c = -1, b = 7, a = -6$

Thus, $S_4(n) = B(4, n) - 6B(3, n) + 7B(2, n) - B(1, n)$

$= \frac{1}{5} n(n+1)(n+2)(n+3)(n+4) - \frac{6}{4} n(n+1)(n+2)(n+3) + \frac{7}{3} n(n+1)(n+2) - \frac{1}{2} n(n+1)$

$= \frac{1}{30} n(n+1)[6(n+2)(n+3)(n+4) - 45(n+2)(n+3) + 70(n+2) - 15]$

$= \frac{1}{30} n(n+1)(6n^3 + 9n^2 + n - 1)$

$= \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1).$

80. $k(k+1)(k+2)^2 = k(k+1)(k+2)(k+3) - k(k+1)(k+2)$

$\Rightarrow \sum_{k=1}^{n} k(k+1)(k+2)^2 = B(4, n) - B(3, n)$

$= \frac{1}{5} n(n+1)(n+2)(n+3)(n+4) - \frac{1}{4} n(n+1)(n+2)(n+3)$

$= \frac{1}{20} n(n+1)(n+2)(n+3)(4n+11)$

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Paragraph for Question Nos. 81 to 85

Let V_r denote the sum of first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is $(2r - 1)$. Let $T_r = V_{r+1} - V_r - 2$;

$Q_r = T_{r+1} - T_r$, $W_r = \dfrac{1}{T_r} + \dfrac{1}{4(r+1)}$ and

$X_r = 3^{Q_r}$ for $r = 1, 2, \ldots$

81. The sum $V_1 + V_2 + \ldots + V_n$ is:
   (a) $\dfrac{1}{12} n(n+1)(3n^2 - n + 1)$
   (b) $\dfrac{1}{12}(n+1)(3n^2 + n + 2)$
   (c) $\dfrac{1}{2} n(2n^2 - n + 1)$
   (d) $\dfrac{1}{3}(2n^3 - 2n + 3)$

82. T_r is always:
   (a) an odd number
   (b) an even number
   (c) a prime number
   (d) a composite number

83. Which one of the following is a correct statement?
   (a) $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 5.
   (b) $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 6.
   (c) $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 11.
   (d) $Q_1 = Q_2 = Q_3 = \ldots$

84. $W_1, W_2, W_3, \ldots$ are in
   (a) A.P. (b) G.P.
   (c) H.P. (d) none of these

85. $X_1, X_2, X_3, \ldots$ are in
   (a) A.P. (b) G.P.
   (c) H.P. (d) none of these.

Ans. 81. (b) 82. (d) 83. (b) 84. (c) 85. (b).

Solution 81. $V_r = \dfrac{r}{2}[2(r) + (r-1)(2r-1)]$

$= \dfrac{1}{2}(2r^3 - r^2 + r)$

$\therefore \quad \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} k^3 - \dfrac{1}{2}\sum_{k=1}^{n} k^2 + \dfrac{1}{2}\sum_{k=1}^{n} k$

$= \dfrac{1}{4} n^2(n+1)^2 - \dfrac{1}{12} n(n+1)(2n+1)$

$+ \dfrac{1}{4} n(n+1)$

$= \dfrac{1}{12} n(n+1)[3n(n+1) - (2n+1) + 3]$

$= \dfrac{1}{12} n(n+1)(3n^2 + n + 2)$

82. $T_r = V_{r+1} - V_r - 2$

$= [(r+1)^3 - r^3] - \dfrac{1}{2}[(r+1)^2 - r^2]$

$+ \dfrac{1}{2}[(r+1) - r] - 2$

$= (3r^2 + 3r + 1) - \dfrac{1}{2}(2r+1) + \dfrac{1}{2} - 2$

$= 3r^2 + 2r - 1 = (3r - 1)(r + 1)$

$\therefore$ T_r is always a composite number.

83. $Q_r = T_{r+1} - T_r = 3\{(r+1)^2 - r^2\} + 2\{(r+1) - r\}$

$= 3(2r + 1) + 2 = 5 + 6r$

$\therefore$ $Q_1, Q_2, \ldots$ forms an A.P. with common difference 6.

84. $W_r = \dfrac{1}{T_r} + \dfrac{1}{4(r+1)} = \dfrac{1}{(3r-1)(r+1)} + \dfrac{1}{4(r+1)}$

$= \dfrac{3}{4(3r-1)}$

$\therefore$ $W_1, W_2, W_3 \ldots$ are in H.P.

85. As Q_1, Q_2, Q_3, are in A.P.

$X_1, X_2, X_3 \ldots$ are in G.P.

Paragraph for Question Nos. 86 to 88

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively of two distinct positive numbers. For $n \geq 2$, let A_{n-1} and H_{n-1} has arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively.

86. Which of the following statements is a correct statement?
   (a) $G_1 > G_2 > G_3 > \cdots$
   (b) $G_1 < G_2 < G_3 < \cdots$
   (c) $G_1 = G_2 = G_3 = \cdots$
   (d) $G_1 < G_3 < G_5 < \cdots$ and $G_2 > G_4 > G_6 > \cdots$

87. Which of the following statements is correct?
   (a) $A_1 > A_2 > \cdots$
   (b) $A_1 < A_2 < A_3 < \cdots$
   (c) $A_1 > A_3 > A_5 > \cdots$ and $A_2 < A_4 < A_6 < \cdots$
   (d) $A_1 < A_3 < A_5 < \cdots$ and $A_2 > A_4 > A_6 \cdots$

88. Which of the following is a statements is correct?
   (a) $H_1 > H_2 > H_3 > \cdots$
   (b) $H_1 < H_2 < H_3 < \cdots$

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(a) $\sum_{n=0}^{\infty} \sin^{2n}\theta \cos^{2n}\theta = \frac{1}{1-\sin^2\theta\cos^2\theta}$

$= \frac{1}{1-\frac{1}{xy}} = \frac{xy}{xy-1}$.

(b) $\sum_{n=0}^{\infty} \tan^{2n}\theta = \frac{1}{1-\tan^2\theta} = \frac{1}{1-\frac{x}{y}} = \frac{y}{y-x}$.

(c) $\sum_{n=0}^{\infty} \sin^{2n}\theta \cos^{4n}\theta = \frac{1}{1-\sin^2\theta\cos^4\theta}$

$= \frac{1}{1-\frac{1}{x^2y}} = \frac{x^2y}{x^2y-1}$.

(d) $\sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{4n}\theta = \frac{xy^2}{xy^2-1}$.

Example 104 Let a_1, a_2, a_3, ... be a geometric progression such that

$$\log_{10}(a_m) = \frac{1}{n} \quad \text{and} \quad \log_{10}(a_n) = \frac{1}{m}$$

for two fixed positive integer m and n, with $m < n$, then

(a) a_{2m+n}	(p) $10^{1/m-1/n}$
(b) a_{mn}	(q) 10
(c) a_{m+n}	(r) $10^{2/n+1/m}$
(d) a_{n-m}	(s) $10^{1/n+1/m}$

Ans.

	p	q	r	s
a			●	
b		●		
c				●
d	●			

Solution

Let $t_r = \log_{10} a_r$, then $t_1, t_2, t_3, \ldots$ are in A.P. Let its common difference be d. Now,

$$t_m - t_n = \frac{1}{n} - \frac{1}{m}$$

$$\Rightarrow \quad (m-n)d = \frac{(m-n)}{mn} \quad \Rightarrow \quad d = \frac{1}{mn}.$$

Now, $t_r - t_m = (r-m)d = \frac{r}{mn} - \frac{1}{n}$

$\Rightarrow \quad t_r = r/mn$

$\Rightarrow \quad a_r = 10^{r/mn}$.

SECTION VI

REASONING TYPE

Example 105 Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

Statement-1: The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.

Statement-2: The numbers b_1, b_2, b_3, b_4 are in H.P.

Ans. (c)

Solution Let a be the first term and r be the common ratio of the G.P. As four numbers are distinct and positive $a > 0$, $r > 0$ and $r \neq 1$.

$$b_1 = a,\ b_2 = a(1+r),\ b_3 = a(1+r+r^2)$$

and $\quad b_4 = a(1+r+r^2+r^3)$.

As $\quad b_2 - b_1 = ar \neq ar^2 = b_3 - b_2$

$\Rightarrow \quad b_1, b_2, b_3, b_4$ are not in A.P.

$$\frac{b_2}{b_1} = 1 + r \neq \frac{1+r+r^2}{1+r} = \frac{b_3}{b_2}$$

$\Rightarrow \quad b_1, b_2, b_3, b_4$ are not in G.P.

As $\quad \frac{1}{b_2} - \frac{1}{b_1} = \frac{-r}{a(1+r)} \neq \frac{-r^2}{a(1+r)(1+r+r^2)} = \frac{1}{b_3} - \frac{1}{b_2}$

b_1, b_2, b_3, b_4 cannot be in H.P.

Example 106 **Statement-1:** There exists no A.P. whose three terms are $\sqrt{3}, \sqrt{5}$ and $\sqrt{7}$.

Statement-2: If t_p, t_q and t_r are three distinct terms of an A.P., then $\frac{t_r - t_p}{t_q - t_p}$ is a rational number.

Ans. (a)

Solution Suppose $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are the pth, qth and rth terms of an A.P. whose common difference is d, then

$$t_r - t_p = (r-p)\,d$$

and $\quad t_q - t_p = (q-p)\,d$

$\Rightarrow \quad \frac{t_r - t_p}{t_q - t_p} = \frac{r-p}{q-p}$ which is a rational numbers.

$\Rightarrow \quad \frac{\sqrt{7}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ is a rational numbers.

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$$\Leftrightarrow \quad \frac{(\sqrt{7}-\sqrt{3})(\sqrt{5}+\sqrt{3})}{5-3} \text{ is rational}$$

$$\Leftrightarrow \quad \frac{\sqrt{35}-\sqrt{15}+\sqrt{21}-3}{2} \text{ is rational}$$

$\Leftrightarrow$ $\sqrt{35}-\sqrt{15}+\sqrt{21}$ is rational, say r.

Now,

$$\sqrt{35}-\sqrt{15}+\sqrt{21} = r$$

$$\Rightarrow \qquad \sqrt{15}-\sqrt{21} = \sqrt{35} - r$$

Squaring both sides, we get

$$15 + 21 - (2)(6)\sqrt{35} = 35 + r^2 - 2r\sqrt{35}$$

$$\Rightarrow \qquad \sqrt{35} = \frac{1-r^2}{12+2r}$$

$\Rightarrow$ $\sqrt{35}$ is rational.

This is a contradiction

Hence, $\sqrt{3}, \sqrt{5}$ and $\sqrt{7}$ cannot be three terms of an A.P.

Example 107 **Statement-1:** $\frac{1^2}{(1)(3)} + \frac{2^2}{(3)(5)} + \ldots$

$$+ \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$$

Because

Statement-2: $\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \ldots + \frac{1}{(2n-1)(2n+1)}$

$$= \frac{1}{2n+1}$$

Ans. (c)

Solution Let

$$t_r = \frac{r^2}{(2r-1)(2r+1)}$$

$$\Rightarrow \qquad 4t_r = \frac{4r^2-1+1}{(2r-1)(2r+1)}$$

$$= 1 + \frac{1}{(2r-1)(2r+1)}$$

$$= 1 + \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$\Rightarrow \qquad 4\sum_{r=1}^{n} t_r = n + \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= n + \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

$$= n + \frac{n}{2n+1} = \frac{2n(n+1)}{2n+1}$$

$$\Rightarrow \qquad \sum_{r=1}^{n} t_r = \frac{n(n+1)}{2(2n+1)}$$

$\therefore$ Statement-1 is true.

But statement-2 is false since

$$\Rightarrow \quad \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}$$

Example 108 **Statement-1:** For $k \geq 0$, let

$$S_k(n) = 1^k + 2^k + \ldots + n^k,$$

then $\qquad S_4(n) = \frac{1}{30} n(n+1)(2n+1)(3n^2+n+1)$

Statement-2: $\binom{k+1}{1} S_k(n) + \binom{k+1}{2} S_{k-1}(n) + \ldots +$

$$\binom{k+1}{k} S_1(n) + \binom{k+1}{k+1} S_0(n) = (n+1)^{k+1} - 1$$

Ans. (d)

Solution We have

$$(r+1)^{k+1} - r^{k+1} = \binom{k+1}{1} r^k + \binom{k+1}{2} r^{k-1}$$

$$+ \ldots + \binom{k+1}{k} r + \binom{k+1}{k+1} r^0$$

$$\Rightarrow \qquad (n+1)^{k+1} - 1 = \sum_{r=1}^{n} \{(r+1)^{k+1} - r^{k+1}\}$$

$$= \binom{k+1}{1} S_k(n) + \binom{k+1}{2} S_{k-1}(n) + \ldots$$

$$+ \binom{k+1}{k} S_1(n) + \binom{k+1}{k+1} S_0(n)$$

$\therefore$ Statement-2 is true.

That statement-1 is false can be checked by putting $n = 4$ in the statement -2.

In fact

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$$S_4(n) = \frac{1}{30}\, n(n+1)(2n+1)(3n^2+3n-1)$$

See solution to Example 79.

Example 109 **Statement-1:** If n is odd,

$1^2 + (3)(2^2) + 3^2 + (3)(4^2) + 5^2 + \ldots$ upto n terms

$$= \frac{1}{6}\, n(n+1)(4n-1)$$

Statement-2: If n is even,

$1^2 + (3)(2^2) + 3^2 + (3)(4^2) + 5^2 + \ldots$ upto n terms

$$= \frac{1}{6}\, n(n+1)(4n+5)$$

Ans. (a)

Solution If $n = 2m$, then statement-2 becomes

$1^2 + (3)(2^2) + 3^2 + (3)(4^2) + 5^2 + \ldots + 3(2m)^2$

$$= \frac{1}{6}\,(2m)(2m+1)(8m+5)$$

$\Rightarrow 1^2 + (3)(2^2) + 3^2 + (3)(4^2) + 5^2 + \ldots + 3(2m-1)^2$

$$= \frac{1}{6}\,(2m)(2m+1)(8m+5) - 3(2m)^2$$

$$= \left(\frac{2m}{6}\right)[(2m+1)(8m+5) - 36m]$$

$$= \frac{1}{3}\, m[16m^2 - 18m + 5]$$

$$= \frac{1}{3}\, m(2m-1)(8m-5)$$

For $n = 2m-1$, we get

$1^2 + (3)(2^2) + 3^2 + (3)(4^2) + 5^2 + \ldots$ upto n terms

$$= \frac{1}{3}\left(\frac{n+1}{2}\right)(n)(4n-1) = \frac{1}{6}\, n(n+1)(4n-1)$$

Example 110 **Statement-1:** If $x > 1$, the sum to infinite number of

$$1 + 3\left(1-\frac{1}{x}\right) + 5\left(1-\frac{1}{x}\right)^2 + 7\left(1-\frac{1}{x}\right)^3 + \cdots$$

is $x^2 - x$

Statement-2: If $0 < y < 1$, the sum of the series

$$1 + 3y + 5y^2 + 7y^3 + \ldots \text{ is } \frac{1+y}{(1-y)^2}$$

Ans. (a)

Solution Let

$$S = 1 + 3y + 5y^2 + 7y^3 + \cdots \quad (1)$$

$$yS = \quad y + 3y^2 + 5y^3 + \cdots \quad (2)$$

Subtracting (2) from (1) we get

$$(1-y)S = 1 + 2y + 2y^2 + 2y^3 + \cdots$$

$$= 1 + \frac{2y}{1-y}$$

$$\Rightarrow \qquad S = \frac{1}{1-y} + \frac{2y}{(1-y)^2} = \frac{1+y}{(1-y)^2}$$

Putting $y = 1 - \frac{1}{x}$, we get

$$S = \frac{1+1-1/x}{(1/x)^2} = x^2 - x$$

EXERCISE

LEVEL 1

SECTION I

SINGLE CORRECT CHOICE TYPE

1. Given a sequence of four numbers such that the first three are in G.P. and the last three are in A.P. with common difference 6. If the first and the fourth number are equal, then common ratio of the G.P. is
   (a) -2 (b) 2 (c) 3 (d) -3
2. The sum of the series
   $$\frac{7}{1^3.2^3} + \frac{19}{2^3.3^3} + \frac{37}{3^3.4^3} + \ldots\ldots \text{ upto } \infty$$
   is
   (a) 3 (b) 2 (c) 1 (d) 1.5
3. $\lim\limits_{n\to\infty}\left[\frac{1^2+2^2+\ldots\ldots+n^2}{5n^3} + \frac{4(1^3+2^3+\ldots.+n^3)}{3n^4} + \frac{1+2+\ldots\ldots+n}{n^2}\right]$
   equals
   (a) 9/10 (b) 1/10 (c) 1/2 (d) 1/3
4. $\lim\limits_{n\to\infty}\left(\frac{1}{n^{100}}\sum_{r=1}^{n} r^{99}\right)$ equals
   (a) $\frac{1}{99}$ (b) $\frac{1}{100}$ (c) $\frac{1}{101}$ (d) 0
5. Let $S_k(n) = \sum_{r=1}^{n} r^k$, then
   $$\lim_{n\to\infty} \frac{S_1(n)S_5(n) - S_3(n)^2}{S_7(n)}$$

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20. Let $S_n = \sum_{r=1}^{n} t_r = \frac{1}{6} n(2n^2 + 9n + 13)$, then $\sum_{r=1}^{n} \sqrt{t_r}$ equals

(a) $\frac{1}{2} n(n + 1)$ (b) $\frac{1}{2} n(n + 3)$

(c) $(n + 1)^2$ (d) n^2

21. Let $a_n = \underbrace{111.....1}_{n \text{ times}}$. The remainder when a_{124} is divided by 271 is

(a) 23 (b) 25 (c) 27 (d) 29

22. Three arithmetic means, three geometric means and three harmonic means are inserted between 1 and 5. The cubic equation whose roots are 3rd A.M., 2nd G.M. and 1st H.M. is

(a) $x^3 - \frac{1}{4}(21 + 4\sqrt{5})\, x^2 + 5x - 5\sqrt{5} = 0$

(b) $x^3 - \frac{1}{4}(21 + 4\sqrt{5})\, x^2 + (5 + \frac{21}{4}\sqrt{5})\, x - 5\sqrt{5} = 0$

(c) $4x^3 - (21 + 4\sqrt{5})\, x^2 + (15 + 21\sqrt{5}) - 20\sqrt{5} = 0$

(d) none of these

23. If t_r denotes the rth term of an A.P., and $t_p = 1/q$, $t_q = 1/p$, then which of the following is a root of the equation

$(p + 2q + 3r)\, x^2 + (q + 2r - 3p)\, x + (r + 2p - 3q) = 0$

(a) t_p (b) t_q (c) t_{pq} (d) t_{p+q}

24. If the pth term of an A.P. is $1/q$ and qth term is $1/p$ and sum of pq terms is $25pq$, then p and q are connected by

(a) $p^2 = 4q^2 + 1$ (b) $p = 4q - 1$

(c) $pq = 4 + p$ (d) none of these

25. If $p(x) = \frac{1 + x^2 + x^4 + ... + x^{2n-2}}{1 + x + x^2 + ... + x^{n-1}}$

is a polynomial in x, then n must be

(a) odd (b) even

(c) greater than 5 (d) less than 5

26. $\sqrt{\underbrace{111.....1}_{200 \text{ digits}} - \underbrace{222.....2}_{100 \text{ digits}}}$ equals

(a) $\sqrt{\underbrace{1313.....13}_{100 \text{ digits}}}$ (b) $\sqrt{\underbrace{33.....3}_{100 \text{ digits}}}$

(c) $\sqrt{\underbrace{2323.....23}_{100 \text{ digits}}}$ (d) none of these

27. Let $S_n(a) = 1 + 2(1 - a) + 3(1 - a)(1 - 2a) + 4(1 - a)(1 - 2a)(1 - 3a) + \ldots$ upto n terms, then $S_{100}\left(\frac{1}{99}\right)$ equals

(a) 100/99 (b) 1/99

(c) 200 (d) 99

28. Let $a_n = \int_0^{\pi/2} \frac{1 - \cos 2nx}{1 - \cos 2x}\, dx$, then $a_1, a_2, a_3, \ldots$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

29. If $S_n = \sum_{r=1}^{n} t_r = \frac{1}{6} n(n + 1)(n + 2)\ \forall n \geq 1$, then

$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{t_r}$ is

(a) 1 (b) 3/2 (c) 2 (d) 5/2

30. If unit's digit of $1^2 + 2^2 + \ldots + n^2$ is 5, then unit's digit of $1^{10} + 2^{10} + \ldots + n^{10}$ is

(a) 1 (b) 3 (c) 5 (d) 7

31. Four geometric means are inserted between $2^{11} - 1$ and $2^{11} + 1$. The product of these geometric means is

(a) $2^{44} - 1$ (b) $2^{44} - 2^{23} + 1$

(c) $2^{44} - 2^{22} + 1$ (d) none of these

32. Let $S_n = \frac{1}{2^2 - 1} + \frac{1}{4^2 - 1} + \ldots + \frac{1}{(2n)^2 - 1}$, then value of S_{25} is

(a) 25/54 (b) 25/53

(c) 25/51 (d) 1/2

33. Suppose $t_r = 1^2 + 2^2 + \ldots + r^2$. Let

$t_1 + t_2 + \ldots + t_n = \frac{1}{12} n(n + 1)(n + 2)k$,

then value of k is

(a) $n + 1$ (b) $2n + 1$

(c) $3n - 1$ (d) n

34. Let $a, b, c > 0$ and a, b, c be in A.P. If a^2, b^2, c^2 are in G.P., then

(a) $a = b = c$ (b) $a^2 + b^2 = c^2$

(c) $a^2 + c^2 = 3b^2$ (d) none of these

35. Sum of the infinite series

$\frac{1}{3} + \frac{3}{(3)(7)} + \frac{5}{(3)(7)(11)} + \frac{7}{(3)(7)(11)(15)} + \ldots$

is

(a) 1/2 (b) 1/3 (c) 1/6 (d) 1/4

36. Let $S = \frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + ...$ upto ∞. Then S is equal to

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33. If a, b, c are in G.P. as well as x_1, x_2, x_3, x_4 are in G.P., then p, q, r are in
(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

34. If x_1, x_2, x_3, x_4 are in G.P., then its common ratio is
(a) $\left(\frac{cr}{ap}\right)^{1/4}$ (b) $\left(\frac{cr}{ap}\right)^{1/3}$
(c) $\sqrt{\frac{cr}{ap}}$ (d) $\sqrt{\frac{ap}{bq}}$

35. If x_1, x_2, x_3, x_4 are in A.P., with common difference d, then
(a) $\frac{a}{p} = \frac{b-4ad}{q} = \frac{4ad^2 - 2bd + c}{r}$
(b) $\frac{a}{p} = \frac{b-ad}{q} = \frac{ad^2 - bd + c}{r}$
(c) $\frac{a}{p} = \frac{b-2ad}{q} = \frac{ad^2 - bd + c}{r}$
(d) none of these

Paragraph for Question Nos. 36 to 40

A man is to receive Rs P_i at the end of ith year for n years. Assume that the rate of interest is Rs r per rupee per year and the interest is compounded annually.

36. If $P_i = P \ \forall \ i$, then present value of income stream is
(a) $\frac{P}{r}\left[1 - (1+r)^{-n}\right]$
(b) $\frac{P}{r}\left[1 + r - (1+r)^{-n}\right]$
(c) $P\left[1 - (1+r)^{-n}\right]$
(d) none of these

37. If $P_i = P \ \forall \ i$ and $n \to \infty$, then present value of the income stream is
(a) P (b) $2P$
(c) P/r (d) $P + P/r$

38. If $P_i = iP \ \forall \ i$, then present value of the income stream is
(a) $\frac{P(1+r)}{r^2}\left[1 - (1+r)^{-n}\right]$
(b) $\frac{P(1+r)}{r^2}\left[1 - (1+r)^{-n}\right] - \frac{nP}{r(1+r)^n}$
(c) $\frac{P(1+r)}{r^2}\left[1 - (1+r)^{-n}\right] - \frac{nP}{(1+r)^n}$
(d) none of these

39. If $P_i = iP \ \forall \ i$ and $n \to \infty$, then present value of the income stream is
(a) $\frac{P(1+r)}{r^2}$ (b) $\frac{P}{r^2}$
(c) rP (d) $r(1-r)P$

40. If $P_i = iP \ \forall \ i$ and $n \to \infty$, then value of r for which present value of income stream is $3P$ is
(a) 0.1 (b) 0.2 (c) 0.3 (d) 0.5

SECTION IV

INTEGER ANSWER TYPE

41. Three numbers a, b, c are chosen from the numbers 1, 2, 3, ... 100. Find the number of ways of choosing, a, b, c so that these are in A.P.

42. Find the largest positive term of the A.P. whose first two terms are 2/5 and 12/23.

43. Find the value of 2551 S where $S = \sum_{k=1}^{50} \frac{k}{k^4 + k^2 + 1}$.

44. Let a = maximum value of the function $f(x) = \frac{1}{8}(6x^2 - x^3 - 16)$ and $r = \lim_{x \to \infty} \frac{1 - \cos x}{x^2}$. If $S = a + ar + ar^2 + \cdots$, find $\sqrt{S}$.

45. If $a = \sum_{n=1}^{\infty} \frac{1}{n^n}$, then find $[a]$, where $[x]$ denoted the greatest integer $\le a$.

46. Let $a_n = \underbrace{11\cdots1}_{n \text{ times}}$. Find the number of primes in the sequence a_2, a_3, a_4, a_5, a_6.

47. Find the sum of the series
$$\sum_{k=1}^{\infty} \frac{6^k}{(3^k - 2^k)(3^{k+1} - 2^{k+1})}.$$

SECTION V

MATRIX MATCH TYPE

48. Let $a_n = \frac{1}{n} \forall \ n \ge 1$ and $[x]$ denotes the greatest integer $\le x$.

(a) $\left[\frac{1}{50} + a_1\right] + \left[\frac{1}{50} + a_3\right] + \cdots \left[\frac{1}{50} + a_{101}\right]$	(p) 2
(b) $[a_1 + a_{100}] + [a_2 + a_{99}] + \cdots + [a_{100} + a_1]$	(q) 1
(c) $[a_1 + a_2 + \cdots + a_{11}]$	(r) 3
(d) $[a_1 + a_2 + \cdots + a_{11} - 2]$	(s) 0

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71. $$a\left(\frac{1}{r}+1+r\right)=\alpha S \quad (1)$$

$$a^2\left(\frac{1}{r^2}+1+r^2\right)=S^2 \quad (2)$$

Divide (2) by (1) to obtain

$$a\left(\frac{1}{r}-1+r\right)=\frac{S}{\alpha} \quad (3)$$

From (2) and (3)

$$2a=S\left(\alpha-\frac{1}{\alpha}\right)=S\left(\frac{\alpha^2-1}{\alpha}\right)$$

Putting this in (2) we get

$$\frac{(\alpha^2-1)^2}{4\alpha^2}\left(\frac{1}{r^2}+1+r^2\right)=1$$

$$\Rightarrow \left(r-\frac{1}{r}\right)^2+3=\frac{4\alpha^2}{(\alpha^2-1)^2} \quad (4)$$

$\Leftrightarrow 3\alpha^4-10\alpha^2+3<0$

$\Leftrightarrow (3\alpha^2-1)(\alpha^2-3)<0$

$\Leftrightarrow 1/3<\alpha^2<3$

But $\alpha^2=1 \Rightarrow a=0$. Not possible.

72. Putting $\alpha^2=2$ in (4)

$(r-1/r)^2=5$

$\Rightarrow r-1/r=\sqrt{5}$, $[\because r>1]$

$\Rightarrow r^2-\sqrt{5}r-1=0$

$\Rightarrow r=\frac{\sqrt{5}+3}{2}$ $[\because r>1]$

73. Put $r=\sqrt{2}$ in (4) and solve.

74. $$a\left(\frac{1}{r}+1+r\right)=\left(\frac{1}{2}\right)(20)$$

$$\Rightarrow \frac{10}{a}=\left(\sqrt{r}-\frac{1}{\sqrt{r}}\right)^2+3\geq 3$$

$\Rightarrow a\leq 10/3$

Also, $$a^2\left(\frac{1}{r^2}+1+r^2\right)=20^2$$

$\Rightarrow a\leq 20/\sqrt{3}$

Thus, $a\leq 10/3$.

75. From (4) in solution of 71

$$\left(r-\frac{1}{r}\right)^2=0 \Rightarrow r=\pm 1.$$

76. $a_1=1$ and

$$a_{n+1}^2=|(1-\alpha)^2+\alpha^2|\,a_n^2$$

$$=\frac{5}{9}a_n^2=\left(\frac{5}{9}\right)^n$$

Now, $A_n=a_n^2<\frac{1}{10}$

$$\Rightarrow \left(\frac{9}{5}\right)^{n-1}>10$$

$\Rightarrow 9^{n-1}>2(5^n)$

$\Rightarrow n\geq 5$.

77. If $\alpha=\frac{1}{2}$, $a_{n+1}^2=\frac{1}{2}a_n^2=\left(\frac{1}{2}\right)^n$

$\therefore A_1=1$ and $A_n=\left(\frac{1}{2}\right)^{n-1}$

$$\Rightarrow \sum_{n=1}^{\infty}A_n=2.$$

78. $a_1=1$, $a_{n+1}^2=(2\alpha^2-2\alpha+1)^n$

$$\therefore \sum_{n=1}^{\infty}A_n=\frac{1}{2\alpha-2\alpha^2}=\frac{8}{3}$$

$\Rightarrow 2\alpha-2\alpha^2=3/8$

$\Rightarrow 16\alpha^2-16\alpha+3=0$

$\Rightarrow \alpha=1/4, 3/4.$

79. Diagonal of $(n+1)$th square $=\sqrt{2}\,a_{n+1}$

Now $a_n=\sqrt{2}\,a_{n+1}$

$\Rightarrow a_n^2=2a_{n+1}^2$

$\Rightarrow a_n^2=2(2\alpha^2-2\alpha+1)a_n^2$

$\Rightarrow \alpha=1/2.$

80. $P_1=4$,

$$P_n=4a_n=4(1-2\alpha+2\alpha^2)^{(n-1)/2}$$

$$=4\left(\frac{5}{8}\right)^{(n-1)/2}$$

$$\therefore \sum_{n=1}^{\infty}P_n=\frac{4\sqrt{8}}{\sqrt{8}-\sqrt{5}}=\frac{8\sqrt{2}\left(\sqrt{8}+\sqrt{5}\right)}{3}$$

$$=\frac{8}{3}\left(4+\sqrt{10}\right).$$

81. Since a, x, y, z and b are in A.P., we have

$$a+b=x+z=2y$$

Also $x+y+z=15 \Rightarrow 2y+y=15$

$\Rightarrow 3y=15$ or $y=5$

$\therefore a+b=10$ (1)

Since a, α, β, γ and b are in H.P., we have

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{\alpha}+\frac{1}{\gamma}=\frac{2}{\beta}$$

Also $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{5}{3}\Rightarrow\frac{3}{\beta}=\frac{5}{3}$ or $\frac{1}{\beta}=\frac{5}{9}$

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CHAPTER 4

LOGARITHM

4.1 EXPONENTIAL FUNCTION

For $x \in \mathbf{R}$, we define

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

It can be proved for every $x \in \mathbf{R}$, the number e^x is a finite number. We call e^x the exponential function.

Properties

1. e^x is defined for each $x \in \mathbf{R}$.
2. $e^x > 0$ for all $x \in \mathbf{R}$.
3. $e^0 = 1$.
4. If $a, b \in \mathbf{R}$, $a < b$ then $e^a < e^b$.
5. $e^x \times e^y = e^{x+y}$ for all $x, y \in \mathbf{R}$.
6. $\frac{e^x}{e^y} = e^{x-y}$ for all $x, y \in \mathbf{R}$.
7. $(e^x)^y = e^{xy}$ for all $x, y \in \mathbf{R}$.
8. For each positive real number x there exists one and only one real number y such that $e^y = x$.
9. e^x is one to one function.

4.2 LOGARITHM

By property 8 above, given a positive real number x, there is one and only one real number y such that $e^y = x$, we call this number y as $\log x$ or $\log_e x$. It is also denoted by $ln\ x$.

In general, the logarithm to the base a ($a > 0$, $a \neq 1$) of the number $x (x > 0)$ is the number y such that $a^y = x$. Thus $y = \log_a x$ means $a^y = x$. For $a = 10$, $\log_{10} x$ is called the *common logarithm* and for $a = e$, $\log_e x$ is known as *natural logarithm.*

Properties

1. $\log_a 1 = 0$, $a > 0$, $a \neq 1$.
2. $\log_a a = 1$, $a > 0$, $a \neq 1$.
3. $\log_e e^x = x\ \forall\ x \in \mathbf{R}$ and $e^{\log_e x} = x\ \forall\ x > 0$ i.e. $\log_e x$ is the inverse function of exponential function e^x.
4. $\log_a (xy) = \log_a x + \log_a y$ for all $x, y > 0$ and $a > 0$, $a \neq 1$.
5. $\log_a (1/x) = -\log_a x\ \forall\ x > 0$, $a > 0$, $a \neq 1$.
6. $\log_a (x/y) = \log_a x - \log_a y$, $x, y > 0$, $a > 0$, $a \neq 1$.
7. If $a \in \mathbf{R}$ and $x > 0$, then we define

$$x^a = e^{a \log_e x} = \sum_{n=0}^{\infty} (a \log_e x)^n$$

8. $\log x^y = y \log x\ \forall\ x > 0$, $y \in \mathbf{R}$.
9. *(Change of base formula)* If $a > 0$, $a \neq 1$ then $\log_a x = \frac{\log_e x}{\log_e a}$. In fact if $a, b, c > 0$ and $a, c \neq 1$ then $\log_a b = \frac{\log_c b}{\log_c a}$.
10. If $0 < x < y$ then $\log x < \log y$ (i.e. $\log x$ is an increasing function). In particular if $x > 1$ then $\log x > 0$ and if $0 < x < 1$ then $\log x < 0$.
11. If $a > 1$ and $0 < x < y$ then $\log_a x < \log_a y$ and $a^x < a^y$. If $0 < a < 1$ and $0 < x < y$ then $\log_a x > \log_a y$ and $a^x > a^y$. See Fig. 4.1 and Fig. 4.2.

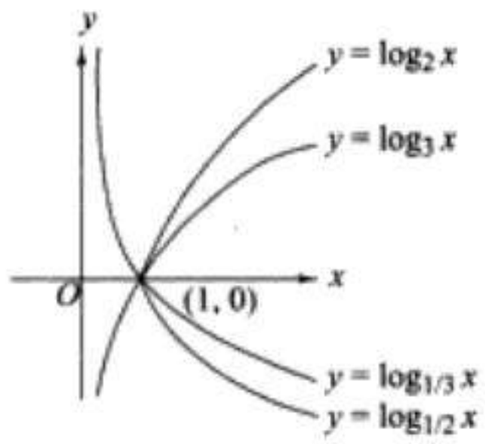

Fig. 4.1

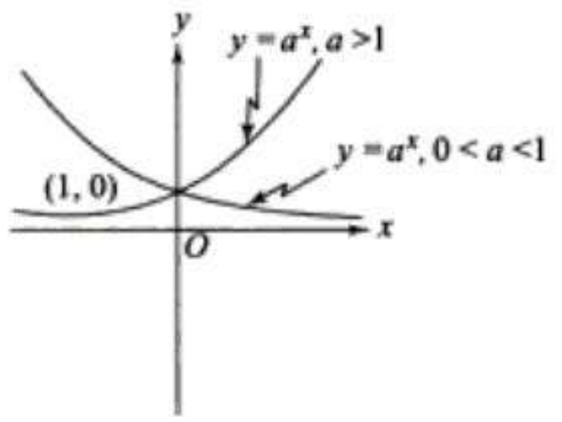

Fig. 4.2

12. For $|x| < 1$, $\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ so

$$\log (1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right) \text{ and}$$

$$\log \left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right]$$

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(a) $\frac{1}{\log_{53!} n}$ (b) 1

(c) $\frac{1}{\log_{n!} 53}$ (d) none of these

Ans. (a)

Solution The given expression is equal to

$$\log_n 2 + \log_n 3 + \ldots + \log_n 53$$

$$= \log_n (2.3 \ldots 53) = \log_n 53! = \frac{1}{\log_{53!} n}$$

Example 20 The number of values of $x \in [0, n\pi]$, $n \in \mathbf{I}$ that satisfy $\log_{|\sin x|} (1 + \cos x) = 2$ is

(a) 0 (b) n

(c) $2n$ (d) none of these

Ans. (a)

Solution The equation is meaningful if $|\sin x| \neq 0, 1$ and $1 + \cos x \neq 0$ so $x \neq k\pi$, $k = 0, 1, \ldots n$, $x \neq (2k + 1)\frac{\pi}{2}$, $k = 0, 1, \ldots n - 1$. Now,

$$\log_{|\sin x|} (1 + \cos x) = 2$$

$$\Leftrightarrow \quad 1 + \cos x = |\sin x|^2 = \sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \quad (1 + \cos x)(\cos x) = 0$$

$$\Leftrightarrow \quad \cos x = 0 \text{ or } \cos x = -1$$

$$\Rightarrow \quad \cos x = 0 \Rightarrow x = (2k + 1)\pi/2.$$

So there is no x which satisfy the given equation.

SECTION II

MULTIPLE CORRECT CHOICE TYPE

Example 21 The set of all x satisfying $4^{x^2+2} - (9)2^{x^2+2} + 8 = 0$ consists of

(a) infinitely many points,

(b) finitely many points from the set of all natural numbers,

(c) finitely many points from the set of all integers

(d) exactly two integers

Ans. (c), (d)

Solution Setting $2^{x^2+2} = t$, the original equation reduces to $t^2 - 9t + 8 = 0 \Rightarrow t = 1, 8$. If $t = 1$ then $2^{x^2+2} = 1 \Rightarrow x^2 + 2 = 0$ but this has no solution. If $2^{x^2+2} = 8$ then $x^2 + 2 = 3 \Rightarrow x = \pm 1$.

Example 22 The equation $x^{\left[(\log_5 x)^2 - \frac{9}{2}\log_5 x + 5\right]} = 5\sqrt{5}$ has

(a) exactly three real solution

(b) at least one real solution

(c) exactly one irrational solution

(d) complex roots

Ans. (a), (b), (c)

Solution Taking logarithm of both the sides, we get

$$\left((\log_5 x)^2 - \frac{9}{2}\log_5 x + 5\right)\log_5 x = \frac{3}{2}$$

$$2(\log_5 x)^3 - 9(\log_5 x)^2 + 10 \log_5 x - 3 = 0$$

$$\Leftrightarrow (\log_5 x - 1)(\log_5 x - 3)(2 \log_5 x - 1) = 0$$

$$\Leftrightarrow \quad \log_5 x = 1, \log_5 x = 3, 2 \log_5 x = 1$$

$$\Leftrightarrow \quad x = 5, 125, \sqrt{5}.$$

Example 23 If $\log_2 (3^{2x-2} + 7) = 2 + \log_2 (3^{x-1} + 1)$ then x equals

(a) 0 (b) 1

(c) 2 (d) none of these

Ans. (b), (c)

Solution The given expression can be written as $\log_2 (3^{2x-2} + 7) = \log_2 4(3^{x-1} + 1)$

$$\Rightarrow \quad 3^{2x-2} + 7 = 4(3^{x-1} + 1)$$

$$\Rightarrow \quad (3^{x-1})^2 - 4(3^{x-1}) + 3 = 0$$

$$\Rightarrow \quad (3^{x-1} - 1)(3^{x-1} - 3) = 0$$

$$\Rightarrow \quad 3^{x-1} = 1, 3^{x-1} = 3$$

$$\Rightarrow \quad x - 1 = 0, x - 1 = 1$$

$$\Rightarrow \quad x = 1, 2.$$

Example 24 Solution of $\log_{x^2+6x+8} \log_{2x^2+2x+3} (x^2 - 2x) = 0$ is

(a) a natural number (b) a negative integer

(c) -1 (d) none of these

Ans. (b), (c)

Solution We must have $x^2 + 6x + 8 > 0$, $2x^2 + 2x + 3 > 0$ and $x^2 + 6x + 8 \neq 1$, $2x^2 + 2x + 3 \neq 1$

$$\Rightarrow \quad x \in (-\infty, -4) \cup (-2, \infty), x \neq -3 \pm \sqrt{2} \qquad (1)$$

and $x^2 - 2x > 0$ i.e. $x \in (-\infty, 0) \cup (2, \infty)$

The given equation can be written as

$$\log_{(2x^2+2x+3)} (x^2 - 2x) = 1$$

$$\Rightarrow \quad x^2 - 2x = 2x^2 + 2x + 3$$

$$\Rightarrow \quad x^2 + 4x + 3 = 0 \Rightarrow x = -1, -3$$

$x = -3$ does not satisfy (1) so $x = -1$.

Example 25 $x^{\log_5 x} > 5$ implies

(a) $x \in (0, \infty)$ (b) $x \in (0, 1/5) \cup (5, \infty)$

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This, we can arrange V and N in $= {}^5P_2 = 20$ ways

Thus, the number of ways in which letters can be arranged is $4 \times 20 = 80$.

Example 28 The number of integral points (integral point means both the coordinates should be integer) that lie exactly in the interior of the triangle with vertices.
(0, 0), (0, 21), (21,0) is

(a) 133 (b) 190 (c) 233 (d) 105

Ans. (b)

Solution The integral points lying on the line $x = 1$ which also lie in the interior of ΔOAB are (1, 1), (1, 2), ..., (1, 19)
$\therefore$ there are 19 points lying on the line $x = 1$

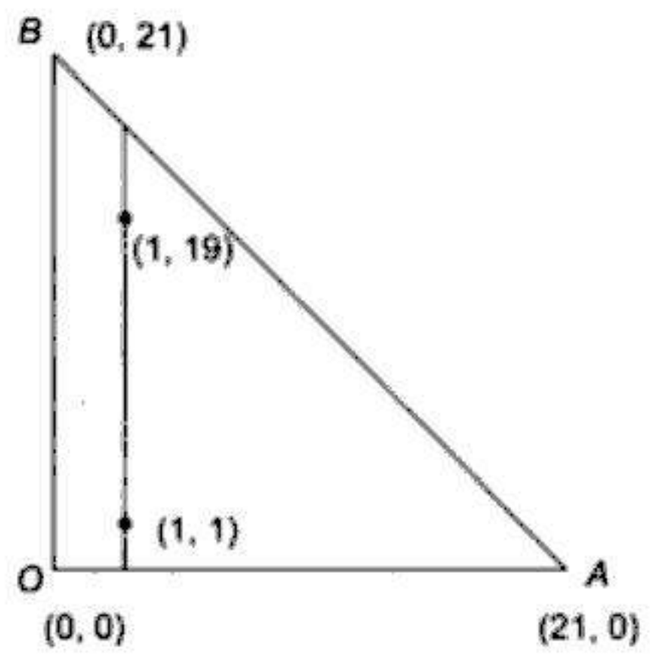

Fig. 5.1

Similarly, the number of integral points on the line $x = 2$ is 18, and on $x = 3$ is 17 and so on. The number of integral points on $x = 19$ is just 1.

Thus, number of integral points lying in the interior of the triangle is

$$19 + 18 + \ldots + 1 = 190.$$

Example 29 The range of function

$$f(x) = {}^{7-x}P_{x-3} \text{ is}$$

(a) {1, 2, 3, 4} (b) {0, 1, 2, 3, 4, 5}
(c) {0, 1, 2, 3} (d) {1, 2, 3, 4, 5, 6}

Ans. (c)

Solution For ${}^{7-x}P_{x-3}$ to be defined, we must have $7 - x \geq 1$ and $x - 3 \geq 0$.

$\Rightarrow$ $3 \leq x \leq 6$

$\therefore$ $x = 3, 4, 5, 6$

Now, $f(3) = {}^4P_0 = 1, f(4) = {}^3P_1 = 3,$
$f(5) = {}^2P_2 = 2$ and $f(6) = 0$

$\therefore$ range of $f = \{0, 1, 2, 3\}$.

Example 30 The number of ways of arranging letters of the word RACHIT so that the vowels are in alphabetical order is

(a) 120 (b) 240 (c) 360 (d) 480

Ans. (c)

Solution We can arrange letters of the word RACHIT in 6! ways. Out of these exactly half, that is, there are $\frac{1}{2}(6!) = 360$ ways to arrange the letters of the word RACHIT so that vowels are in alphabetical order.

Example 31 If ${}^{n-1}C_r = (K^2 - 3)({}^nC_{r+1})$, then K belongs to

(a) $[-\sqrt{3}, \sqrt{3}]$ (b) $(-\infty, -2)$
(c) $(2, \infty)$ (d) $(\sqrt{3}, 2]$

Ans. (d)

Solution ${}^{n-1}C_r = (K^2 - 3)({}^nC_{r+1})$

$$\Rightarrow \quad K^2 - 3 = \frac{(n-1)!}{r!(n-1-r)!} \cdot \frac{(r+1)!(n-r-1)!}{n!}$$

$$= \frac{r+1}{n}$$

But $0 \leq r \leq n-1 \Rightarrow 1 \leq r+1 \leq n$

$$\Rightarrow \quad \frac{1}{n} \leq \frac{r+1}{n} \leq 1 \Rightarrow \frac{1}{n} \leq K^2 - 3 \leq 1$$

$$\Rightarrow \quad \frac{1}{n} + 3 \leq K^2 \leq 4$$

$$\Rightarrow \quad 3 < K^2 \leq 4 \qquad \left[\because \frac{1}{n} > 0\right]$$

$$\Rightarrow \quad -2 \leq K < -\sqrt{3} \text{ or } \sqrt{3} < K \leq 2$$

Thus, $K \in (\sqrt{3}, 2]$

Example 32 A five digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is

(a) 216 (b) 240 (c) 600 (d) 3125

Ans. (a)

Solution The sum of the numerals 0, 1, 2, 3, 4 and 5 is 15. We know that a five digit number is divisible by 3 if and only if the sum of its digits is divisible by 3. Therefore, we should not use either 0 or 3 while forming the five digit numbers. If we do not use 0, then the remaining digits can be arranged in ${}^5P_5 = 5! = 120$ ways. If we do not use 3, then the remaining digits can be arranged in ${}^5P_5 - {}^4P_4 = 5! - 4! = 120 - 24 = 96$ ways to obtain a five digit number. Thus, the total number of such 5 digit numbers is $120 + 96 = 216$.

Example 33 The number of ways in which a mixed double game can be arranged from amongst 9 married couples if no husband and wife play in the same game is

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Example 72 The number of solutions of the system of congruences

$$x \equiv -3 \pmod{49}$$

and $$x \equiv 2 \pmod{11}$$

is

(a) 0 (b) 1
(c) 2 (d) infinite

Example 73 The number of values of $x \in \mathbf{I}$ for which $2x \equiv 1 \pmod{2008}$ is

(a) 0 (b) 1
(c) 2 (d) infinite.

Example 74 The smallest values of $x \geq 100$ for which $29x \equiv 1 \pmod{17}$ is

(a) 111 (b) 112
(c) 113 (d) 114

Example 75 If $hcf\,(a, 5) = 1$, then

(a) $a^4 \equiv 1 \pmod 5$ (b) $a^5 \equiv 1 \pmod 5$
(c) $a^6 \equiv 1 \pmod 5$ (d) $a^7 \equiv 1 \pmod 5$

Ans. 71. (b), 72. (d), 73. (a), 74. (b), 75. (a)

Solution

71. $100 \equiv 2 \pmod 7$

$\Rightarrow 100^3 \equiv 2^3 \equiv 1 \pmod 7$

$\Rightarrow 100^{72} \equiv (100^3)^{24} \equiv 1 \pmod 7$

Thus, $100^{73} \equiv 2 \pmod 7$

72. $x \equiv -3 \pmod{49}$

$\Rightarrow x = -3 + 49$ m where $m \in \mathbf{I}$.

Putting this in the second equation, we obtain

$-3 + 49m = 2 \pmod{11}$

$\Rightarrow \quad 5m = 5 \pmod{11}$

$\Rightarrow \quad m \equiv 1 \pmod{11}$

$\Rightarrow \quad m = 1 + 11t$ where $t \in \mathbf{I}$.

$\therefore \quad x = 46 + 539s$ where $s \in \mathbf{I}$.

73. 2008 cannot divide $2x - 1$ as $2x - 1$ is odd

74. Write the equation as $5x \equiv -1 \pmod{17}$ and note that $102 + m \equiv m \pmod{17}$

$\therefore 111 \equiv 9, 112 \equiv 10, 113 \equiv 11, 114 \equiv 12 \pmod{29}$

Now check $x \equiv 10$ satisfies $5x \equiv -1 \pmod{29}$

75. As $hcf\,(a, 5) = 1$, there exists $k \in \mathbf{N}$, $1 \leq k \leq 4$, such that $a \equiv k \pmod 5$. But for $1 \leq k \leq 4$, $k^4 \equiv 1 \pmod 5$

Paragraph for Question Nos. 76 to 80

For $n \in \mathbf{N}$, let $\sigma(n)$ denote the sum of all divisors of n. A number is said to be perfect if $\sigma(n) = 2n$.

Example 76 If $n \in \mathbf{N}$ and $\{d_1, d_2, \ldots, d_k\}$ is the set of all its divisors, then $\sum_{j=1}^{k} \frac{1}{d_j}$ is equal to

(a) $\sigma(n)$ (b) $n\ \sigma\,(n)$
(c) $\sigma(n)/n$ (d) none of these

Example 77 If p is a prime, then $(p - 1)\ \sigma\,(p^n)$ equals

(a) p^{n+1} (b) $p^{n+1} - 1$
(c) $p^{n+1} + 1$ (d) none of these

Example 78 If p, q are two distinct primes, then $\sigma\,(pq)$ equals

(a) $\sigma(p) + \sigma(q)$ (b) $\sigma(p)\ \sigma(q)$
(c) $q\sigma(p) + p\sigma(q)$ (d) none of these

Example 79 If n is a perfect number, then $\sum_{d|n} \frac{1}{d}$ equals

(a) 1 (b) 2
(c) n (d) none of these

Example 80 The number of natural numbers n for which $\sigma(n) = 10$, is

(a) 0 (b) 1
(c) 2 (d) 3

Ans. 76. (c), 77. (b), 78. (b), 79. (b), 80. (a)

Solution

76. Use $d \left| n \Leftrightarrow \frac{n}{d} \right| n$

Thus, $\sum_{j=1}^{k} \frac{1}{d_j} = \sum_{j=1}^{n} \frac{d_j}{n} = \frac{1}{n}\ \sigma(n)$

77. $(p - 1)\ \sigma\,(p^n)$

$= (p - 1)(1 + p + p^2 + \ldots + p^n)$

$= p^{n+1} - 1$

78. $\sigma(pq) = 1 + p + q + pq = (1 + p)(1 + q)$

79. Using Example 76, we get

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n} = \frac{2n}{n} = 2$$

80. As $\sigma(n) \leq n + 1$, $n \geq 9$. But for $1 \leq k \leq 9$, $\sigma(k) \neq 10$

SECTION IV

INTEGER ANSWER TYPE

Example 81 If ${}^{n+2}C_8 : {}^{n}P_4 = 57 : 16$ find n.

Ans. 19.

Solution ${}^{n+2}C_8 : {}^{n}P_4 = 57 : 16$

$$\Rightarrow \quad \frac{(n+2)!}{8!\,(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad (n+2)(n+1)n(n-1) = \frac{57}{16} \times 8! = 143640$$

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(b) The number of ways
$= {}^{12}C_3 + {}^{12}C_4 + {}^{12}C_5 + \ldots + {}^{12}C_{12}$
$= 2^{12} - ({}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2) = 4096 - (1 + 12 + 66)$
$= 4017.$

(c) The prime factorization of 210 is $2 \cdot 3 \cdot 5 \cdot 7$. We can assign 2 to any of x_1, x_2, x_3 or x_4. That is, 2 can be assigned in 4 ways. Similarly, 3(5) [7] can be assigned in 4 ways. A variable which is not assigned any prime will be assigned value 1. Thus, the number of solutions of the given equation is (4) (4) (4) (4) = 256.

(d) Note that we cannot use 0 at any place. The required number of five digit numbers is equal to the number of ways of selecting 5 digits out of 9

$$^9C_5 = \frac{9\times8\times7\times6}{4\times3\times2} = 126.$$

Example 98 Consider all possible permutations of the letters of the word R A C H I T I H C A R

(a) The number of words containing the word ACHIT is	(p) 56700
(b) The number of words beginning with RA and ending with AR is	(q) 630
(c) The number of words in which vowels occur at the odd places is	(r) 45360
(d) The number of words in which the word IIT appears is	(s) 2520

Ans.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

Solution (a) We can permute ACHIT, R, R, I, H, C, A in $\frac{7!}{2!} = 2520$ ways.

(b) We can permute C, C, H, H, I, I, T at 7 places in $\frac{7!}{2!2!2!} = 630$ ways.

(c) 4 places out of 6 odd places in ${}^6C_4$ ways and arrange A, A, I, I in $\frac{4!}{2!2!}$. Therefore, vowels can be arranged in $\left({}^6C_4\right)\left(\frac{4!}{2!2!}\right)$ ways. Remaining letters R, R, C, C, H, H, T can be arranged $\frac{7!}{2!2!2!}$ ways.

Therefore, the desired number of ways

$$= (15)\ \frac{4!}{2!2!}\ \frac{7!}{2!2!2!} = 56700$$

(d) The number of words in which IIT appears
= number of ways of arranging
IIT, R, R, A, A, C, C, H, H

$$= \frac{9!}{2!2!2!2!} = 45360$$

SECTION VI

REASONING TYPE

Example 99 **Statement-1:** The expression

$$\binom{40}{r}\binom{60}{0} + \binom{40}{r-1}\binom{60}{1} + \cdots$$

attains maximum value when $r = 50$.

Statement-2: $\binom{2n}{r}$ is maximum when $r = n$.

Ans. (a)

Solution We have

$$\frac{\binom{2n}{r}}{\binom{2n}{r+1}} = \frac{(2n)!}{r!(2n-r)!}\ \frac{(r+1)!(2n-r-1)!}{(2n)!}$$

$$= \frac{r+1}{2n-r}$$

Since for $0 \le r \le n-1$, $\frac{r+1}{2n-r} < 1$, we get

$$\binom{2n}{0} < \binom{2n}{1} < \ldots < \binom{2n}{n-1} < \binom{2n}{n}$$

Also, as $\binom{2n}{r} = \binom{2n}{2n-r}$

$$\binom{2n}{2n} < \binom{2n}{2n-1} < \ldots < \binom{2n}{n+1} < \binom{2n}{n}$$

Thus, $\binom{2n}{r}$ is maximum when $r = n$.

Next,

$$\binom{40}{r}\binom{60}{0} + \binom{40}{r-1}\binom{60}{1} + \ldots$$

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22. Out of 10 white, 8 black and 6 red balls, the number of ways in which one or more balls can be selected is given by
(a) 681 (b) 691 (c) 679 (d) 692

23. A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q$ contains exactly two elements is
(a) $9 \times {}^nC_2$ (b) $3^n - {}^nC_2$
(c) $2 \times {}^nC_n$ (d) ${}^nC_2 \cdot 3^{n-2}$

24. Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. The number of words which have at least one of their letters repeated is
(a) 69760 (b) 30240
(c) 99748 (d) none of these

25. The number of ways in which we can select four numbers from 1 to 30 so as to exclude every selection of four consecutive numbers is
(a) 27378 (b) 27405
(c) 27397 (d) none of these

26. If ${}^nC_3$, ${}^nC_5$ and ${}^nC_6$ are in A.P., the value of n can be
(a) 14 (b) 11 (c) 9 (d) 5

27. The letters of the word SURITI are written in all possible orders and these words are written out as in a dictionary. Then the rank of the word SURITI is
(a) 224 (b) 245 (c) 307 (d) 315

28. A candidate is required to answer 6 out of 10 questions which are divided into two groups, each containing 5 questions. He is not permitted to attempt more than 4 questions from either group. The number of different ways in which the candidate can choose six questions is
(a) 50 (b) 150 (c) 200 (d) 250

29. Let $A = \{1, 2, 3 \ldots n\}$, $B = \{a, b, c\}$, then the number of functions from A to B that are onto is
(a) $3^n - 3(2^n - 1)$ (b) $3^n - 2^n$
(c) $3(2^n - 1)$ (d) $(n-1)!$

30. If $\alpha = {}^mC_2$, where $m \geq 2$, then the value of ${}^\alpha C_2$ is given by
(a) ${}^{m+1}C_4$ (b) ${}^{m-1}C_4$
(c) ${}^{m+2}C_4$ (d) $3\,({}^{m+1}C_4)$

31. There are n white and n black balls marked 1, 2, $\cdots$, n. The number of ways in which we can arrange these balls in a row so that neighbouring balls are of different colours is
(a) $n!$ (b) $(2n)!$ (c) $2(n!)^2$ (d) $\dfrac{(2n)!}{(n!)^2}$.

32. The number of squares which we can form on a chessboard is
(a) 64 (b) 160 (c) 224 (d) 204

33. The number of 5 digit numbers that contain 7 exactly once is
(a) $(41)(9^3)$ (b) $(37)(9^3)$
(c) $(7)(9^4)$ (d) $(41)(9^4)$

34. The units digit of $17^{2(89)} + 11^{2(88)} \cdot 7^{2(88)}$ is
(a) 1 (b) 8 (c) 2 (d) 9

35. If $\dfrac{1}{{}^4C_n} = \dfrac{1}{{}^5C_n} + \dfrac{1}{{}^6C_n}$, then value of n is
(a) 3 (b) 4
(c) 1 (d) none of these

36. At an election there are five candidates and three members are to be elected, and a voter may vote for any number of candidates not greater than the number to be elected. The number of ways in which the person can vote is
(a) 25 (b) 30 (c) 35 (d) $2^5 - 2^3$

37. If n is odd and ${}^nC_0 < {}^nC_1 < {}^nC_2 < \ldots < {}^nC_r$, then maximum possible value of r is
(a) $\dfrac{1}{2}(n+1)$ (b) $\dfrac{1}{2}(n-1)$
(c) n (d) none of these

38. The number of arrangements of the letters of the word BANANA in which two N's do not appear adjacently is
(a) 40 (b) 60 (c) 80 (d) 100

39. The number of positive integral solutions of $x_1 x_2 x_3 x_4 x_5 = 840$ is
(a) 3125 (b) 6125 (c) 6250 (d) 7250

40. The number of ways we can put 5 different balls in 5 different boxes such that at most three boxes is empty, is equal to
(a) $5^5 + 5$ (b) $5^5 - 10$
(c) $5^5 - 5$ (d) none of these

41. The number of ordered pairs of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$ is
(a) 2 (b) 4 (c) 6 (d) 8

42. The number of the divisors of 20! is
(a) 4140 (b) 41040 (c) 4204 (d) 81650

43. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an English disctionary. The number of words that appear before the word COCHIN is
(a) 360 (b) 192
(c) 96 (d) 48

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93. A man is allowed to take at most to steps either forward or backward.
Statement-1: The number of ways in which he will be just two steps away from his initial position is 572.
Statement-2: The number of ways of in which he will be just two steps forward from his initial position is 31 ($^5P_5$).

LEVEL 2

SECTION I

SINGLE CORRECT CHOICE TYPE

1. The number of 5-digit numbers of distinct digits in which the middle digit is the largest is
(a) $\sum_{r=1}^{9} {}^rP_4$ (b) $\sum_{r=4}^{9} {}^9P_r$
(c) 10078 (d) none of these

2. Let $m = 2^{20} - 1$, $n = 2^{110} - 1$, then $hcf(m, n)$ is
(a) 41 (b) 205 (c) 1025 (d) 1023

3. The ten's digit of $23^{41} \times 25^{81}$ is
(a) 2 (b) 3 (c) 5 (d) 7

4. Let $n = 100^{50} - 43$, the sum of the digits in the decimal notation of n is
(a) 890 (b) 893 (c) 895 (d) 897

5. The number of non-negative integral solutions of $3x + y + z \le 25$ is
(a) 825 (b) 1225 (c) 1215 (d) 1325

6. If $[x]$ denote the greatest integer $\le x$ and $n = 100$, the value of
$$\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\left[\frac{n+8}{16}\right]+\ldots$$
is
(a) 50 (b) 75
(c) 100 (d) 125
$\left[\text{Hint : use } \left[\frac{x}{2}\right]+\left[\frac{x+1}{2}\right]=[x]\right]$

7. The number of zeros at the end of $(5^{100} - 1)!$ is
(a) 100 (b) $\frac{1}{4}(5^{100} - 1)$
(c) $\frac{1}{4}(5^{100} - 26)$ (d) $\frac{1}{4}(5^{100} - 401)$

8. The last digit of
$(1! + 2! + \ldots + 2007!)^{8(8)}$ is
(a) 1 (b) 2 (c) 7 (d) 9

9. The remainder when
$(1 + 2! + \ldots + 2007!)^{500}$
is divided by 7
(a) 1 (b) 2 (c) 3 (d) 4

10. The number of ways of choosing three distinct numbers from the set {0, 1, 2, ..., 29} so that sum of the numbers is 30, is
(a) 75 (b) 80 (c) 85 (d) 98

11. If $1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{23} = \frac{a}{23!}$, then remainder when a is divided by 49 is
(a) 0 (b) 1
(c) 2 (d) 3

12. Let $[x]$ = greatest integer $\le x$. Two rational numbers x and y which satisfy $x + y = 2$ and $2x^{2007} - 3y^{2006} + x^{2008} + (x - y)\log_2 13 = 0$, then $x - y$ equals
(a) 0 (b) 1 (c) 2 (d) – 1

13. Find the number of ways of choosing two distinct natural numbers ≤ 100, such that they differ by at most ten is
(a) 935 (b) 940 (c) 945 (d) 950
[Hint: Use the following diagram.

$a + b + c = 100 - 2$, $a, c \ge 0$, $0 \le b \le 9$]

14. If $\sum_{r=0}^{n} \frac{1}{{}^nC_r} = a$, and
$$S = \sum\sum_{0 \le i \le j \le n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j}\right)$$, then S equals
(a) $na/4$ (b) $n^2a/2$
(c) $na/2$ (d) $na^2/2$

15. The number of ways of distributing 10 identical objects among 8 persons is
(a) $^{10}C_8$ (b) $^{17}C_7$
(c) 10^8 (d) 8^{10}

16. Let $S = 1! + 4! + 7! + \ldots + 400!$, then ten's digit of S is
(a) 1 (b) 2 (c) 6 (d) 7

17. A leaf is torn from a book. If sum of the numbers of the remaining pages is 4,851 the missing pages are
(a) 51, 52 (b) 49, 50
(c) 77, 78 (d) 37, 38

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done is
(a) 216 (b) 240
(c) 600 (d) 3125 [1989]

17. Eighteen guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the sitting arrangements can be made. [1991]

18. There are four balls of different colours and four boxes of the same colours as those of the balls. The number of ways in which the balls could be placed one each in a box, such that a ball does not go to a box of its own colour, is _____ [1992]

19. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1\ \forall\ k \geq n$, show that $b_n = {}^{2n+1}C_{n+1}$ [1992]

20. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots is
(a) 15 (b) 9
(c) 7 (d) 8 [1994]

21. A committee of 12 is to be formed from nine women and eight men. In how many ways can this be done if at least five women have to be included in a committee?
In how many of these committees
(a) the women are in majority?
(b) the men are in majority? [1994]

22. Let n and k be positive integers such that $n \geq \frac{k(k+1)}{2}$. The number of solutions $(x_1, x_2, \ldots, x_k)$, $x_1 \geq 1, x_2 \geq 2, \ldots, x_k \geq k$, all integers, satisfying $x_1 + x_2 + \ldots + x_k = n$, is _____. [1996]

23. In how many ways three girls and nine boys can be seated in two vans each having numbered seats, 3 in front and 4 at the back? How many seating arrangements are possible if 3 girls sit together in a back row on adjacent seats? Now, if all seating arrangements are equally likely, what is the probability of 3 girls sitting together in a back row on adjacent seats? [1996]

24. Number of divisors of the form $4n + 2$ $(n \geq 0)$ of integers 240 is
(a) 4 (b) 8
(c) 10 (d) 3 [1998]

25. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals
(a) $(n-1)a_n$ (b) $n a_n$
(c) $\frac{1}{2} n a_n$ (d) none of these [1998]

26. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible is
(a) 6 (b) 7
(c) 8 (d) 9 [1998]

27. In a college of 300 students, every student reads 5 newspapers and every newspaper is read by 60 students. The number of newspapers is
(a) at least 30 (b) at most 20
(c) exactly 25 (d) none of the above [1998]

28. For $2 \leq r \leq n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$
(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$
(c) $\binom{n+2}{r}$ (d) $\binom{n+2}{r-1}$ [2000]

29. How many different nine digit number can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions?
(a) 16 (b) 36
(c) 60 (d) 180 [2000]

30. For any positive integer m, n (with $n \geq m$), let $\binom{n}{m} = {}^nC_m$. Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \ldots + \binom{m}{m} = \binom{n+1}{m+1}$$

Hence or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \ldots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$$ [2000]

31. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals
(a) 5 (b) 7
(c) 6 (d) 4 [2001]

32. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then the number of onto function from E to F is
(a) 14 (b) 16 (c) 12 (d) 8

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15. For the first place we have nine choice, for each of the next four 10 choices. At this stage add the numbers already written and the choose digit for the unit's place is 5 ways.

16. A_k is obtained by putting $x = -k$ every where in the expression except in $x + k$. Therefore

$$A_k = \frac{m!}{(-k)\ldots(-k+k-1)(-k+k+1)\ldots(-k+m)}$$

$$= \frac{(-1)^k\ m!}{(k!)\,(m-k)!} = (-1)^k\,({}^mC_k)$$

17. Consider the product
$(1 + 1)\ (1 + 2)\ \ldots\ (1 + n)$

18. Consider the product

$$(1 + 1)\left(1+\frac{1}{2}\right)\ldots\left(1+\frac{1}{n}\right)$$

19. $(M, M), (A, A), (T, T), E, H, I, C, S$
We can arrange 4 distinct letters in ${}^8P_4$ ways, 2 identical and 2 distinct in $({}^3C_1)({}^7C_2)\frac{4!}{2!}$ ways and

2 pairs of identical letters in $({}^3C_2)\ \frac{4!}{2!2!}$ ways

20. $({}^{mn}C_n)\ ({}^{mn-n}C_n)\ \ldots\ ({}^nC_n)$

21. ${}^nC_2 - n = 90$

22. $(10 + 1)\ (8 + 1)\ (6 + 1) - 1$

23. Choose two elements ${}^nC_2$ ways, and rest of the $(n - 2)$ elements in 3^{n-2} ways.

24. $10^5 - {}^{10}P_5$

25. ${}^{30}C_4 - 27$

26. $2({}^nC_5) = {}^nC_4 + {}^nC_6 \quad \Rightarrow 2 = \frac{{}^nC_4}{{}^nC_5} + \frac{{}^nC_6}{{}^nC_5}$

27. The letters of SURITI can be arranged in 6!/2! = 720/2 = 360 ways. The alphabetic order is I, R, S, T and U. The number of words beginning I is ${}^5P_5$ = 5! = 120, and those beginning R number 5!/2! = 120/2 = 60. Then come words beginning SI, numbering 4! = 24, and SR, numbering 4!/2! = 12. We then get ${}^3P_3 = 3! = 6$ words beginning SUI, after which come the words SURIIT and SURITI. Thus the rank of SURITI is
$120 + 60 + 24 + 12 + 6 + 2 = 224.$

28. The number of ways the candidate can choose questions under the given conditions is enumerated below.

Group 1	4	3	2
Group 2	2	3	4
No. of ways	$({}^5C_4)\,({}^5C_2)$	$({}^5C_3)\,({}^5C_3)$	$({}^5C_2)\,({}^5C_4)$

From this we see that the required number is
$({}^5C_4)\,({}^5C_2) + ({}^5C_3)\,({}^5C_3) + ({}^5C_2)\,({}^5C_4)$
$= (5)\,(10) + (10)\,(10) + (10)\,(5) = 200.$

29. Total number of functions = 3^n
Number of functions with exactly two elements in range is $({}^3C_2)\,(2^n - 2)$ and with exactly one element in range is 3.

30. $\alpha = \frac{1}{2}\,m(m-1)$ and ${}^nC_2 = \frac{1}{2}\,\alpha(\alpha - 1)$

31. The balls must be of alternate colours.

32. For a square of size $k \times k$, we have to choose $(k + 1)$ consecutive horizontal and $(k + 1)$ consecutive vertical lines from the chess board, and this can be done in $(9 - k) \times (9 - k)$ ways.

33. If 7 is used at first place, the number of numbers is 9^4 and for any other four place it is 8×9^3.

34. Unit's digit of $17^{2009} - 7^{2009}$ is 0.

35. $1 = \frac{{}^4C_n}{{}^5C_n} + \frac{{}^4C_n}{{}^6C_n} \Rightarrow n = 2$

36. ${}^5C_1 + {}^5C_2 + {}^5C_3 = 25$

37. If n is odd, ${}^nC_r$ is maximum when $r = \frac{1}{2}\,(n-1)$ or
$\frac{1}{2}\,(n+1)$

38. BAAA can be arranged in 4 ways and two N's can be adjusted in ${}^7C_2$ ways.

39. Use $= 840 = 2^3 \times 3 \times 5 \times 7$
2's can be assigned in ${}^3C_3 + {}^3C_2 + {}^3C_1$ ways and rest of the three digits in $5 \times 5 \times 5$ ways.

40. Total number of ways – number of ways in which exactly four boxes remain empty.

41. $(x + 3)^2 + y^2 = 13 \Rightarrow x + 3 = \pm 2,\ y = \pm 3$ or $x + 3 = \pm 3,\ y = \pm 2$

42. $20! = 2^{18} \times 3^8 \times 5^4 \times 7^2 \times 11 \times 13 \times 17 \times 19$

43. All the words beginning with CC, CH, CI and CN occur before COCHIN.

44. $({}^3C_1)\ \frac{5!}{3!} = 60$

45. $100! = 2^{97} \times 3^{48} \times 5^{24} \times \ldots$

46. $\frac{a_n}{a_{n+1}} = \frac{n+1}{10} < 1$ for $1 \le n \le 8$.

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Thus, the number of solutions is

$$18 + (17 + 15 + \ldots + 3 + 1) = 99$$

93. Required number of ways
= number of non-negative integral solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 6$
$= {}^{6+5-1}C_{5-1} = {}^{10}C_4 = \dfrac{10!}{4!6!}$
= The number of different ways of arranging 6A's and 4B's.

94. The man will be just two steps forward from his initial position (in at most 10 steps) if his movement is as follows:

2F or 3F; 1B or 4F, 2B or 5F, 3B or 6F, 4B

Thus, required number of ways

$= {}^{2}C_0 + {}^{4}C_1 + {}^{6}C_2 + {}^{8}C_3 + {}^{10}C_4 = 286$

$\therefore$ Statement-2 is false.

Also, the number of ways in which the man is just two steps backwards = 286.

$\therefore$ required number of ways = 2(286) = 572.

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$$2S = {}^{59}C_0 + {}^{59}C_1 + \ldots + {}^{59}C_{59} = 2^{59}$$

$$\Rightarrow \quad S = 2^{58}$$

Example 23 The sum of the rational terms in the expansion of $\left(2^{1/5} + \sqrt{3}\right)^{20}$ is

(a) 71 (b) 85
(c) 97 (d) none of these

Ans. (d)

Solution $(r + 1)$th term of $\left(2^{1/5} + \sqrt{3}\right)^{20}$ is

$$t_{r+1} = {}^{20}C_r (2^{1/5})^{20 - r} (3^{1/2})^r = {}^{20}C_r \, 2^{4 - r/5} \, (3)^{r/2}$$

As 2 and 3 are relatively prime $2^{4 - r/5}$ and $3^{r/2}$ will be rational if both $4 - r/5$ and $r/2$ are integers, i.e. if $r/5$ and $r/2$ are integers. This is possible if $r = 0, 10, 20$.

Thus, the sum of rational terms is

$${}^{20}C_0(2^4)\,(3^0) + {}^{20}C_{10}\,(2^2)\,(3^5) + {}^{20}C_{20}\,(2^0)\,(3^{10})$$

As this is more than 71, 85 or 97, the answer is (d).

Example 24 If sum of the coefficients of the first, second and third terms of the expansion of $\left(x^2 + \frac{1}{x}\right)^m$ is 46, then the coefficient of the term that does not contain x is

(a) 84 (b) 92 (c) 98 (d) 106

Ans. (a)

Solution We are given

$${}^mC_0 + {}^mC_1 + {}^mC_2 = 46 \Rightarrow 2m + m(m - 1) = 90$$

$$\Rightarrow \quad m^2 + m - 90 = 0 \Rightarrow m = 9 \text{ as } m > 0$$

Now, $(r + 1)$th term of $\left(x^2 + \frac{1}{x}\right)^m$ is

$${}^mC_r (x^2)^{m - r} \left(\frac{1}{x}\right)^r = {}^mC_r \, x^{2m - 3r}$$

For this to be independent of x is

$$2m - 3r = 0 \Rightarrow r = 6$$

$\therefore$ term independent of x is ${}^9C_6 = 84$.

Example 25 Value of $\sum_{k=1}^{n} (C_k)\,(C_{k-1})$ is

(a) ${}^{2n}C_n$ (b) $\frac{1}{2}\left({}^{2n+2}C_{n+1}\right) - {}^{2n}C_n$
(c) ${}^{2n}C_{n+2}$ (d) none of these

Ans. (b)

Solution We have

$$C_0 x + C_1 x^2 + \ldots + C_n x^{n+1} = x(1 + x)^n \qquad (1)$$

and

$$C_0 + C_1 \frac{1}{x} + \ldots + C_n \frac{1}{x^n} = \left(1 + \frac{1}{x}\right)^n \qquad (2)$$

Now,

$$\sum_{k=1}^{n} (C_k)\,(C_{k-1})$$

= constant term in

$$\left[C_0 + C_1 \frac{1}{x} + \cdots + C_n \frac{1}{x^n}\right] \times \left[C_0 x + C_1 x^2 + \cdots + C_n x^{n+1}\right]$$

= constant term in $\left(1 + \frac{1}{x}\right)^n (1 + x)^n x$

= coefficient of x^{n-1} in $(1 + x)^{2n}$

= ${}^{2n}C_{n-1}$

Also, $\frac{1}{2}\left({}^{2n+2}C_{n+1}\right) - {}^{2n}C_n = {}^{2n}C_{n-1}$

Example 26 Value of

$$S = 1 \times 2 \times 3 \times 4 + 2 \times 3 \times 4 \times 5 + \ldots + n\,(n + 1)\,(n + 2)\,(n + 3)$$

is

(a) $\frac{1}{5}\,n(n + 1)\,(n + 2)\,(n + 3)\,(n + 4)$
(b) $\frac{1}{5!}\left({}^{n+3}C_5\right)$
(c) $\frac{1}{5}\left({}^{n+4}C_4\right)$
(d) none of these

Ans. (a)

Solution $k(k + 1)\,(k + 2)\,(k + 3)$

$$= 4!\,\frac{(k + 3)!}{(k - 1)!4!} = 4!\,\left({}^{k+3}C_4\right)$$

Thus, $S = 4!\,\left({}^4C_4 + {}^5C_4 + \ldots + {}^{n+3}C_4\right)$

= 4! [Coefficient of x^4 in $(1 + x)^4 + (1 + x)^5 + \ldots + (1 + x)^{n+3}$]

= 4! [Coefficient of x^4 in $(1 + x)^4 \, \frac{(1 + x)^n - 1}{1 + x - 1}$]

= 4! [Coefficient of x^5 in $(1 + x)^{n+4}$]

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Example 38 The remainder when 2^{2003} is divided by 17 is

(a) 1 (b) 2
(c) 8 (d) none of these

Ans. (c)

Solution We have $2^{2000} = (2^4)^{500} = (17 - 1)^{500}$

$= {}^{500}C_0\ 17^{500} - {}^{500}C_1\ 17^{499} + \cdots - {}^{500}C_{499}17 + (-1)^{500}$

$= 17\ m + 1$, where m is some positive integer.

$\Rightarrow \quad 2^{2003} = 8(2^{2000}) = 8(17\ m + 1) = 17(8\ m) + 8$

This show that the required remainder is 8.

Example 39 The interval in which $x\ (> 0)$ must lie so that the greatest term in the expansion of $(1 + x)^{2n}$ has the greatest coefficient is

(a) $\left(\frac{n-1}{n}, \frac{n}{n-1}\right)$ (b) $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$

(c) $\left(\frac{n}{n+2}, \frac{n+2}{n}\right)$ (d) none of these

Ans. (b)

Solution Greatest coefficient in the expansion of $(1 + x)^{2n}$ is ${}^{2n}C_n$. We are given ${}^{2n}C_n\ x^n$ is the greatest term.

$\therefore \quad {}^{2n}C_{n-1}\ x^{n-1} < {}^{2n}C_n\ x^n$

and $\quad {}^{2n}C_{n+1}\ x^{n+1} < {}^{2n}C_n\ x^n$

$\Rightarrow \quad \frac{{}^{2n}C_{n-1}}{{}^{2n}C_n} < x < \frac{{}^{2n}C_n}{{}^{2n}C_{n+1}}$

$\Rightarrow \quad \frac{(2n)!}{(n-1)!(n+1)!}\frac{n!n!}{(2n)!} < x < \frac{(2n)!}{n!n!}\frac{(n+1)!(n-1)!}{(2n)!}$

$\Rightarrow \quad \frac{n}{n+1} < x < \frac{n+1}{n}.$

Example 40 The largest term in the expansion of $(3 + 2x)^{50}$, where $x = 1/5$, is

(a) 5th (b) 6th (c) 8th (d) 9th

Ans. (b)

Solution Greatest term in the expansion of $(x + y)^n$ is kth term where $k = \left[\frac{(n+1)y}{x+y}\right]$

In the present case

$$k = \left[\frac{(50+1)(2x)}{3+2x}\right] = \left[\frac{(51)(2/5)}{3+2/5}\right] = 6$$

Thus, 6th term is the largest term.

SECTION II

MULTIPLE CORRECT CHOICE TYPE

Example 41 Values of x for which the sixth term of the expansion of

$E = \left(3^{\log_3\sqrt{9^{|x-2|}-1}} + 7^{(1/5)\log_7[(4).3^{|x-2|}-9]}\right)^7$ is 567, are

(a) 1 (b) 2
(c) 3 (d) none of these

Ans. (a), (c)

Solution

Put $\quad y = 3^{\log_3\sqrt{9^{|x-2|}}} \Rightarrow \log_3 y = \log_3\sqrt{9^{|x-2|}}$

$\Rightarrow \quad y = \sqrt{9^{|x-2|}} = 3^{|x-2|}$

Next, put $\quad z = \left(7^{(1/5)\log_7[(4).3^{|x-2|}-9]}\right)$

$\Rightarrow \quad \log_7 z = \frac{1}{5}\log_7[(4)3^{|x-2|} - 9]$

$= \log_7[(4)\ 3^{|x-2|} - 9]^{1/5}$

$\Rightarrow \quad z = [(4)3^{|x-2|} - 9]^{1/5}$

Now, $E = (y + z)^7$ and 6th term is given by

$$t_6 = {}^7C_5\ y^{7-5}\ z^5 = 21\left(3^{|x-2|}\right)^2\left\{(4)\,3^{|x-2|} - 9\right\}$$

$\Rightarrow \quad 567 = 21\{3^{2|x-2|}\}\ \{(4)\ 3^{|x-2|} - 9\}$

$\Rightarrow \quad 27 = \{(4)\ 3^{3|x-2|} - (9)\ (3^{2|x-2|})$

$\Rightarrow \quad 27 = (4)\ 3^{3|x-2|} - (9)\ 3^{2|x-2|}$

$\Rightarrow \quad 4u^3 - 9u^2 - 27 = 0$ where $u = 3^{|x-2|}$

Note that $u = 3$ satisfies this equation*.

$\therefore \quad 3^{|x-2|} = 3 \Rightarrow |x - 2| = 1 \Rightarrow x - 2 = \pm 1$

$x = 2 \pm 1 = 3$ or 1.

Example 42 If $C_r = \binom{n}{r}$, then sum of the series

$S = C_0^2 + \frac{1}{2}C_1^2 + \frac{1}{3}C_2^2 + \cdots$ upto $(n + 1)$ terms is:

(a) $\frac{1}{n+1}\binom{2n+1}{n+1}$ (b) $\frac{1}{2(n+1)}\binom{2n+2}{n+1}$

(c) $\frac{1}{n+1}\binom{2n+1}{n}$ (d) $\frac{(2n+1)!}{(n+1)!^2}$

* From available answer, possible values of u are $3^{|3-2|} = 3^{|1-2|} = 3$ for the choices (a) and (c) and $3^{|2-2|} = 3^0 = 1$ for the choice (b).

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But $$\sum_{k=1}^{n}(-1)^{k-1} C_k = 1 - \sum_{k=0}^{n}(-1)^k \cdot C_k = 1 - 0 = 1,$$

$$\sum_{k=1}^{n}(-1)^{k-1} k C_k = -\sum_{k=0}^{n}(-1)^k k C_k = 0$$

and $$\sum_{k=1}^{n}(-1)^{k-1} k^2 C_k = -\sum_{k=0}^{n}(-1)^k k^2 C_k = 0$$

Substituting these in (1), we get

$$\sum_{k=1}^{n}(-1)^{k-1} C_k (a - k)^2 = a^2$$

Example 53 If $x \in \mathbf{R}$, and

$$S = 1 - C_1 \frac{1+x}{1+nx} + C_2 \frac{1+2x}{(1+nx)^2} - C_3 \frac{1+3x}{(1+nx)^3}$$

+ ...upto (n + 1) terms

then S

(a) equals x^2 (b) equals 1
(c) equals 0 (d) is independent of x

Ans. (c), (d)

Solution Putting $\frac{1}{1+nx} = y$, we can write

$$S = [C_0 - C_1 y + C_2 y^2 - \ldots + (-1)^n C_n y^n]$$
$$- xy [C_1 - 2C_2 y + 3C_2 y^2 - \ldots + (-1)^{n-1} nC_n y^{n-1}]$$
$$= (1 - y)^n + xy \frac{d}{dt} \{(1 - t)^n\}]_{t=y}$$
$$= (1 - y)^n - nxy (1 - y)^{n-1}$$
$$= (1 - y)^{n-1} [1 - y - nxy]$$
$$= (1 - y)^{n-1} [1 - (1 + nx)y] = 0$$

Example 54 If

$$(x^{2006} + x^{2008} + 2)^{2010}$$
$$= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \quad (1)$$

then value of

$$a_0 - \frac{1}{2} a_1 - \frac{1}{2} a_2 + a_3 - \frac{1}{2} a_4 - \frac{1}{2} a_5 + a_6 + \ldots$$

is

(a) less than 2 (b) greater than 0
(c) equals 2 (d) none of these

Ans. (a), (b)

Solution Let $\omega \neq 1$ be a cube root of unity

Put $x = \omega, \omega^2$ in (1), so that

$$1 = a_0 + a_1 \omega + a^2 \omega^2 + \ldots \quad (2)$$

and $$1 = a_0 + a_1 \omega^2 + a_2 \omega + \ldots \quad (3)$$

Adding (2) and (3) and using $\omega + \omega^2 = -1$, we get

$$2 = 2a_0 - a_1 - a_2 + 2a_3 + \ldots$$

Example 55 Let $(1 + \sqrt{2})^n = x_n + y_n \sqrt{2}$ where x_n, y_n are integers, then

(a) $x_n^2 - 2y_n^2 = (-1)^n$
(b) $x_n + 2y_n - x_{n+1} = 0$
(c) $x_n^2 - 2y_n^2 = 1$
(d) $y_{n+1} = x_n + y_n$

Ans. (a), (b), (d)

Solution We have

$$(1 + \sqrt{2})^n = x_n + y_n \sqrt{2} \quad (1)$$

$$\Rightarrow \quad (1 - \sqrt{2})^n = x_n - y_n \sqrt{2} \quad (2)$$

From (1) and (2) we get

$$(1 + \sqrt{2})^n (1 - \sqrt{2})^n = x_n^2 - 2y_n^2$$

$$\Rightarrow \quad (-1)^n = x_n^2 - 2y_n^2$$

Next, $x_{n+1} + y_{n+1} \sqrt{2} = (1 + \sqrt{2})^{n+1}$

$$= (1 + \sqrt{2}) (x_n + \sqrt{2} y_n)$$
$$= (x_n + 2y_n) + \sqrt{2} (x_n + y_n)$$

Thus, $x_{n+1} = x_n + 2y_n$ or $x_{n+1} - x_n - 2y_n = 0$

Example 56 11th term in the expansion of

$$\left(3 - \sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{20}$$

is

(a) an irrational number (b) a rational number
(c) a positive integer (d) a negative integer

Ans. (b), (c)

Solution We have

$$\frac{17}{4} + 3\sqrt{2} = \frac{1}{4} (9 + 8 + 12\sqrt{2})$$
$$= \frac{1}{4} (3 + 2\sqrt{2})^2$$

$$\therefore \quad 3 - \sqrt{\frac{17}{4} + 3\sqrt{2}} = 3 - \frac{1}{2} (3 + 2\sqrt{2}) = \frac{3}{2} - \sqrt{2}$$

Thus, 11th term in the expansion of

$\left(3 - \sqrt{\frac{17}{4} + 3\sqrt{2}}\right)^{20}$ is

$$^{20}C_{10} \left(\frac{3}{2}\right)^{20-10} \left(-\sqrt{2}\right)^{10}$$

which is a positive integer and hence a rational number.

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$$\Rightarrow\ 2\sum_{r=0}^{n}\frac{r}{C_r} = n\ a_n \Rightarrow \sum_{r=0}^{n}\frac{r}{C_r} = \frac{1}{2}n\ a_n$$

Similarly, $\sum_{r=0}^{n}\frac{n-r}{C_r} = \frac{1}{2}n\,a_n$

Using $rC_r = nC_{r-1}$, we get

$$\sum_{r=1}^{n}\frac{1}{{}^{r}C_r} = \frac{1}{n}\sum_{r=1}^{n}\frac{1}{C_{r-1}} = \frac{1}{n}\sum_{r=0}^{n}\frac{1}{C_r} = \frac{1}{n}a_{n-1}$$

Similarly, $\sum_{r=0}^{n-1}\frac{1}{(n-r)C_r} = \frac{1}{n}a_{n-1}$

Example 83 Remainder when N is divided by A.

(a) $N = 99^{100}$ and $A = 10$	(p) 7
(b) $N = 2^{2007} + 2008$ and $A = 9$	(q) 0
(c) $N = 9^{2009} - 8(2008) - 9$ and $A = 64$	(r) 1
(d) $N = 7^{1010}$ and $A = 1000$	(s) 19

Ans.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

Solution

(a) $99^{100} = (10-1)^{100}$

$= 10^{100} - {}^{100}C_1\,(10)^{99} + {}^{100}C_2\,(10)^{98} - \cdots - {}^{100}C_{99}\,(10) + 1$

$= 10\,m + 1$

for some $m \in \mathbf{N}$.

(b) $2^3 = 9 - 1 \Rightarrow 2^{2007} = (9-1)^{669}$

$= 9m - 1$ for some $m \in \mathbf{N}$

Thus, when $2^{2007} + 2008$ is divided by 9, the remainder is 0.

(c) Use $(1+8)^n - 8(n-1) - 9$

$= 1 + 8n + 64\,[{}^{n}C_2 = {}^{n}C_3\,8 + \cdots + {}^{n}C_n\,8^{n-2}] - 8n - 1$

$= 64\,m$ for some $m \in \mathbf{N}$.

(d) As $7^2 = 50 - 1$, we get

$7^{100} = (1-50)^{50} = {}^{50}C_0 + {}^{50}C_1\,(-50) + {}^{50}C_2\,(50)^2 + 50^3\,m$

where m is some positive integer.

$\therefore\ 7^{100} = 1 + 2500\,(25 \times 49 - 1) + 125000\,m$

$= 1 + 1000\,[4 + m]$

where m is some positive integer.

SECTION VI

REASONING TYPE

Example 84 Let $n \in \mathbf{N}$ and C_r stand for $\binom{n}{r}$, $r = 0, 1, 2, \cdots n$, and let $A = \sum_{r=0}^{n}\frac{1}{C_r}$

Statement-1: If $S = \sum\sum_{0\le i<j\le n}\left(\frac{i}{C_i} + \frac{j}{C_j}\right)$ then $S = \frac{n^2 A}{2}$

Statement-2: If $T = \sum\sum_{0\le i<j\le n}\left(\frac{1}{C_i} + \frac{1}{C_j}\right)$ then $T = nA$

Ans. (a)

Solution Note that

$$T = \sum\sum_{i<j}\left(\frac{1}{C_i} + \frac{1}{C_j}\right) = \sum_{i=0}^{n-1}\frac{n-i}{C_i} + \sum_{j=1}^{n}\frac{j}{C_j}$$

$= nA$

Using $C_i = C_{n-i}$ for $0 \le i \le n$, we get

$$S = \sum\sum_{0\le i<j\le n}\left(\frac{n-i}{C_i} + \frac{n-j}{C_j}\right) = nT - S$$

$\Rightarrow\quad 2S = nT = n^2 A \quad \Rightarrow \quad S = n^2 A/2$

Example 85 Let $m, n \in \mathbf{N}$ and $C_r = \binom{n}{r}$ for $0 \le r \le n$.

Statement-1: $\frac{1}{m!}C_0 + \frac{n}{(m+1)!}C_1 + \frac{n(n-1)}{(m+2)!}C_2 + \cdots + \frac{n(n-1)\cdots 2.1}{(m+n)!}C_n$

$$= \frac{(m+n+1)(m+n+2)\cdots(m+2n)}{(m+n)!} \qquad (1)$$

Statement-2: For $r \ge 0$,

$$\binom{m}{r}C_0 + \binom{m}{r-1}C_1 + \binom{m}{r-2}C_2 + \cdots + \binom{m}{0}C_r = \binom{m+n}{r}$$

Ans. (a)

Solution $\binom{m}{r}C_0 + \binom{m}{r-1}C_1 + \binom{m}{r-2} + \cdots + \binom{m}{0}C_r$

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29. Let $a_n = \sum_{k=0}^{n} (1 + 2^k)C_k$, then which of the following equals a_{n+1}
    (a) $5a_n - 6a_{n-1}$ (b) $5a_n + 6a_n$
    (c) $2a_n + 3a_{n-1}$ (d) none of these
30. If $\log x = \log_{10} x$, the values of x for which 4th term in the expansion of
$$\left[\sqrt{x^{1/(1+\log x)}} + x^{1/12}\right]^6$$
is 200 are
    (a) 0.001, 0.1 (b) 0.0001, 10
    (c) 0.01, 100 (d) 0.1, 10000
31. If $\sum_{r=1}^{n} r(r+1)\frac{C_r}{C_{r-1}} = 77$, then n equals
    (a) 5 (b) 6
    (c) 7 (d) 11
32. If n is an odd natural number, then number of zeros at the end of $99^n + 1$ is
    (a) 2 (b) n
    (c) $2n$ (d) none of these
33. Value of $S = \sum\sum_{0 \le r < s \le n} (C_r + C_s)^2$ is
    (a) $2^{2n}(n+1)({}^{2n}C_n)$
    (b) $2^{2n} + (n+1)({}^{2n}Cn)$
    (c) $2^{2n-1} + (n+1)({}^{2n}Cn)$
    (d) none of these
34. Value of $S = \sum\sum_{0 \le r < s \le n} (r+s)(C_r + C_s)^2$
    (a) $n[(n-1)({}^{2n}C_n) + 2^{2n}]$
    (b) $n[(n+1)({}^{2n}C_n) + 2^{2n}]$
    (c) $n[2^{2n} - n({}^{2n}C_n)]$
    (d) none of these
35. Value of $S = \sum_{k=1}^{\infty}\sum_{r=0}^{k} \frac{1}{3^k}({}^kC_r)$ is
    (a) 2 (b) $\frac{2}{3}$
    (c) $\frac{1}{3}$ (d) none of these
36. Value of
$$S = {}^nC_r + 3({}^{n-1}C_r) + 5({}^{n-2}C_r) + \cdots \text{ upto } (n-r+1) \text{ terms is}$$
    (a) ${}^{n+2}C_{r+2}$
    (b) ${}^{n+2}C_{r+2} + {}^{n+1}C_{r+2}$
    (c) ${}^{n+2}C_{r+1}$
    (d) ${}^{n+2}C_{r+2} + {}^{n+1}C_r$
37. If n is a multiple of 3 and $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x_r$ and
$$\sum_{r=0}^{n} (-1)^r a_r ({}^nC_r) = k({}^nC_{n/3})$$
then value of k is
    (a) 0 (b) 1
    (c) 2 (d) none of these
38. If the middle term of $(1 + x)^{2n}$ is also the largest term then x lies in the interval
    (a) $\left[1 - \frac{1}{n+1}, 1 + \frac{1}{n}\right]$ (b) $\left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$
    (c) [0, 1] (d) $[-1/n, 1/n]$
39. Sum of the series
$$S = \sum_{r=0}^{n} (-1)^r ({}^nC_r)\left[\frac{2^r}{3^r} + \frac{8^r}{3^{2r}} + \frac{26^r}{3^{3r}} + \cdots \text{upto} \infty\right]$$
is
    (a) $\frac{2}{3^n - 1}$ (b) $\frac{1}{3^n + 1}$
    (c) $\frac{1}{3^n - 1}$ (d) $\frac{2}{3^n + 1}$
40. The ratio of the coefficient of x^{15} to the term independent of x in the expansion of $\left(x^2 + \frac{2}{x}\right)^{15}$ is
    (a) 1 : 8 (b) 1 : 12
    (c) 1 : 16 (d) 1 : 32

SECTION II

MULTIPLE CORRECT CHOICE TYPE

41. For all $n \in \mathbf{N}$, $3^{2n+2} - 8n - 9$ is divisible by
    (a) 8 (b) 16 (c) 32 (d) 64
42. If the third term in the expansion of $[(1/x) + x^{\log_{10} x}]^5$ is 1000, then x is equal to
    (a) 100 (b) 10 (c) 1 (d) $1/\sqrt{10}$.
43. If the third term in the expansion of $(x + x^{\log_{10} x})^5$ is 10^6, then x can be
    (a) $10^{-1/3}$ (b) 10 (c) $10^{-5/2}$ (d) 10^2.
44. Positive integer (s) which is (are) greater than $(1 + 0.0001)^{10000}$ is (are)
    (a) 3 (b) 4 (c) 5 (d) 7
45. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \cdots + a_{20}x^{20}$, then
    (a) $a_1 = 20$ (b) $a_2 = 210$
    (c) $a_4 = 8085$ (d) $a_{20} = 2^2 \times 3^7 \times 7$.

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(a) 0 (b) 1
(c) 9 (d) none of these

8. If the middle term of $\left(x+\frac{1}{x}\sin^{-1}x\right)^8$ is $\frac{35\pi^4}{8}$, then value of x can be
(a) 1/2 (b) $\sqrt{3}/2$
(c) $1/\sqrt{2}$ (d) 1

9. If n is odd, then value of
$$S=\frac{C_1}{C_0}-2\frac{C_2}{C_1}+3\frac{C_3}{C_2}-\cdots \text{ upto } n \text{ terms}$$
is
(a) $\frac{1}{2}(n+1)$ (b) $\frac{1}{3}(n+1)$
(c) $\frac{1}{2}n(n+1)$ (d) none of these

10. The expression
$$S=n^2\left(\frac{C_0}{C_1}\right)+(n-1)^2\left(\frac{C_1}{C_2}\right)+\cdots+1^2\left(\frac{C_{n-1}}{C_n}\right)$$
equals
(a) $\frac{1}{6}n(n+1)(2n+1)$
(b) $\frac{1}{6}n(n+1)(n+2)$
(c) $\frac{1}{3}n(n+1)(n+2)$
(d) none of these

11. If $S_n=\frac{1}{n+1}\sum_{i=0}^{n}\binom{n}{i}$, then $2S_{n+1}-S_n$ equals
(a) $\frac{2}{(n+1)}$ (b) $\frac{2}{(n+2)}$
(c) $\frac{n}{(n+2)}$ (d) $\frac{(n+1)}{(n+2)}$

12. If $a, b, p \in \mathbf{N}$ and p is prime, then
$$(a+b)^p-a^p-b^p$$
is divisible by
(a) p (b) $\frac{1}{2}p(p-1)$
(c) p^2 (d) none of these

13. $\frac{C_0}{n(n+1)}-\frac{C_1}{(n+1)(n+2)}+\frac{C_2}{(n+2)(n+3)}-\cdots$ upto $(n+1)$ terms equals
(a) $\int_0^1 x^n(1-x)^n\,dx$
(b) $\int_0^1 x^{n-1}(1-x)^{n+1}\,dx$
(c) $\int_0^1 x^{n+1}(1-x)^n\,dx$
(d) none of these

14. $\frac{C_0}{n(n+2)}-\frac{C_1}{(n+1)(n+3)}+\frac{C_2}{(n+2)(n+4)}\cdots$ upto $(n+1)$ terms equals
(a) $\int_0^1 x^n(1-x)^n\,dx$
(b) $\int_0^1 x^{n-1}(1-x)^{n+1}\,dx$
(c) $\int_0^1 x^{n+1}(1-x)^n\,dx$
(d) none of these

15. $C_0^2+\frac{1}{2}C_1^2+\frac{1}{3}C_2^2+\cdots+\frac{1}{n+1}C_n^2$ equals
(a) $\frac{1}{n}({}^{2n}C_n)$ (b) $\frac{1}{n+1}({}^{2n+1}C_n)$
(c) $\frac{1}{n}({}^{2n}C_{n+1})$ (d) none of these

16. If p is an odd prime and $[x]$ denotes the greatest integer $\le x$, then
$$[(2+\sqrt{3})^p]-2^{p+1}$$
is divisible by
(a) 2 (b) p
(c) $p-1$ (d) none of these

17. For $r=0, 1, \ldots, 10$, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$. Then
$\sum_{r=1}^{10}A_r(B_{10}B_r-C_{10}A_r)$ is equal to
(a) $B_{10}-C_{10}$ (b) $A_{10}(B_{10}^2-C_{10}A_{10})$
(c) 0 (d) $C_{10}-B_{10}$

18. If $n\ge 2$, value of $S=C_0-3C_1+5C_2-7C_3+\cdots$ upto $(n+1)$ terms is
(a) 0 (b) ${}^{2n}C_n$
(c) $(-1)^n$ (d) none of these

19. The hundred's digit of 3^{100} is
(a) 0 (b) 1 (c) 2 (d) 7

20. The thousand's digit of 3^{100} is
(a) 0 (b) 1 (c) 2 (d) 7

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$${}^{n+1}C_1 + {}^{n+1}C_2\, s_1 + {}^{n+1}C_3\, s_2 + \cdots + {}^{n+1}C_{n+1}\, s_n = 2^n S_n$$

[1984]

8. Find the sum of the series:

$$\sum_{r=0}^{n} (-1)^r ({}^nC_r)\left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \cdots \text{ upto } m \text{ terms}\right]$$

[1985]

9. Prove that

$$\sum_{r=0}^{n} k^2 ({}^nC_k) = n(n+1)2^{n-2} \quad \forall\ n \geq 1.$$

[1986]

10. If C_r stands for ${}^nC_r$ then the sum of the series

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 + \cdots + (-1)^n (n+1)C_n^2]$$

where n is an even positive integer is equal to

(a) 0 (b) $(-1)^{n/2}(n+1)$
(c) $(-1)^n (n+2)$ (d) $(-1)^n n$
(e) none of these

[1986]

11. Let $R = (5\sqrt{5} + 11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+1}$ [1988]

12. Show that

$${}^mC_0\, {}^nC_k + {}^mC_1\, {}^nC_{k-1} + \cdots + {}^mC_k\, {}^nC_0 = {}^{m+n}C_k$$

where m, n, k are positive integers, and ${}^pC_q = 0$ for $p < q$.

[1989]

13. Prove that

$$C_0 - (2^2)C_1 + (3^2)C_2 - \cdots + (-1)^n (n+1)^2 C_n = 0,$$

$n > 2$ where $C_r = {}^nC_r$. [1989]

14. The expression $(x + (x^3-1)^{1/2})^5 + (x - (x^3-1)^{1/2})^5$ is a polynomial of degree

(a) 5 (b) 6
(c) 7 (d) 8

[1992]

15. If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ and $a_k = 1\ \forall\ k \geq n$, show that $b_n = {}^{2n+1}C_{n+1}$ [1992]

16. Let n be a positive integer

$$(1 + x + x^2)^n = a_0 + a_1 x + \cdots + a_{2n} x^{2n}$$

show that

$$a_0^2 - a_1^2 + a_2^2 - \cdots + a_{2n}^2 = a_n$$

[1994]

17. The sum of the rational terms in the expansion of $\left(\sqrt{2} + 3^{1/5}\right)$ is _____.

[1997]

18. Prove that

$$\frac{3!}{2(n+3)} = \sum_{r=0}^{n} (-1)^r \left(\frac{{}^nC_r}{{}^{r+3}C_r}\right)$$

[1997]

19. If in the expansion of $(1+x)^m (1-x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively, then m is

(a) 6 (b) 9
(c) 12 (d) 24

[1999]

20. In the binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of the 5th and 6th terms is zero. Then $\frac{a}{b}$ equals

(a) $\frac{n-5}{6}$ (b) $\frac{n-4}{5}$
(c) $\frac{5}{n-4}$ (d) $\frac{6}{n-5}$ [2001]

21. The coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is

(a) ${}^{12}C_6 + 3$ (b) ${}^{12}C_6 + 1$
(c) ${}^{12}C_6$ (d) ${}^{12}C_5 + 2$

[2003]

22. If n and k are positive integers, show that

$$2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \cdots + (-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$$

where $\binom{n}{k}$ stands for ${}^nC_k$. [2003]

23. If $\binom{n}{r}$ stands for ${}^nC_r$, the value of

$$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \cdots + \binom{30}{20}\binom{30}{30}$$

is

(a) $\binom{30}{10}$ (b) $\binom{30}{15}$
(c) $\binom{31}{11}$ (d) $\binom{60}{30}$

[2005]

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17. $S = \sum_{k=0}^{n} {}^nC_k \sin(kx) \cos(n-k)x$ (1)

Writing the terms in the reverse order

$$S = \sum_{k=0}^{n} {}^nC_{n-k} \sin(n-k)x \cos(kx)$$

$$= \sum_{k=0}^{n} {}^nC_k \sin(n-k)x \cos(kx) \quad (2)$$

Adding (1) and (2), we get

$$2S = \sum_{k=0}^{n} {}^nC_k \{\sin(kx)\cos(n-k)x + \sin(n-k)x\cos(kx)\}$$

$$2S = \sum_{k=0}^{n} ({}^nC_k)\sin(nx) = 2^n \sin(nx)$$

$$\Rightarrow \quad S = 2^{n-1}\sin(nx)$$

18. $S_1 - S_2 = \sum_{k=0}^{n} {}^nC_k [\cos(kx)\cos(n-k)x - \sin(kx)\sin(n-k)x]$

$$= \sum_{k=0}^{n} {}^nC_k \cos(k+n-k)x$$

$$= 2^n \cos(nx)$$

19. Put $x = 1$

20. Since $2^{\log_{\sqrt{2}}(x^{3/2})} = \sqrt{2}^{2\log_{\sqrt{2}}(x^{3/2})} = x^3$

We consider the expansion of $\left(x^3 + \frac{1}{x^3}\right)^{11}$

$$t_{r+1} = {}^{11}C_r (x^3)^{11-r} \frac{1}{x^{3r}}$$

$$= {}^{11}C_r \, x^{33-6r}$$

For coefficient of x^9, we set

$$33 - 6r = 9 \quad \Rightarrow \quad r = 4$$

Thus, desired coefficient is ${}^{11}C_4$

21. Using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, we can write the given equation as

$${}^{2n+1}C_r = {}^{2n+1}C_{r^2-182}$$

$$\Rightarrow \quad r = r^2 - 182 \text{ or } r + r^2 - 182 = 2n + 1$$

As the last equation is not possible

$$r = r^2 - 182 \quad \Rightarrow \quad r = 14$$

Since $n \geq r - 1$ we get least possible value of n is 13.

22. $S = \frac{n!}{(m+n)!} [{}^{m+n}C_m \, {}^nC_0 + {}^{m+n}C_{m+1} \, {}^nC_1 + \cdots + {}^{m+n}C_{m+n} \, {}^nC_n]$

$$= \frac{n!}{(m+n)!} [{}^{m+n}C_m \, {}^nC_n + {}^{m+n}C_{m+1} \, {}^nC_{n-1} + \cdots + {}^{m+n}C_{m+n} \, {}^nC_0]$$

$$= \frac{n!}{(m+n)!} {}^{(m+2n)}C_{m+n}$$

23. $2({}^nC_r) = {}^nC_{r-1} + {}^nC_{r+1}$

$$\Rightarrow 2 = \frac{{}^nC_{r-1}}{{}^nC_r} + \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{r}{n-r+1} + \frac{n-r}{r+1}$$

$$\Rightarrow 2(nr - r^2 + r + n - r + 1) = r^2 + (n-r)^2 + n$$

$$n + 2 = n^2 - 4nr + 4r^2 = (n-2r)^2$$

24. $A_k = \frac{{}^nC_k}{{}^{n+1}C_{k+1}} = \frac{k+1}{n+1}$

$$\Rightarrow \quad \sum_{k=0}^{n} A_k = \frac{(n+1)(n+2)}{2(n+1)} = \frac{n+2}{2}$$

25. $1 - 2x + 3x^2 - 4x^3 - \cdots = (1+x)^{-2}$

$$\Rightarrow \quad (1 - 2x + 3x^2 - \cdots)^{-n} = (1+x)^{2n}$$

Thus, coefficient of x^n is ${}^{2n}C_n$

26. For $n \geq 3$,

$$\frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n < 3 \leq n$$

$$\Rightarrow \quad (n+1)^n < n^{n+1}$$

27. 6^{th} term is indendent of x. Thus

$$I(\alpha) = {}^{10}C_5 [(\sin^{-1}\alpha)(\cos^{-1}\alpha)]^5$$

$$= {}^{10}C_5 \left[\frac{\pi^2}{16} - \left(\sin^{-1}\alpha - \frac{\pi}{4}\right)^2\right]^5$$

Thus, $|I(\alpha)| \leq {}^{10}C_5 \left(\frac{\pi^2}{16}\right)^5$

28. $C_1 a^{n-1} + {}^2C_2 \, x a^{n-2} + {}^3C_3 \, x^2 a^{n-3} + \cdots + n \, C_n \, x^{n-1}$

$$= \frac{d}{dt}\{(a+t)^n\}]_{t=x} = n(a+x)^{n-1}$$

Multiplying by x and putting $a = 1 - x$, we get, the sum of the series as nx

29. $a_n = \sum_{k=0}^{n} C_k + \sum_{k=0}^{n} C_k (2^k) = 2^n + 3^n$

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= the number of ways of selecting n persons from n men and n women and appointing a men's leader and a women's leader

$$= n^2\ (^{2n-2}C_{n-2})$$

Thus, $S(2) = \frac{n^2}{2}\ (^{2n}C_n) - n^2\ (^{2n-2}C_{n-2})$

56. Sum of coefficients the integral powers of x is
$= {}^{40}C_0 + {}^{40}C_2\ (2^2) + {}^{40}C_4\ (2^4) + \cdots + {}^{40}C_{40}\ (2^{40})$
$= \frac{1}{2}\ [(1+2)^{40} + (1-2)^{40}] = \frac{1}{2}\ (3^{40}+1)$
For the other part, use sum of all the powers of x is 3^{40}.

57. $2S_n(x) = [{}^nC_0 \sin(0x)\cos(nx) + {}^nC_n \sin(nx)\cos(0x)]$
$+ [{}^nC_1 \sin x \cos(n-1)x + {}^nC_{n-1} \sin(n-1)x \cos x]$
$+ [{}^nC_2 \sin 2x \cos(n-2)x + {}^nC_{n-2} \sin(n-2)x \cos 2x]$
$+ \cdots$ upto $(n+1)$ terms
$= {}^nC_0 [\sin(0x)\cos(nx) + \sin(nx)\cos(0x)]$
$+ {}^nC_1 [\sin x \cos(n-1)x + \sin(n-1)x \cos x] +$
$ {}^nC_2 [\sin(2x)\cos(n-2)x + \sin(n-2)x \cos 2x] + \cdots$
$+ {}^nC_n [\sin(nx)\cos(0x) + \sin(0x)\cos(nx)]$
$= [{}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n] \sin(nx) = 2^n \sin(nx)$
$\Rightarrow \quad S_n(x) = 2^{n-1} \sin(nx)$

58. We have $k\ (^{2n}C_k) = 2n\ (^{2n-1}C_{k-1})$

Thus, $S(n) = \sum_{k=1}^{n} 2n(^{2n-1}C_{k-1})$

$= (2n)2^{2n-2} = n(2^{2n-1})$ [See Theory]

$S(2n) = (2n)2^{2n-1} = n(2^{2n})$ [See Theory]

For $S(3n)$ and $S(4n)$ use ${}^nC_r = 0\ \forall\ r > n$.

59. $S(n, r)$ = coefficient of x^r in
$(1 + x + x^2 + \cdots)(C_0 - C_1 x + C_2 x^2 - \cdots + (-1)^n C_n x^n)$
= Coefficient of x^r in $(1-x)^{-1}(1-x)^n$
$= (-1)^r\ (^{n-1}C_r)$
Now, $(-1)^r\ (^{n-1}C_r) = 28$
This is possible if $n = 9$, $r = 2$ or $n = 9$ or $r = 6$.

60. $3^2 = 9 = 10 - 1$
$\Rightarrow \quad 3^{100} = (10-1)^{50} = 10m + 1$ where $m \in \mathbf{N}$.
$17^2 = 289 = 290 - 1$
$\Rightarrow \quad 17^{50} = (290-1)^{25} = 10m - 1$ where $m \in \mathbf{N}$
$\therefore$ Last digit of 17^{50} is 9
Similarly, last digit of 7^{50} is 9 and last digit of 19^{50} is 1

61. $S = {}^{30}C_r$ which is maximum when $r = 15$.

62. $S = {}^{30}C_r$ is least when $r = 0$

63. $A_n = {}^nC_0\ {}^nC_{n-1} + {}^nC_1\ {}^nC_{n-2} + \cdots + {}^nC_{n-1}\ {}^nC_0$
$= {}^{2n}C_{n-1}$

Now, $\frac{A_{n+1}}{A_n} = \frac{(2n+2)!}{n!(n+2)!}\ \frac{(n-1)!(n+1)!}{(2n)!}$

$= \frac{(2n+2)(2n+1)}{n(n+2)} = \frac{15}{4} \Rightarrow n = 2, 4$

64. Put $m = n$, $r = n$

65. Use ${}^nC_r = {}^nC_{n-r}$

66. $E = (x+2)^n - (x+1)^n$

67. $E = \frac{1}{x}\ [(2x+1)^n - (x+1)^n]$

68. $E = \frac{1}{x}\ [(x+2)^{n+1} - 2^{n+1}]$

69. m = coefficient of x^r in E, where
$E = (x+2)^r + (x+2)^{r+1} + \cdots + (x+2)^n$
$= (x+2)^r\ \frac{[(x+2)^{n-r+1} - 1]}{x+2-1}$
$= \frac{(x+2)^{n+1} - (x+2)^r}{x+1}$

70. Sum up the series using A.G.P.

71. $(1+x)^{41}\ (1 - x + x^2)^{40} = (1+x)\ (1-x^3)^{40}$

72. Use formula for $(a+b)^n - (a-b)^n$

73. $2^n = 4096 = 2^{12} \quad \Rightarrow \quad n = 12$
Greatest coefficient $= {}^{12}C_6$

74. Use $2x^2 + x - 3 = (x-1)(2x+3)$

75. ${}^8C_5\ (x^{-8/3})^3\ (x^2 \log_{10} x)^5 = 5600$
$\Rightarrow \quad x^2\ (\log_{10} x)^5 = 100$

76. Use multinomial expansion.

77. $5^{99} = 5(26-1)^{49} = 5\ (26m - 1)$
where m is some integer.

78. Use $1 + x - 2x^2 = (1-x)(1+2x)$

79. $b = \sum_{r=1}^{50} a_r$

But $a_r = r^2\ \frac{(50-r+1)}{r} = r\ (51 - r)$

80. $E = (x - 4 + 5)^{100} = (x+1)^{100}$
Thus, coefficient of $x^2 = {}^{100}C_2 = 4950$

81. (a) ${}^{656}C_r\ (7^{1/3})^{654-r}\ (11^{1/9})^r$
will free from radical sign
$\Leftrightarrow$ r is a multiple of 9 and $0 \le r \le 656$

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where A_{13} = cofactor of (1,3)th element in A i.e.

$$A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

8. If $A = \text{diag}\ (\lambda_1, \lambda_2, \cdots, \lambda_n)$ then A^{-1} exists if $\lambda_i \neq 0$ $\forall\ i$ and
   $A^{-1} = \text{diag}\ (\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_n^{-1})$
   Also, $A^m = \text{diag}\ (\lambda_1^m, \lambda_2^m \ldots \lambda_n^m)$ if $m \in \mathbf{N}$
9. If a square matrix A satisfies the equation $a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r = 0$, then A is invertible if $a_0 \neq 0$ and its inverse is given by

$$A^{-1} = \frac{1}{a_0} [a_1 I + a_2 A + \cdots + a_r A^{r-1}]$$

7.4 SOME DEFINITIONS AND RESULTS

A square matrix is said to be *symmetric* if $A' = A$ and *skew symmetric* if $A' = -A$.
A square matrix A is *Hermitian* if $A^* = A$ and *skew Hermitian* if $A^* = -A$.
Results

1. The main diagonal elements of a skew symmetric matrix are zeros, i.e. $a_{ii} = 0\ \forall\ i$.
2. Determinant of a skew-symmetric matrix of odd order is zero and determinant of a skew-symmetric matrix of even order is a perfect square.
3. Every square matrix can be written uniquely as a sum of symmetric and a skew symmetric matrix, i.e. if A is a square matrix, then there exists a symmetric matrix P and a skew symmetric matrix Q such that
   $$A = P + Q$$
   In fact
   $$P = \frac{1}{2}(A + A') \text{ and } Q = \frac{1}{2}(A - A').$$
4. For every square matrix A, matrices $A + A'$, AA' and $A'A$ are symmetric and matrices $A - A'$ and $A' - A$ are skew symmetric.
   A square matrix is said to be *orthogonal* if
   $$AA' = A'A = I.$$
   In other words A is orthogonal matrix if and only if $A^{-1} = A'$.
   A is said to be unitary if $A^*A = I$

Results

1. If A is an orthogonal matrix, then $|A| \neq 0$.
   In fact $|A| = \pm 1$. and $\bar{A}$, A', A^* are orthogonal also.
2. If A and B are two orthogonal matrices, then AB and BA are both orthogonal matrices.

A square matrix A is said to be a nilpotent matrix if there exists a positive integer m such that $A^m = O$.
The least positive integer m satisfying the condition $A^m = O$ is called the *index* of the nilpotent matrix.

A square matrix A is said to be *idempotent* matrix if $A^2 = A$ and is said to be *involutory* if $A^2 = I$.
If $A = (a_{ij})_{n \times n}$ is a square matrix, the trace of A denoted by tr (A) is sum of all the main diagonal elements, i.e.

$$\text{tr}\ (A) = a_{11} + a_{22} + \ldots + a_{nn}$$

7.5 RANK OF A MATRIX

If $A = (a_{ij})_{m \times n}$ is a matrix, and B is its submatrix of order r, then $|B|$, the determinant is called as *r-rowed minor of A*.
Definition Let $A = (a_{ij})_{m \times n}$ be a matrix. A positive integer r is said to be of rank of A if

(i) A possesses at least one r-rowed minor which is different from zero; and
(ii) every $(r + 1)$ rowed minor of A is zero.
From (ii), it automatically follows that all minors of higher order are zeros.

Result The rank of a matrix does not change when the following *elementary row operations* are applied to the matrix.

(a) Two rows are interchanged $(R_i \leftrightarrow R_j)$
(b) A row is multiplied by a non-zero constant, $(R_i \to kR_i$, with $k \neq 0)$
(c) A constant multiple of another row is added to a given row $(R_i \to R_i + kR_j)$ where $i \neq j$

Note : The arrow $\to$ means "replaced by".
Note that the application of these elementary row operations does not change a singular matrix to a non-singular matrix or a non-singular matrix to a singular matrix. Therefore, the order of the largest non-singular square submatrix is not affected by application of any of the elementary row operations. Thus, the rank of a matrix does not change by application of any of the elementary row operations.
A matrix obtained from a given matrix by applying any of the elementary row operations is said to be equivalent to it.
If A and B are two equivalent matrices, we write $A \sim B$.
Note that if $A \sim B$, then $\rho(A) = \rho(B)$.
By using the elementary row operations, we shall try to transform the given matrix in the following form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & 0 \cdots & * \end{pmatrix}$$

where * stands for zero or non-zero element. That is, we shall try to make a_{ii} as 1 and all the elements below a_{ii} as zero.

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$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = A_{\alpha+\beta}$$

Example 4 Let A and B be two 2×2 matrices. Consider the statements

(i) $AB = O \Rightarrow A = O$ or $B = O$

(ii) $AB = I_2 \Rightarrow A = B^{-1}$

(iii) $(A + B)^2 = A^2 + 2AB + B^2$

Then

(a) (i) is false, (ii) and (iii) are true
(b) (i) and (iii) are false, (ii) is true
(c) (i) and (ii) are false, (iii) is true
(d) (ii) and (iii) are false, (i) is true

Ans. (b)

Solution (i) is false.

If $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Thus, $AB = O$ but neither $A = O$ nor $B = O$

(iii) is false since matrix multiplication is not commutative.

(ii) is true as the product AB is an identity matrix, if and only B is inverse of the matrix A.

Example 5 Let A, B be two $n \times n$ matrices such that $A + B = AB$, then

(a) $AB = BA$ (b) $AB = I_n$
(c) $A = I_n$ or $B = I_n$ (d) none of these.

Ans. (a)

Solution $A + B = AB \Rightarrow I_n - A - B + AB = I_n$

$\Rightarrow$ $(I_n - A)(I_n - B) = I_n$

$\Rightarrow$ $I_n - A$ is invertible and its inverse is $I_n - B$.

$\therefore$ $(I_n - B)(I_n - A) = I_n \Rightarrow B + A = BA$.

Thus, $AB = A + B = B + A = BA$.

Example 6 If A and B are symmetric matrices, then $AB - BA$ is a

(a) symmetric matrix
(b) skew symmetric matrix
(c) diagonal matrix
(d) null matrix

Ans. (b)

Solution We are given $A' = A$, $B' = B$

Now $(AB - BA)' = (AB)' - (BA)'$

$= B'A' - A'B'$

$= BA - AB$

$= -(AB - BA)$

i.e. $(AB - BA)' = -(AB - BA)$

Hence $AB - BA$ is a skew symmetric matrix.

Example 7 If $D = \text{diag}(d_1, d_2, \ldots, d_n)$ where $d_i \neq 0$, for $i = 1, 2, \ldots, n$, then D^{-1} is equal to

(a) D
(b) $2D$
(c) diag $(d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1})$
(d) Adj D

Ans. (c)

Solution See Theory.

Example 8 The inverse of a symmetric matrix (if it exists) is

(a) a symmetric matrix
(b) a skew symmetric matrix
(c) a diagonal matrix
(d) none of these

Ans. (a)

Solution Let A be an invertible symmetric matrix.

We have $AA^{-1} = A^{-1}A = I_n$

$\Rightarrow$ $(AA^{-1})' = (A^{-1}A)' = (I_n)'$

$\Rightarrow$ $(A^{-1})'A' = A'(A^{-1})' = I_n$

$\Rightarrow$ $(A^{-1})'A = A(A^{-1})' = I_n$

$\Rightarrow$ $(A^{-1})' = A^{-1}$ [inverse of a matrix is unique]

Example 9 The inverse of a skew symmetric matrix (if it exists) is

(a) a symmetric matrix
(b) a skew symmetric matrix
(c) a diagonal matrix
(d) none of these

Ans. (b)

Solution We have $A' = -A$

Now, $AA^{-1} = A^{-1}A = I_n$

$\Rightarrow$ $(AA^{-1})' = (A^{-1}A)' = (I_n)'$

$\Rightarrow$ $(A^{-1})'A' = A'(A^{-1})' = I_n$

$\Rightarrow$ $(A^{-1})'(-A) = (-A)(A^{-1})' = I_n$

Thus, $(A^{-1})' = -(A^{-1})$

[inverse of a matrix is unique]

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Now, $AX = O \Rightarrow A^{-1}(AX) = O$

$\Rightarrow \quad (A^{-1}A)X = O$ or $IX = O$ or $X = O$

$\therefore \quad x = y = z = 0$ is the only solution of the system of equations.

Example 29 If $A = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$, then A^{100} is equal to

(a) $\begin{pmatrix} 1 & 0 \\ 50 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ (1/2)^{100} & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 25 & 0 \end{pmatrix}$ (d) none of these

Ans. (a)

Solution $A^2 = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2(1/2) & 1 \end{pmatrix}$

$$A^3 = A^2 A = \begin{pmatrix} 1 & 0 \\ 2(1/2) & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3(1/2) & 1 \end{pmatrix}$$

Continuing in this way, we get

$$A^{100} = \begin{pmatrix} 1 & 0 \\ 100(1/2) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 50 & 1 \end{pmatrix}$$

In fact $A^{2n} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \forall\, n \in N$

Example 30 If

$$\begin{pmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{pmatrix}\begin{pmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{pmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

then

(a) $a = b = 1$

(b) $a = \cos 2\theta,\ b = \sin 2\theta$

(c) $a = \sin 2\theta,\ b = \cos 2\theta$

(d) $a = 1,\ b = \sin 2\theta$

Ans. (b)

Solution We have

$$\begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \frac{1}{1+\tan^2\theta}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \cos^2\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix}$$

$$= \cos^2\theta \begin{bmatrix} 1-\tan^2\theta & -2\tan\theta \\ 2\tan\theta & 1-\tan^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

$\therefore \quad a = \cos 2\theta,\ b = \sin 2\theta$

SECTION II

MULTIPLE CORRECT CHOICE TYPE

Example 31 Let A and B be two matrices different from I such that $AB = BA$ and $A^n - B^n$ is invertible for some positive integer n. If $A^n - B^n = A^{n+1} - B^{n+1} = A^{n+2} - B^{n+2}$, then

(a) $I - A$ is singular

(b) $I - B$ is singular

(c) $A + B = AB + I$

(d) $(I - A)(I - B)$ is non-singular

Ans. (a), (b), (c)

Solution $A^{n+2} - B^{n+2} = (A + B)(A^{n+1} - B^{n+1}) - AB(A^n - B^n)$

$\Rightarrow \quad A^n - B^n = (A + B)(A^n - B^n) - AB(A^n - B^n)$

$\Rightarrow \quad I = A + B - AB \quad [\because A^n - B^n$ is not invertible]

$\Rightarrow \quad (I - A)(I - B) = O.$

As $A, B \neq I$, we get

$I - A$ and $I - B$ are signular matrices.

Example 32 If A is matrix of size $n \times n$ such that $A^2 + A + 2I = O$, then

(a) A is non-singular (b) $A \neq O$

(c) $|A| \neq 0$ (d) $A^{-1} = -\frac{1}{2}(A + I).$

Ans. (a), (b), (c), (d)

Solution We have $A(A + I) = -2I$

$\Rightarrow \quad |A(A + I)| = |-2I| \Rightarrow |A|\,|A + I| = (-2)^n \neq 0.$

Thus, $|A| \neq 0$. Also, $A\left\{-\frac{1}{2}(A + I)\right\} = I$

$\Rightarrow \quad A^{-1} = -\frac{1}{2}(A + I).$

Clearly $A \neq O$ for otherwise $|A| = 0$.

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SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos. 46 to 50

Two $n \times n$ square matrices A and B are said to be similar if there exists a non-singular matrix P such that

$$P^{-1} A P = B$$

Example 46 If A and B are two singular matrices, then

(a) det (A) = det (B)

(b) det (A) + det (B) = 0

(c) det $(AB) \neq 0$

(d) none of these

Example 47 If A and B are similar matrices such that det $(AB) = 0$, then

(a) det $(A) = 0$ and det $(B) = 0$

(b) det $(A) = 0$ or det $(B) = 0$

(c) $A = O$ and $B = O$

(d) $A = O$ or $B = O$

Example 48 If A and B are similar matrices such that det $(A) = 1$, then

(a) det $(B) = 1$ (b) det (A) + det $(B) = 0$

(c) det $(B) = -1$ (d) none of these

Example 49 If A and B are similar and B and C are similar, then

(a) AB and BC are similar

(b) A and C are similar

(c) $A + C$ and B are similar

(d) none of these

Example 50 If A and B are two non-singular matrices, then

(a) A is similar to B (b) AB is similar to BA

(c) AB is similar to $A^{-1}B$ (d) none of these

Ans. 46 (a), 47. (a), 48. (a), 49. (b), 50. (b)

Solution As A and B are similar matrices there exists a non-singular matrix P such that

$$A = P^{-1} BP$$

$$\Rightarrow \det(A) = \det(P^{-1} BP)$$

$$= \det(P^{-1}) \det(B) \det(P)$$

$$= \frac{1}{\det(P)} \det(B) (\det P)$$

$$= \det B$$

Thus, det $(A) = 0 \Leftrightarrow$ det $(B) = 0$ and det $(A) = 1 \Leftrightarrow$ det $(B) = 1$

Next, $A = P^{-1} BP$, $B = Q^{-1} C Q$,

$\Rightarrow \quad A = P^{-1} (Q^{-1} C Q)P = (QP)^{-1} C QP$

Thus, A is similar to C

Also, as $AB = B^{-1}(BA)B$, we get AB is similar to BA.

Paragraph for Question Nos. 51 to 55

Let A be a square matrix of order $n \times n$. A constant λ is said to be characteristic root of A if there exists a $n \times 1$ matrix X such that

$$AX = \lambda X$$

Example 51 If λ is a characteristic root of A, then

(a) $A - \lambda I = O$ (b) $A - \lambda I$ is singular

(c) $A - \lambda I$ is non-singular (d) none of these

Example 52 If a constant λ is such that $A - \lambda I$ is non-singular, then

(a) $\lambda = 0$ (b) $\lambda \neq 0$

(c) λ is not a characteristic root of A

(d) none of these

Example 53 If 0 is a characteristic root of A, then

(a) A is non-singular (b) A is singular

(c) $A = O$ (d) $A = I_n$

Example 54 If λ is a characteristic root of A and $n \in \mathbf{N}$, then λ^n is a characteristic root of

(a) A^n (b) A^{n-1}

(c) A^{-n} (d) $A - A^n + A^{-n}$

Example 55 Let P be a non-singular matrix, then which of the following matrices have the same characteristic roots.

(a) A and AP (b) A and PA

(c) A and $P^{-1} AP$ (d) none of these

Ans. 51. (b), 52. (c), 53. (b), 54. (a), 55. (c)

Solution Since $X \neq O$ is such that $(A - \lambda I)X = O$, $|A - \lambda I| = 0 \Leftrightarrow A - \lambda I$ is singular. If $A - \lambda I$ is non-singular the then equation $(A - \lambda I)X = O \Rightarrow X = O$

If $\lambda = 0$, we get $|A| = 0 \Rightarrow$ A is singular.

We have $A^2 X = A(AX) = A(\lambda X) = \lambda(AX)$

$= \lambda^2 X$,

$A^3 X = A(A^2 X) = A(\lambda^2 X)$

$= \lambda^2(AX) = \lambda^2(\lambda X) = \lambda^3 X$

Continuing in this way, we obtain

$$A^n X = \lambda^n X \ \forall \ n \in \mathbf{N}.$$

Also, $|P^{-1} AP - \lambda I| = |P^{-1} (A - \lambda I)P|$

$= |P^{-1}| \, |A - \lambda I| \, |P| = |A - \lambda I|$

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We have

$$(AB)B^m = AB^{m+1} = B^{m+1}A \quad \text{[using (5)]}$$
$$= B(B^m A)$$
$$= B(AB^m) = (BA)B^m \quad \text{[using (4)]}$$

Thus,

$$(AB)B^m = (BA)B^m$$

As B^m is an invertible matrix, we can cancel B^m from both the sides to obtain

$$AB = BA$$
$\Rightarrow$ $A^{-1}BA = B$
and $B^{-1}AB = A$ etc.

Example 72 Match the expression/statements on the left with the expression on the right.

(a) The minimum value of $\frac{x^2+2x+4}{x+2}$, $x > -2$ (p) 0

(b) Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew symmetric, and $(A + B)(A - B) = (A - B)(A + B)$. If $(AB)' = (-1)^k AB$, then possible value of k are (q) 1

(c) Let $a = \log_3\log_3 2$. A integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than (r) 2

(d) If $\binom{n-1}{r} = (k^2 - 3)\binom{n}{r+1}$, then possible integral value of k. (s) 3

Ans.

	p	q	r	s
a			●	
b	●	●		●
c		●		
d			●	

Solution (a) Put $x + 2 = t$, then $t > 0$. Also

$$\frac{x^2+2x+4}{x+2} = \frac{(t-2)^2+2(t-2)+4}{t}$$
$$= \frac{t^2-2t+4}{t} = t + \frac{4}{t} - 2$$
$$= \left(\sqrt{t} - \frac{2}{\sqrt{t}}\right)^2 + 2 \geq 2.$$

$\therefore$ least value of $\frac{x^2+2x+4}{x+2}$ is 2

which is attained when $x = 0$.

(b) $(A + B)(A + B) = (A + B)(A - B)$
$\Rightarrow AB = BA$.
Now, $(AB)' = B'A' = (-B)A = -BA = -AB$
$\therefore (-1)^k = -1 \Rightarrow k = 1, 3$.

(c) $1 < 2^{(-k+3^{-a})} < 2$
$\Rightarrow 0 < -k + 3^{-a} < 1$
$\Rightarrow k < 3^{-a} < k + 1$.
$\Rightarrow k < (\log_3 2)^{-1} < k + 1$
$\Rightarrow k < \log_2 3 < k + 1$
$\Rightarrow 2^k < 3 < 2^{k+1} \Rightarrow k = 1$.

(d) $k^2 - 3 = \frac{(n-1)!}{r!(n-r-1)!} \frac{(r+1)!(n-r-1)!}{n!}$
$= \frac{r+1}{n}$

But $0 < \frac{r+1}{n} \leq 1$.

Thus, $0 < k^2 - 3 \leq 1 \Rightarrow k = 2$.

SECTION VI

REASONING TYPE

Example 73 Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denote by $tr(A)$, the sum of diagonal entries of A. Assume that $A^2 = I$.
Statement-1: If $A \neq I$ and $A \neq -I$, then $\det(A) = -1$.
Statement-2: If $A \neq I$ and $A \neq -I$, then $tr(A) \neq 0$.
Ans. (c)

Solution Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Now, $A^2 = I \Rightarrow \det(A^2) = 1$
$\Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$.
Also, $A^2 = I \Rightarrow A = A^{-1}$
$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If $\det A = 1$, then

$a = d, b = -b, c = -c \Rightarrow a = d, b = c = 0$.

In this case $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

$|A| = 1 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$.

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33. If α, β, γ are three real numbers and

$$A = \begin{bmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & 1 \end{bmatrix}$$

then

(a) A is symmetric (b) A is orthogonal
(c) A is singular (d) A is not invertible.

34. Let $C_k = {}^nC_k$ for $0 \le k \le n$ and

$$A_k = \begin{pmatrix} C_{k-1}^2 & 0 \\ 0 & C_k^2 \end{pmatrix} \text{ for } k \ge 1, \text{ and } A_1 + A_2 + \ldots + A_n$$

$$= \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \text{ then}$$

(a) $k_1 = k_2$ (b) $k_1 + k_2 = {}^{2n}C_{2n} + 1$
(c) $k_1 = {}^{2n}C_n - 1$ (d) $k_2 = {}^{2n}C_{n+1}$

35. If $A(\theta) = \begin{pmatrix} \sin\theta & i\cos\theta \\ i\cos\theta & \sin\theta \end{pmatrix}$, then

(a) $A(\theta)$ is invertible for all $\theta \in \mathbf{R}$
(b) $A(\theta)^{-1} = A(-\theta)$
(c) $A(\theta)^{-1} = A(\pi - \theta)$
(d) $A(\theta) + A(\pi + \theta)$ is a null matrix

36. D is a 3×3 diagonal matrix. Which of the following statements is not true?

(a) $D' = D$
(b) $AD = DA$ for each matrix of size 3×3
(c) D^{-1} if exists is a diagonal matrix
(d) none of these.

37. If A is an invertible matrix, then which of the followings are true:

(a) $A \ne O$ (b) Adj. $A \ne O$
(c) $|A| \ne 0$ (d) $A^{-1} = |A|$ Adj.A.

38. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then

(a) $A^3 = I$ (b) $A^{-1} = A^2$
(c) $A^n = A \ \forall \ n \ne 4$ (d) none of these

39. If D_1 and D_2 are two 3×3 diagonal matrices, then

(a) $D_1 D_2$ is diagonal matrix
(b) $D_1 D_2 = D_2 D_1$
(c) $D_1^2 + D_2^2$ is a diagonal matrix
(d) none of these

40. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then

(a) $A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \ \forall \ n \in \mathbf{N}$.

(b) $\lim_{n\to\infty} \frac{1}{n} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

(c) $\lim_{n\to\infty} \frac{1}{n^2} A^{-n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) none of these

41. Let A, B and C be 2×2 matrices with entries from the set of real numbers. Define * as follows:

$$A * B = \frac{1}{2}(AB + BA), \text{ then}$$

(a) $A * B = B * A$ (b) $A * A = A^2$
(c) $A * (B + C) = A * B + A * C$
(d) $A * I = A$

42. With A, B, C as in Question No. 41, define o as follows:

$$AoB = \frac{1}{2}(AB - BA) \text{ then}$$

(a) $AoA = O$ (b) $AoI = O$
(c) $Ao(B + C) = AoB + AoC$
(d) $AoB = BoA$

43. Let $A = \begin{pmatrix} 0 & \sin\alpha & \sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha\sin\beta \\ -\sin\alpha\sin\beta & -\cos\alpha\cos\beta & 0 \end{pmatrix}$,

then

(a) $|A|$ is independent of α and β
(b) A^{-1} depends only on α
(c) A^{-1} depends only on β
(d) none of these

44. Let A and B are two matrices such that $AB = BA$, then for every $n \in \mathbf{N}$,

(a) $A^nB = BA^n$
(b) $(AB)^n = A^nB^n$
(c) $(A + B)^n = {}^nC_0A^n + {}^nC_1 A^{n-1} B + {}^nC_2 A^{n-2} B + \ldots + {}^nC_n B^n$.
(d) $A^{2n} - B^{2n} = (A^n - B^n)(A^n + B^n)$

45. If A and B are square matrices of the same order such that

$$A^2 = A, B^2 = B, AB = BA = O, \text{ then}$$

(a) $AB^2 = O$ (b) $(A + B)^2 = A + B$
(c) $(A - B)^2 = A - B$ (d) none of these.

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15. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $f(\lambda) = \det(A - \lambda I)$, then

(a) $f(\lambda) = \lambda^2 - (a + d)\lambda + ad - bc$
(b) $f(A) = 0$
(c) $A^2 = 0$ implies $A^r = 0 \ \forall \ r \geq 2$
(d) none of these

16. Let A be as in problem 15 and $A^k = O$ for some $k \geq 3$; then

(a) $A = O$ (b) $A^2 = O$
(c) A is singular (d) A is non-singular

17 If $a^2 + b^2 + c^2 = -2$, and

$$A = \begin{bmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{bmatrix}$$, then A is

non-singular if

(a) $x = 0$ (b) -1
(c) 1 (d) 2

18. Let A, B and C be 2×2 matrices with entries from the set of real numbers. Define * as follows:

$$A * B = \frac{1}{2}(AB' + A'B)$$

(a) $A * B = B * A$
(b) $A * A = A^2$
(c) $A * (B + C) = A * B + A * C$
(d) $A * I = A + A'$

19. With A, B, C as in Question No. 18, define o as follows:

$$AoB = \frac{1}{2}(AB' - BA'), \text{ then}$$

(a) $AoA = O$ (b) $AoI = O$
(c) $Ao(B + C) = AoB + AoC$
(d) none of these

20. Let A be a symmetric matrix such that $A^5 = O$ and $B = I + A + A^2 + A^4$, then B is

(a) symmetric (b) singular
(c) non-singular (d) skew symmetric

SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos. 21 to 25

A square matrix A is said to be an idempotent matrix if $A^2 = A$.

21. If A is a non-singular idempotent matrix, then

(a) $A = I_n$ (b) $A = O$
(c) $A + A' = O$ (d) none of these

22. If A and B are idempotent matrices and $AB = BA$, then which of the following is an idempotent matrix

(a) $A + B$ (b) AB
(c) $A - B$ (d) $B - A$

23. Suppose A is an idempotent matrix, and $B = I - A$, then

(a) B is an idempotent (b) $AB = I_n$
(c) $BA' = I_n$ (d) none of these

24. If A is an idempotent matrix and $B = I - A$, then which of the following is not true

(a) $AB = O$ (b) $BA = O$
(c) $B^2 = I$ (d) $AB = I_n$

25. If A, B and $A + B$ are idempotent matrices, then

(a) $AB = BA = O$ (b) $AB = BA = I_n$
(c) $AB = BA \neq O$ (d) $AB = BA \neq I_n$

Paragraph for Question Nos. 26 to 30

Let $A = (a_{ij})_{n \times n}$ be a matrix such that $a_{ij} \in \mathbf{C}$. Define trace of A, denoted by $tr(A)$ as follows:

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Let A, B and C be three $n \times n$ matrices such that $C'C = I_n$ and $\lambda, \mu \in \mathbf{C}$.

26. $\text{tr}(\lambda A + \mu B)$ equals

(a) $\lambda \ \text{tr}(A) + \mu \ \text{tr}(B)$ (b) $(\lambda + \mu) \ \text{tr}(A + B)$
(c) $\lambda\mu \ \text{tr}(A + B)$ (d) $(\lambda + \mu) \ \text{tr}(AB)$

27. $\text{tr}(AB)$ equals

(a) $\text{tr}(A) \ \text{tr}(B)$ (b) $\text{tr}(BA)$
(c) $\text{tr}(A) + \text{tr}(B)$ (d) none of these

28. $\text{tr}(AA')$ is

(a) positive (b) negative
(c) non-negative (d) non-positive

29. $\text{tr}(C'A\,C)$ equals

(a) $\text{tr}(A)$ (b) $\text{tr}(CA)$
(c) $\text{tr}(AC')$ (d) $\text{tr}(C)$

30. $\text{tr}(ABC)$ equals

(a) $\text{tr}(CBA)$ (b) $\text{tr}(BAC)$
(c) $\text{tr}(ACB)$ (d) $\text{tr}(ABC)$

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SECTION VI

73. (d) 74. (a) 75. (a)
76. (b) 77. (b)

LEVEL 2

SECTION I

1. (a) 2. (d) 3. (b)
4. (a) 5. (c) 6. (a)
7. (a) 8. (b) 9. (b)
10. (c) 11. (b) 12. (c)

SECTION II

13. (a), (b) 14. (a) 15. (a), (b), (c)
16. (b), (c) 17. (a), (b), (d)
18. (a), (c), (d) 19. (b), (c) 20. (c)

SECTION III

21. (a) 22. (b) 23. (a)
24. (a) 25. (a) 26. (a)
27. (b) 28. (c) 29. (a)
30. (a)

SECTION IV

31. 7 32. 11 33. 1
34. 3 35. 3 36. 100
37. 25

SECTION V

38.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

39.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

SECTION VI

40. (a) 41. (b)
42. (c) 43. (b)

PAST YEARS' IIT QUESTIONS

1. (d) 2. 4 3. (b)
5. (a) 6. (d) 7. $\frac{125}{3}$ $(\text{unit})^2$
8. (c) 9. (b) 10. (a)
11. (a) 12. (b) 13. (b)

HINTS AND SOLUTIONS

LEVEL 1

1. $|A|^3 = |A^3| = -27 \Rightarrow |A| = -3$
2. Use $AA^{-1} = I$
3. Use $A = -A'$
4. Set $|A_t| = 0$
5. $A^2 = I$
   $\Rightarrow a^2 + b^2 + c^2 = 1$ and $bc + ca + ab = 0$
   Also, $|A^2| = 1 \Rightarrow |A| = \pm 1$.
   But $|A| = 3abc - (a^3 + b^3 + c^3)$
   $= -(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$
   $= -(a + b + c)$
   As $a + b + c > 0$, we get $|A| = -1$.
   $\therefore a^3 + b^3 + c^3 = 7$
6. Let $A = \begin{pmatrix} 0 & -b & -c \\ b & 0 & -a \\ c & a & 0 \end{pmatrix}$, then $tr(A^2) = 0$

   $-2(a^2 + b^2 + c^2) = 0$

   $\Rightarrow a = b = c = 0$.

   $\therefore A = 0$
7. Use $A + X = B - X \Rightarrow X = \frac{1}{2}(B - A)$
8. Calculate directly
9. $A^2 = O$
10. $|A| = 1$ for each x.
11. Calculate directly
12. If $A = O$, $\text{tr}(A) = 0$. Suppose $A \neq O$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

    then $|A| = 0$ and $A^2 - (a + d)A + ad - bc = 0$

    $\Rightarrow a + d = 0$
13. Use $A^2 - (a + d)A + ad - bc = 0$ to obtain $ad - bc = 0$
14. Use $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$
15. See definition.
16. Calculate directly.
17. Use $A = \omega B$.

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CHAPTER 8

DETERMINANTS

8.1 EVALUATION OF DETERMINANTS

A determinant of order two is written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad (a_{ij} \in \mathbf{C} \ \forall \ i, j)$$

and is equal to $a_{11}\, a_{22} - a_{12}\, a_{21}$.

A determinant of order three is written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (a_{ij} \in \mathbf{C} \ \forall \ i, j)$$

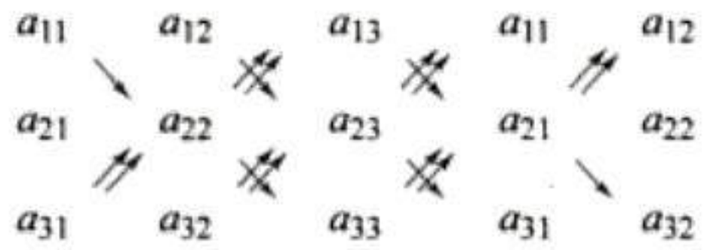
and is equal to

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{12}\,(a_{21}\, a_{33} - a_{23}\, a_{31}) + a_{13}(a_{21}\, a_{32} - a_{22}\, a_{31})$$

$$= a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{32}\, a_{21} - a_{13}\, a_{31}\, a_{22} - a_{32}\, a_{23}\, a_{11} - a_{12}\, a_{21}\, a_{33}$$

A determinant of order 3 can also be evaluated by using the following diagram, due to Sarrus:



The product of the three terms on each of the three single arrows are prefixed by a positive sign and the product of the three terms on each of the three double arrows are prefixed by a negative sign.

Remark

This method does not work for higher order determinants.

8.2 MINORS AND COFACTORS

MINOR

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n. Then the minor M_{ij} of the element a_{ij} of the matrix A is the determinant of the square sub-matrix of order $(n - 1)$ obtained by deleting ith row and jth column of the matrix A.

Cofactor

Let $A = (a_{ij})_{n \times n}$ be a square matrix of order n. Then the cofactor of the element a_{ij} of the matrix A is denoted by C_{ij} and is equal to $(-1)^{i+j} M_{ij}$ where M_{ij} is the minor of the element a_{ij} of the matrix A.

Note that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\, M_{11} - a_{12}\, M_{12} + a_{13}\, M_{13}$$

$$= a_{11}\, C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

then $\Delta = a_{i1}\, C_{i1} + a_{i2}\, C_{i2} + a_{i3}\, C_{i3} \qquad i = 1, 2, 3$

$$= a_{1j}\, C_{1j} + a_{2j}\, C_{2j} + a_{3j}\, C_{3j} \qquad j = 1, 2, 3$$

Also $a_{i1}\, C_{j1} + a_{i2}\, C_{j2} + a_{i3}\, C_{j3} = 0 \qquad i \neq j$

and $a_{1i}\, C_{1j} + a_{2i}\, C_{2j} + a_{3i}\, C_{3j} = 0 \qquad i \neq j$

The above results remain true for determinants of every order.

8.3 PROPERTIES OF DETERMINANTS

1. Reflection Property

The determinant remains unaltered if its rows are changed into columns and the columns into rows.

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Example 6 If

$$f(x)=\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$$

then $f(500)$ is equal to

(a) 0 (b) 1 (c) 500 (d) -500

Ans. (a)

Solution Taking x common from R_2 and $x(x-1)$ common from R_3, we get

$$f(x)=x^2(x-1)\begin{vmatrix} 1 & x & x+1 \\ 2 & x-1 & x+1 \\ 3 & x-2 & x+1 \end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$, we get

$$f(x)=x^2(x-1)\begin{vmatrix} 1 & x & 1 \\ 2 & x-1 & 2 \\ 3 & x-2 & 3 \end{vmatrix}=0$$

Thus, $f(500)=0$.

Example 7 The number of positive integral solutions of the equation $\Delta = 12$, where

$$\Delta=\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

is

(a) 3 (b) 15 (c) 4 (d) $2^3 3^2$

Ans. (a)

Solution Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = 2\begin{vmatrix} y+z & z & y \\ z+x & z+x & x \\ x+y & x & x+y \end{vmatrix}$$

Using $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} y+z & -y & -z \\ z+x & 0 & -z \\ x+y & -y & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & -y & -z \\ x & 0 & -z \\ x & -y & 0 \end{vmatrix}$$

[using $C_1 \to C_1 + C_2 + C_3$]

$$= 2xyz\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 2xyz\begin{vmatrix} 2 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

[using $C_1 \to C_1 + C_2 + C_3$]

Taking 2 common and applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\Delta = 4xyz\begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 4xyz$$

Now, $4xyz = 12 \Rightarrow xyz = 3$

The number 3 can be assigned to any of x, y, z. Therefore, the number of positive integral solutions of $\Delta = 12$ is 3.

Example 8 If $a_1, a_2, a_3, \ldots\ldots\ldots, a_9$ are in H.P. and $a_4 = 5$, $a_5 = 4$, then value of the determinant

$$\Delta=\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_6 & a_7 & a_8 \end{vmatrix}$$

equals

(a) $\frac{1}{20}$ (b) $\frac{50}{21}$ (c) $\frac{3}{20}$ (d) $\frac{7}{90}$

Ans. (b)

Solution Let $a_i = \frac{1}{a+id}$ for $1 \le i \le 9$.

We are given $5 = \frac{1}{a+4d}$ and $4 = \frac{1}{a+5d}$.

$\Rightarrow$ $a+4d = \frac{1}{5}, a+5d = \frac{1}{4}$

$\Rightarrow$ $a = 0, d = \frac{1}{20}$.

Thus, $a_k = \frac{20}{k}$ for $1 \le k \le 9$, and hence

$$\Delta = 20^3\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{vmatrix} = 20^3\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{3}{40} & \frac{1}{12} \\ \frac{1}{7} & \frac{3}{56} & \frac{4}{63} \end{vmatrix}$$

[Using $C_2 \to C_2 - \frac{1}{2}C_1$ and $C_3 \to C_3 - \frac{1}{3}C_1$]

$$= 20^3\left[\frac{3}{40}\times\frac{4}{63} - \frac{1}{12}\times\frac{3}{56}\right]$$

$$= 20^3 \times \frac{1}{15\times 224} = \frac{50}{21}.$$

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Solution We have

$$\bar{Z} = \begin{vmatrix} \bar{2} & \overline{3+i} & -\bar{3} \\ \overline{3-i} & \bar{0} & -\overline{1+i} \\ -\bar{3} & -\overline{1-i} & \bar{4} \end{vmatrix} = \begin{vmatrix} 2 & 3-i & -3 \\ 3+i & 0 & -1-i \\ -3 & -1+i & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3+i & -3 \\ 3-i & 0 & -1+i \\ -3 & -1-i & 4 \end{vmatrix} = Z \text{ [reflection property]}$$

$\therefore$ Z is purely real.

Example 22 If

$$\Delta_1 = \begin{vmatrix} a_1^2+b_1+c_1 & a_1a_2+b_2+c_2 & a_1a_3+b_3+c_3 \\ b_1b_2+c_1 & b_2^2+c_2 & b_2b_3+c_3 \\ c_3c_1 & c_3c_2 & c_3^2 \end{vmatrix}$$

and $\Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\dfrac{\Delta_1}{\Delta_2}$ is equal to

(a) $a_1\ b_2\ c_3$

(b) $a_1\ a_2\ a_3$

(c) $a_3\ b_2\ c_1$

(d) $a_1\ b_1\ c_1 + a_2\ b_2\ c_2 + a_3\ b_3\ c_3$

Ans. (a)

Solution Taking c_3 common from R_3 and applying $R_2 \to R_2 - R_3$ and $R_1 \to R_1 - R_3$. We obtain

$$\Delta_1 = c_3 \begin{vmatrix} a_1^2+b_1 & a_1a_2+b_2 & a_1a_3+b_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Taking b_2 common from R_2 and applying $R_1 \to R_1 - R_2$ we get

$$\Delta_1 = b_2\ c_3 \begin{vmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1\ b_2\ c_3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3\ \Delta_2$$

$$\Rightarrow \qquad \frac{\Delta_1}{\Delta_2} = a_1b_2c_3$$

Example 23 For $a, b, c, x, y, z \in \mathbf{R}$, if

$$\Delta_1 = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} \text{ and}$$

$$\Delta_2 = \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix} \text{ then}$$

(a) $\Delta_1 = \Delta_2$ (b) $\Delta_1 + \Delta_2 = 0$

(c) $\Delta_1 = 2\Delta_2$ (d) $\Delta_1 = -2\Delta_2$

Ans. (a)

Solution We can write the determinant Δ_1 as product of two determinants as follows:

$$\Delta_1 = \begin{vmatrix} x^2 & -2x & 1 \\ y^2 & -2y & 1 \\ z^2 & -2z & 1 \end{vmatrix} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

We interchange C_1 and C_3 in the first determinant and absorb the resulting negative sign in C_2 to get

$$\Delta_1 = \begin{vmatrix} 1 & 2x & x^2 \\ 1 & 2y & y^2 \\ 1 & 2z & z^2 \end{vmatrix} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix} = \Delta_2$$

Example 24 Let

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

then value of $\int_{\pi/4}^{\pi/2} f(x)dx$ is

(a) 0 (b) $\pi/48$

(c) $-\dfrac{\pi}{2} - \dfrac{\pi}{15\sqrt{2}}$ (d) none of these

Ans. (d)

Solution Applying $R_2 \to R_2 - R_3$, we get

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ -\sin^2 x & 0 & 0 \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

$$= -(-\sin^2 x) \begin{vmatrix} \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

$$= \sin^2 x\ [\cos x \operatorname{cosec}^2 x - \cos^2 x\ (\sec^2 x + \cot x \operatorname{cosec} x)]$$

$$= \cos x - \sin^2 x - \cos^3 x = \cos x \sin^2 x - \frac{1}{2}(1 - \cos^2 x)$$

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$$\Delta = (a+b+c-x)\begin{vmatrix} 1 & 1 & 1 \\ c & b-x & a \\ b & a & c-x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we obtain

$$\Delta = -x\begin{vmatrix} 1 & 0 & 0 \\ c & b-c-x & a-c \\ b & a-b & c-b-x \end{vmatrix} \quad [\because a+b+c=0]$$

Expanding along R_1, we get

$$\Delta = -x[(b-c-x)(c-b-x) - (a-b)(a-c)]$$
$$= x[(a-b)(a-c) - (x+b-c)(x-b+c)]$$
$$= x[a^2 - ab - ac + bc - x^2 + b^2 + c^2 - 2bc]$$
$$= x[a^2 + b^2 + c^2 - bc - ca - ab - x^2]$$

$\Delta = 0$ implies $x=0$ or $x^2 = a^2+b^2+c^2-bc-ca-ab$.

Now $x^2 = a^2 + b^2 + c^2 - bc - ca - ab$

$$= a^2 + b^2 + c^2 - \frac{1}{2}[(a+b+c)^2 - a^2 - b^2 - c^2]$$

$$= \frac{3}{2}(a^2 + b^2 + c^2) \qquad [\because a+b+c=0]$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$$

Example 37 The determinant

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$$

is equal to zero if

(a) a, b, c are in A.P.

(b) a, b, c are in G.P.

(c) a, b, c are in H.P.

(d) α is a root of $ax^2 + 2bx + c = 0$.

Ans. (b), (d).

Solution Applying $R_3 \to R_3 - \alpha R_1 - R_2$, we get

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ 0 & 0 & -(a\alpha^2+2b\alpha+c) \end{vmatrix}$$

$$= (b^2 - ac)(a\alpha^2 + 2b\alpha + c)$$

by expanding along R_3. Now $\Delta = 0$ if either $b^2 - ac = 0$, or $a\alpha^2 + 2b\alpha + c = 0$, that is, if either a, b and c are in G.P., or α is a root of $ax^2 + 2bx + c = 0$.

Example 38 If

$$\Delta(x) = \begin{vmatrix} x^2-5x+3 & 2x-5 & 3 \\ 3x^2+x+4 & 6x+1 & 9 \\ 7x^2-6x+9 & 14x-6 & 21 \end{vmatrix}$$

$= ax^3 + bx^2 + cx + d$, then

(a) $a = 0$ (b) $b = 0$

(c) $c = 0$ (d) d = 47

Ans. (a), (b), (c), (d)

Solution Differentiate both the sides with respect to x

$$\Delta'(x) = \begin{vmatrix} 2x-5 & 2x-5 & 3 \\ 6x+1 & 6x+1 & 9 \\ 14x-6 & 14x-6 & 21 \end{vmatrix} + \begin{vmatrix} x^2-5x+3 & 2 & 3 \\ 3x^2+x+4 & 6 & 9 \\ 7x^2-6x+9 & 14 & 21 \end{vmatrix}$$

$$= 0 + 0 = 0$$

$\Rightarrow$ $\Delta(x)$ is a constant.

Thus, $a = 0$, $b = 0$ and $c = 0$.

For value of d, put $x = 0$

Example 39 If $\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix}$

$$= (x-y)(y-z)(z-x)(1/x + 1/y + 1/z)$$

then

(a) $k = -3$ (b) $k = -1$

(c) $k = 1$ (d) $k = 3$

Ans. (b)

Solution We can write the given expressions as

$$x^k y^k z^k \Delta_1 = \frac{1}{xyz}(x-y)(y-z)(z-x)(yz+zx+xy)$$

where $\Delta_1 = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

$\Rightarrow$ $x^{k+1} y^{k+1} z^{k+1} \Delta_1$

$$= (x-y)(y-z)(z-x)(yz+zx+xy)$$

Note that degree of Δ_1 is 5.

$\therefore$ degree of LHS $= 3k + 3 + 5 = 3k + 8$

Also degree of RHS = 5

$\therefore$ $3k + 8 = 5$ or $k = -1$.

Example 40 If

$$\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$$, then,

a factor of Δ is

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Ans. (a), (b)

Solution The given system of equations will have a non-trivial solution if

$$\Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 + 7R_1$, we get

$$\Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta + 4\sin 3\theta & 0 & 7 \\ 2 + 7\sin 3\theta & 0 & 14 \end{vmatrix} = 0$$

Expanding along C_2, we get

$$\Delta = 7[2\ (\cos\ 2\theta + 4\ \sin\ 3\theta) - (2 + 7\ \sin\ 3\theta)] = 0$$

$$\Rightarrow \quad 2\cos\ 2\theta + \sin\ 3\theta - 2 = 0$$

$$\Rightarrow \quad \sin\ \theta(2\ \sin\ \theta - 1)\ (2\ \sin\ \theta + 3) = 0$$

As $2 \sin \theta + 3$ can never be zero, we get

$$\sin\ \theta = 0 \quad \text{or} \quad \sin\ \theta = 1/2 = \sin\ (\pi/6)$$

$$\Rightarrow \quad \theta = n\pi \quad \text{or} \quad \theta = m\pi + (-1)^m\ \pi/6\ (n, m \in \mathbf{I})$$

SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos. 51 to 55

Let

$$\Delta(x) = \begin{vmatrix} 2x^3 - 3x^2 & 5x + 7 & 2 \\ 4x^3 - 7x & 3x + 2 & 1 \\ 7x^3 - 8x^2 & x - 1 & 3 \end{vmatrix}$$

$$= a_0 + a_1\ x + \cdots + a_4\ x^4$$

To evaluate a_i we differentiate $\Delta(x)$ i times w.r.t. x and put $x = 0$ or divide $\Delta(x)$ by x^4 put $1/x = t$, differentiate $(4 - i)$ time w.r.t. t and put $t = 0$.

Example 51 a_0 equals

(a) 0 (b) 1 (c) 2 (d) 3

Example 52 a_1 equals

(a) 0 (b) 61 (c) 161 (d) 191

Example 53 a_4 equals

(a) 41 (b) – 43 (c) – 41 (d) 43

Example 54 a_3 equals

(a) – 70 (b) – 73 (c) – 74 (d) 0

Example 55 a_2 equals

(a) 46 (b) – 43 (c) 41 (d) 0

Ans. 51 (a), 52 (c), 53 (b), 54 (b) 55 (c)

Solution 51. $\Delta(0) = 0$

52. $$\Delta'(x) = \begin{vmatrix} 6x^2 - 6x & 5x + 7 & 2 \\ 12x^2 - 7 & 3x + 2 & 1 \\ 21x^2 - 16x & x - 1 & 3 \end{vmatrix} + \begin{vmatrix} 2x^3 - 3x^2 & 5 & 2 \\ 4x^3 - 7x & 3 & 1 \\ 7x^3 - 8x^2 & 1 & 3 \end{vmatrix}$$

$$a_1 = \Delta(0) = 161$$

53. $$\frac{\Delta(x)}{x^4} = \begin{vmatrix} 2 - 3/x & 5 + 7/x & 2 \\ 4 - 7/x^2 & 3 + 2/x & 1 \\ 7 - 8/x & 1 - 1/x & 3 \end{vmatrix}$$

Taking limit as $x \to \infty$, we get

$$a_4 = \begin{vmatrix} 2 & 5 & 2 \\ 4 & 3 & 1 \\ 7 & 1 & 3 \end{vmatrix} = -43$$

54. Put $1/x = t$ in $\Delta(x)/x^4$ and write

$$\Delta_1(t) = \begin{vmatrix} 2 - 3t & 5 + 7t & 2 \\ 4 - 7t^2 & 3 + 2t & 1 \\ 7 - 8t & 1 - t & 3 \end{vmatrix}$$

$$\Delta_1'(t) = \begin{vmatrix} -3 & 5 + 7t & 2 \\ -14t & 3 + 2t & 1 \\ -8 & 1 - t & 3 \end{vmatrix} + \begin{vmatrix} 2 - 3t & 7 & 2 \\ 4 - 7t^2 & 2 & 1 \\ 7 - 8t & -1 & 3 \end{vmatrix}$$

$$\Rightarrow \quad a_3 = \Delta_1'(0) = -73$$

55. As $a_0 = 0$ we get

$$a_1 + a_2x + a_3x^2 + a_4x^3$$

$$= \begin{vmatrix} 2x^2 - 3x & 5x + 7 & 2 \\ 4x^2 - 7 & 3x + 2 & 1 \\ 7x^2 - 8x & x - 1 & 3 \end{vmatrix}$$

Differentiating we get

$$a_2 + 2a_3x + 3a_4x^2$$

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Ans. 0

Solution We have

$$\sum_{r=1}^{n}\Delta_r = \begin{vmatrix} \sum_{r=1}^{n}(2r+1) & 6n(n+2) & f(n) \\ \sum_{r=1}^{n}2^{r-1} & 3\cdot 2^{n+1}-6 & g(n) \\ \sum_{r=1}^{n} r(n-r+1) & n(n+1)(n+2) & h(n) \end{vmatrix}$$

But $\sum_{r=1}^{n}(2r+1) = 2\sum_{r=1}^{n} r + n = 2\cdot n\frac{(n+1)}{2} + n = n(n+2)$,

$$\sum_{r=1}^{n} 2^{r-1} = 2^n - 1,$$

and $\sum_{r=1}^{n} r(n-r+1) = (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2$

$$= \frac{(n+1)n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6}n(n+1)(n+2)$$

Thus,

$$\sum_{r=1}^{n}\Delta_r = \begin{vmatrix} n(n+2) & 6n(n+2) & f(n) \\ 2^n-1 & 6(2^n-1) & g(n) \\ \frac{1}{6}n(n+1)(n+2) & n(n+1)(n+2) & h(n) \end{vmatrix} = 0$$

[$\because$ C_1 and C_2 are proportional]

Example 71 If $0 \le \theta \le \pi$, and the system of equations

$$x = (\sin\theta)\,y + (\cos\theta)\,z$$
$$y = z + (\cos\theta)\,x$$
$$z = (\sin\theta)\,x + y$$

has a nontrivial solution, find $4\theta/\pi$.

Ans. 3

Solution We rewrite the system of equations as follows:

$$-x + (\sin\theta)\,y + (\cos\theta)\,z = 0$$
$$(\cos\theta)\,x - y + z = 0$$
$$(\sin\theta)\,x + y - z = 0$$

As this system of equations has a non-trivial solution

$$\begin{vmatrix} -1 & \sin\theta & \cos\theta \\ \cos\theta & -1 & 1 \\ \sin\theta & 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow -1 + \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta + 1 = 0$

$\Rightarrow \sin 2\theta = -1 \Rightarrow 2\theta = 3\pi/2$

$\Rightarrow 4\theta/\pi = 3$

Example 72 If the system of equations

$$ax + hy + g = 0 \quad (1)$$
$$hx + by + f = 0 \quad (2)$$

and $ax^2 + 2hxy + by^2 + 2gx + 2fy + c + t = 0$ (3)

has a unique solution, and

$$\frac{abc + 2fgh - af^2 - bg^2 - ch^2}{h^2 - ab} = 8, \text{ find } t.$$

Ans. 8

Solution Since we should be able to solve (1) and (2) for x and y, we must have $ab - h^2 \neq 0$. We can rewrite (3) as

$$x(ax+hy+g) + y(hx+by+f) + gx + fy + c + t = 0 \quad (4)$$

Since any solution of (1) and (2) must be a solution of (3), the given system of equations can be rewritten as

$$ax + hy + g = 0,\ hx + by + f = 0$$

and $gx + fy + c + t = 0$

Since the above system of equations is consistent, we must have

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+t \end{vmatrix} = 0$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} + \begin{vmatrix} a & h & 0 \\ h & b & 0 \\ g & f & t \end{vmatrix} = 0$$

$\Rightarrow [abc + 2fgh - af^2 - bg^2 - ch^2] + t(ab - h^2) = 0$

$\Rightarrow t = \dfrac{abc + 2fgh - af^2 - bg^2 - ch^2}{h^2 - ab}$

Example 73 Find the number of values of t for which the system of equations

$$(a+2t)x + by + cz = 0$$
$$bx + (c+2t) + az = 0$$
$$cx + ay + (b+2t)\,z = 0$$

has non-trivial solutions.

Ans. 3

Solution The given system of equations will have a non-trivial solution if

$$\Delta = \begin{vmatrix} a+2t & b & c \\ b & c+2t & a \\ c & a & b+2t \end{vmatrix} = 0$$

Clearly Δ is a cubic polynomial in t and has 3 roots.

Example 74 Find the number of real roots of the equation

$$\Delta(x) = \begin{vmatrix} 6-5x & x-2 & 4-4x \\ x-2 & 2-x & 0 \\ 4-4x & 0 & 3-5x \end{vmatrix} = 0$$

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$\therefore$ $\Delta(x) = 0$ has only one real root, viz. -6.
It lies in $[-7, 7]$ and $[-7, 0]$.

(d) Denote the determinant by $\Delta(x)$.
Applying $C_1 \to C_1 + C_2 + C_3$, we obtain

$$\Delta(x) = (7-2x)\begin{vmatrix} 1 & 4-x & 2 \\ 1 & 2 & 4-x \\ 1 & 4-x & 2-x \end{vmatrix}$$

Use $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$.

$$\Delta(x) = (7-2x)\begin{vmatrix} 1 & 4-x & 2 \\ 0 & x-2 & 2-x \\ 0 & 0 & -x \end{vmatrix}$$

$$= (7-2x)(x-2)(-x)$$

Thus, real roots of $\Delta(x) = 0$ is are 0, 2 and 7/2 which lie in $[-7, 7]$, $[0, 4]$ and $[0, 6]$.

SECTION VI

REASONING TYPE

Example 80 Let $\Delta_n = \begin{vmatrix} a_n & a_{n+3} & a_{n+6} \\ a_{n+1} & a_{n+4} & a_{n+7} \\ a_{n+2} & a_{n+5} & a_{n+8} \end{vmatrix}$

Statement-1: If $a_k > 0\ \forall\ k \geq 1$ and $a_1, a_2, a_3, \ldots$ are in G.P. then

$$\Delta_n = 0\ \forall\ n \geq 1.$$

Statement-2: If $a_1, a_2, a_3 \ldots$ are in A.P. then

$$\Delta_n = 0\ \forall\ n \geq 1.$$

Ans. (b)
Solution Let $a_k = ar^{k-1}\ \forall\ k \geq 1$, then

$$\Delta_n = a_n a_{n+3} a_{n+6}\begin{vmatrix} 1 & 1 & 1 \\ r & r & r \\ r^2 & r^2 & r^2 \end{vmatrix} = 0$$

Next, if $a_k = b + (k-1)d$, then using $C_2 \to C_3 - C_1$ and $C_2 \to C_2 - C_1$, we get

$$\Delta_n = \begin{vmatrix} a_n & 3d & 3d \\ a_{n+1} & 3d & 3d \\ a_{n+2} & 3d & 3d \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are identical]

Example 81 Let $f(x) = \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2$.

Statement-1: If $\sin 2x = 1$, then $f(x) = 2/3$
Statement-2: $f(x) = 0$ if $\sin x = \cos x$
Ans. (d)
Solution Multiplication of two determinants leads us

$$f(x) = \begin{vmatrix} 1 & -y & y \\ -y & 1 & y \\ y & y & 1 \end{vmatrix}^2$$

where $y = \sin x \cos x$
Using $C_1 \to C_1 - C_2$, $C_2 \to C_2 + C_3$, we get

$$f(x) = (1+y)^2\begin{vmatrix} 1 & 0 & y \\ -1 & 1 & y \\ 0 & 1 & 1 \end{vmatrix} = (1+y)^2\begin{vmatrix} 1 & 0 & y \\ 0 & 1 & 2y \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (1+y)^2(1-2y)$$

When $\sin 2x = 1$, $y = 1/2$ and $f(x) = 0$
When $\sin x = \cos x$, $1 - 2y = 1 - 2\sin^2 x = \cos 2x = 0$
$\therefore$ $f(x) = 0$.

Example 82 Suppose $x > 0$, $y > 0$, $z > 0$ and

$$\Delta(a, b, c) = \begin{vmatrix} x\log 2 & 3 & 15 + \log(a^x) \\ y\log 3 & 5 & 25 + \log(b^y) \\ z\log 5 & 7 & 35 + \log(c^z) \end{vmatrix}$$

Statement-1: $\Delta(8, 27, 125) = 0$

Statement-2: $\Delta\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right) = 0$

Ans. (b)
Solution Using $\log(b^c) = c\log b$ and applying $C_3 \to C_3 - 5C_2$ we get

$$\Delta(a, b, c) = \begin{vmatrix} x\log 2 & 3 & x\log a \\ y\log 3 & 5 & y\log b \\ z\log 5 & 7 & z\log c \end{vmatrix}$$

$\Delta(8, 27, 125) = \Delta(2^3, 3^3, 5^3) = 0$ as in this case C_1 and C_3 are proportional. Similarly,
$\Delta(1/2, 1/3, 1/5) = \Delta(2^{-1}, 3^{-1}, 5^{-1}) = 0$

Example 83 Let $a \neq 0$, $p \neq 0$ and

$$\Delta = \begin{vmatrix} a & b & c \\ 0 & p & q \\ p & q & 0 \end{vmatrix}$$

Statement-1: If the equations

$$ax^2 + bx + c = 0 \text{ and } px + q = 0$$

have a common root, then $\Delta = 0$.

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(a) a (p) 1/2
(b) b (q) $-5/4$
(c) $f(2)$ (r) 1/4
(d) $f'(0)$ (s) 1/8

82. If $0 \le x \le 1$ and $f(x) = \begin{vmatrix} x & 1 & 1 \\ -1 & x & 1 \\ -1 & -1 & x \end{vmatrix}$, then $f(x)$ has

(a) least value (p) 4
(b) greatest value (q) 1
(c) a maximum at (r) 0
(d) a minimum at (s) -1

83. Let

$$f(x) = \begin{vmatrix} 2\cot x & -1 & 0 \\ 1 & \cot x & -1 \\ 0 & 1 & 2\cot x \end{vmatrix}$$

then

(a) $\int_{\pi/6}^{\pi/3} f(x)dx$ (p) -16
(b) $f(\pi/4)$ (q) $-16/3$
(c) $f'(\pi/4)$ (r) 8
(d) $f(\pi/2)$ (s) 0

84. Let $\Delta_1 = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$,

$$\Delta_2 = \begin{vmatrix} 1 & y & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} x & x+y & x+2y \\ -x & x & 0 \\ 0 & -x & x \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & 1 & 1 \end{vmatrix}$$

then

(a) Δ_1 (p) $3x^2(x+y)$
(b) Δ_2 (q) xy
(c) Δ_3 (r) $-2(x^3+y^3)$
(d) Δ_4 (s) $(x-y)(y-1)(1-x)$

85. If a, b, c are distinct, match the right side with degree of the polynomial on the left side.

(a) $\begin{vmatrix} x & a^2 & b \\ x^2 & b^2 & b \\ x^3 & c^2 & c \end{vmatrix}$ (p) 0

(b) $\begin{vmatrix} x^3+a^3 & x^2+a^2 & x+a \\ x^3+b^3 & x^2+b^2 & x+b \\ x^3+c^3 & x^2+c^2 & x+c \end{vmatrix}$ (q) 1

(c) $\begin{vmatrix} 1 & x & x^2 \\ x^3 & x^4 & x^5 \\ x^6 & x^7 & x^8 \end{vmatrix}$ (r) 2

(d) $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix}$ (s) 3

86. Match the expression on the left with the appropriate value on the right

(a) If $f(x) = \begin{vmatrix} \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(\beta+\gamma) & \sin(\gamma+\alpha) & \sin(\alpha+\beta) \end{vmatrix}$ (p) -1

and $f(0) = 0$, then $\sum_{j=-5}^{5} f(j)$ equals

(b) If $\sin 2x = 1$, then (q) 0

$$f(x) = \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2$$ equals

(c) If $f(x) = \begin{vmatrix} \sin x & \sec x & x^2-1 \\ \operatorname{cosec} x & x\sin x & \cos x \\ \tan x & x\tan x & x^2+1 \end{vmatrix}$ (r) 2

then $\int_{-\pi/3}^{\pi/3} f(x)\,dx$ equals

(d) Maximum value of the determinant (s) 3

$$\begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$$

SECTION VI

REASONING TYPE

87. **Statement-1:** If

$$\Delta(x) = \begin{vmatrix} \sin x - \cos x & \sin\left(x-\frac{\pi}{3}\right) & \cos\left(x-\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}-x\right) & \tan\left(x-\frac{\pi}{4}\right) & \sec\left(x-\frac{\pi}{4}\right) \\ \cos\left(\frac{2\pi}{3}+x\right) & \sec\left(\frac{2\pi}{3}+x\right) & \cot\left(x+\frac{\pi}{4}\right) \end{vmatrix}$$

then $\Delta\left(\frac{\pi}{4}\right) = 0$

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28. Let $f(x) = a + bx + cx^2$ and $\omega \neq 1$ be a cube root of unity, and

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

then factors of Δ are

(a) $f(1)$ (b) $f(\omega)$

(c) $f(\omega^2)$ (d) none of these.

29. The system of equations

$$(1 + \lambda)x_1 + x_2 + x_3 = 1$$
$$x_1 + (1 + \lambda)x_2 + x_3 = \lambda$$
$$x_1 + x_2 + (1 + \lambda)x_3 = \lambda^2$$

(a) is in consistent if $\lambda = 0$ or $\lambda = -3$

(b) has a unique solution if $\lambda(\lambda + 3) \neq 0$

(c) has infinite number of solution for $\lambda = -3$

(d) none of these

30. Let $|a_1| > |a_2| + |a_3|$, $|b_2| > |b_1| + |b_3|$ and $|c_3| > |c_1| + |c_2|$ and let

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

then

(a) $\Delta \neq 0$ (b) $\Delta = 0$

(c) $|\Delta| > 0$ (d) none of these

SECTION III

COMPREHESION TYPE

Paragraph for Question Nos. 31 to 34

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

and $c_{ij} = (-1)^{i+j}$ (determinant obtained by deleting ith row and jth column), then

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = \Delta^2.$$

31. If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 7$ and

$\Delta = \begin{vmatrix} x^3 - 1 & 0 & x - x^4 \\ 0 & x - x^4 & x^3 - 1 \\ x - x^4 & x^3 - 1 & 0 \end{vmatrix}$, then

(a) $\Delta = 7$ (b) $\Delta = 343$

(c) $\Delta = -49$ (d) $\Delta = 49$

32. Suppose $a, b, c \in \mathbf{R}$, $a + b + c > 0$, $A = bc - a^2$, $B = ca - b^2$ and $C = ab - c^2$,

and $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = 49$,

then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ equals

(a) -7 (b) 2401 (c) -2401 (d) 7

33. If $a^3 + b^3 + c^3 - 3abc = -5$ and A, B, C are as in Question 32, value of $aA + bB + cC$ is

(a) -5 (b) 5 (c) 25 (d) -25

34. If A, B, C are as in Question 32, $3ABC - A^3 - B^3 - C^3 = \pi^2$ and

$$\Delta = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

then $\sin^3(\Delta) + \cos^3(\Delta)$ equals

(a) 1 (b) -1 (c) 0 (d) none of these

Paragraph for Question Nos. 35 to 39

Sometimes it is easier for us to evaluate a determinant if we can express it as product of two determinant. For instance, if

$S_k = a^k + b^k + c^k$, then

$\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix}$ can be written as Δ^2

where $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

which is quite easy to evaluate.

35. If $a_1 > a_2 > a_3$, $b_1 > b_2 > b_3$ and $a_i b_j \neq 1$ for $1 \leq i, j \leq 3$, then the determinant

$$\Delta = \begin{vmatrix} \frac{1 - a_1^3 b_1^3}{1 - a_1 b_1} & \frac{1 - a_1^3 b_2^3}{1 - a_1 b_2} & \frac{1 - a_1^3 b_3^3}{1 - a_1 b_3} \\ \frac{1 - a_2^3 b_1^3}{1 - a_2 b_1} & \frac{1 - a_2^3 b_2^3}{1 - a_2 b_2} & \frac{1 - a_2^3 b_3^3}{1 - a_2 b_3} \\ \frac{1 - a_3^3 b_1^3}{1 - a_3 b_1} & \frac{1 - a_3^3 b_2^3}{1 - a_3 b_2} & \frac{1 - a_3^3 b_3^3}{1 - a_3 b_3} \end{vmatrix}$$

is

(a) positive (b) non-negative

(c) negative (d) non-positive

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degree 3, 4 and 5 respectively, then show that

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(a) & B(a) & C(a) \\ A'(a) & B'(a) & C'(a) \end{vmatrix}$$

is divisible by $f(x)$. [1984]

15. Three lines $px + qy + r = 0$, $qx + ry + p = 0$ and $rx + py + q = 0$ are concurrent if:
(a) $p + q + r = 0$
(b) $p^2 + q^2 + r^2 = pq + rq + rp$
(c) $p^3 + q^3 + r^3 = 3pqr$
(d) none of these [1985]

16. True/False

If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

then the two triangles with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (a_1, b_1), (a_2, b_2), (a_3, b_3) must be congruent. [1985]

17. If $f_r(x)$, $g_r(x)$, $h_r(x)$, $r = 1, 2, 3$, are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, $r = 1, 2, 3$

and $$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

then $F'(x)$ at $x = a$ is _____. [1985]

18. Show that

$$\begin{vmatrix} {}^{x}C_{r} & {}^{x}C_{r+1} & {}^{x}C_{r+2} \\ {}^{y}C_{r} & {}^{y}C_{r+1} & {}^{y}C_{r+2} \\ {}^{z}C_{r} & {}^{z}C_{r+1} & {}^{z}C_{r+2} \end{vmatrix} = \begin{vmatrix} {}^{x}C_{r} & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^{y}C_{r} & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^{z}C_{r} & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

[1985]

19. The determinant

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$$

is equal to zero, if
(a) a, b, c are in A.P.
(b) a, b, c are in G.P.
(c) a, b, c are in H.P.
(d) α is a root of $ax^2 + bx + c = 0$
(e) $(x - a)$ is a factor of $ax^2 + 2bx + c$
[1986]

20. Consider the system of linear equations in x, y, z:

$$(\sin 3\theta)x - y + z = 0$$
$$(\cos\theta)x + 4y + 3z = 0$$
$$2x + 7y + 7z = 0$$

Find the values of θ for which the system has a non-trivial solution. [1986]

21. Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$

Then $\int_0^{\pi/2} f(x)\,dx =$ _____. [1987]

22. Let α_1, α_2 and β_1, β_2 be the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$, respectively. If the system of equations

$$\alpha_1 y + \alpha_2 z = 0$$
$$\beta_1 y + \beta_2 z = 0$$

has a non-trivial solution, show that

$$\frac{b^2}{q^2} = \frac{ac}{pr}$$ [1987]

23. The value of θ lying between $\theta = 0$ and $\theta = \frac{\pi}{2}$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0 \text{ are}$$

(a) $\frac{7\pi}{24}$ (b) $\frac{5\pi}{24}$
(c) $\frac{11\pi}{24}$ (d) $\frac{\pi}{24}$ [1988]

24. The value of the determinant

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$$ is _____. [1988]

25. Let

$$\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2 - 3n \end{vmatrix}$$

show that

$$\sum_{a=1}^{n} \Delta_a = c,\ a \text{ constant.}$$ [1989]

26. Let the three digit numbers A28, 3B9 and 62C,

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SECTION VI

87. (a) 88. (d) 89. (b)
90. (a) 91. (c)

LEVEL 2

SECTION I

1. (b) 2. (d) 3. (a)
4. (d) 5. (b) 6. (d)
7. (b) 8. (c) 9. (d)
10. (c) 11. (a) 12. (b)
13. (c) 14. (b) 15. (a)
16. (c) 17. (c) 18. (a)
19. (c) 20. (b)

SECTION II

21. (a), (b), (c) 22. (a), (b), (c)
23. (a), (b), (c), (d) 24. (a), (b)
25. (a), (b), (c) 26. (a), (b), (c), (d)
27. (a), (c), (d) 28. (a), (b), (c)
29. (a), (b) 30. (a), (c)

SECTION III

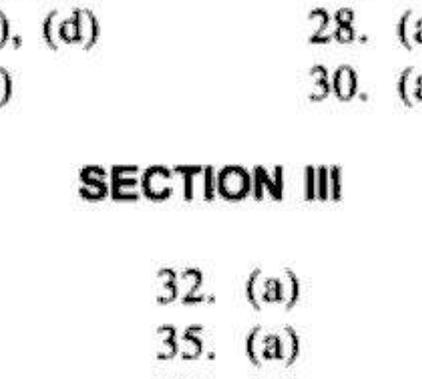
31. (d) 32. (a) 33. (a)
34. (b) 35. (a) 36. (d)
37. (d) 38. (d) 39. (a)

SECTION IV

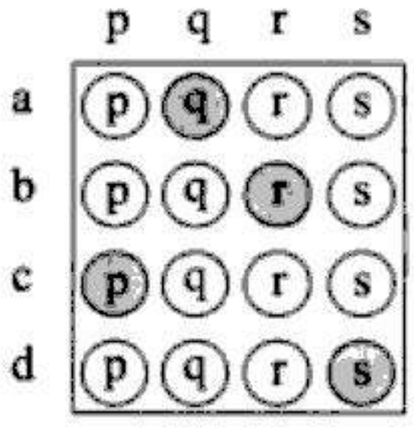
40. 0 41. 1 42. 5
43. 57 44. 17 45. 9
46. 0 47. 12

SECTION V

48.



49.



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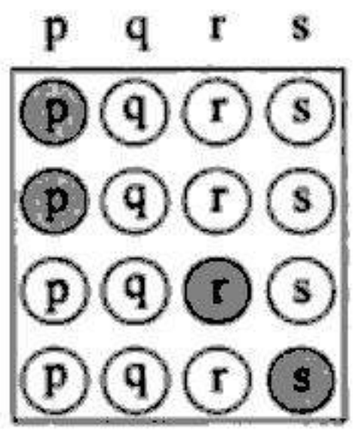
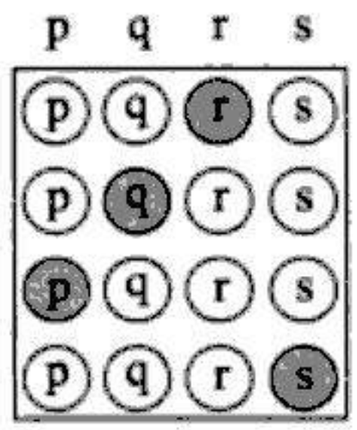


51.



SECTION VI

52. (b) 53. (d)

PAST YEARS' IIT QUESTIONS

1. $k = 33/2 \quad x = 5\lambda\backslash 2,\ y = \lambda\backslash 3,\ z = \lambda,\ \lambda = 6$
2. (d)
3. $x = 1,\ y = 0,\ z = 0,\ w = 0$
4. $m < -15/2$ or $m > 30$
5. 0
6. $\{-1, 2\}$
7. (b)
10. False
11. 2, 7,
12. $x = (4 - 5k)/7,\ y = (13k - 9)/7$
    $z = k$ where $k \in R$.
13. 0
15. (a), (b), (c)
16. False
17. 0
19. (b), (e)
20. $\theta = n\pi$ or $\theta = m\pi + (-1)^m \pi/6,\ n, m \in I$.
21. $-(15\pi + 32)/60$
23. (a), (c)
24. 0
27. 2
29. 0
30. $\lambda = \sin 2\alpha + \cos 2\alpha;\ \alpha = m\pi$ or $\alpha = k\pi + \pi/4$ where $m, k \in I$.
32. (d)
33. $$\frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$
34. (d) 35. (d) 36. 0
37. (b) 38. (d) 39. (a)

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But $\cos(2\pi/3) = \cos(\pi - \pi/3) = -\cos(\pi/3) = -1/2$.
and $\cos(4\pi/3) = \cos(\pi + \pi/3) = -\cos(\pi/3) = -1/2$.

44. First take a, b, c common from C_1, C_2, C_3 and then from R_1, R_2, R_3

$$\Delta(x) = 3a^2b^2c^2x^2(3 + x)$$

45. Expanding the given determinant along C_1, we get

$$f(x) = 2\cos x \begin{vmatrix} 2\cos x & 1 \\ 1 & 2\cos x \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 2\cos x \end{vmatrix}$$

$$= (2\cos x)(4\cos^2 x - 1) - 2\cos x = 8\cos^3 x - 4\cos x$$

46. Multiplying C_3 by $\sin\beta$, the given determinant is

$$\Delta = \frac{1}{\sin\beta}\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha\sin^2\beta \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha\sin^2\beta \\ 0 & -\sin\beta & \sin\beta\cos\beta \end{vmatrix}$$

Then $C_3 \to C_3 + (\cos\beta)C_2$ gives

$$\Delta = \frac{1}{\sin\beta}\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha(\cos^2\beta + \sin^2\beta) \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha(\cos^2\beta + \sin^2\beta) \\ 0 & -\sin\beta & 0 \end{vmatrix}$$

$$= \frac{1}{\sin\beta}\begin{vmatrix} \cos\alpha & \sin\alpha\cos\beta & \sin\alpha \\ -\sin\alpha & \cos\alpha\cos\beta & \cos\alpha \\ 0 & -\sin\beta & 0 \end{vmatrix}.$$

Finally, expanding along R_3, the determinant is

$$\Delta = \frac{1}{\sin\beta}(\sin\beta)\begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}$$

$$= \cos^2\alpha + \sin^2\alpha = 1.$$

47. By the law of sines we can write $\sin A = ak$, $\sin B = bk$ and $\sin C = ck$, so that the given determinant equals

$$\begin{vmatrix} a^2 & bak & cak \\ bak & 1 & \cos A \\ cak & \cos A & 1 \end{vmatrix} = a^2\begin{vmatrix} 1 & bk & ck \\ bk & 1 & \cos A \\ ck & \cos A & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos A \\ \sin C & \cos A & 1 \end{vmatrix}.$$

After $C_2 \to C_2 - (\sin B)\,C_1$ and
$C_3 \to C_3 - (\sin C)\,C_1$, this becomes

$$a^2\begin{vmatrix} 1 & 0 & 0 \\ \sin B & 1 - \sin^2 B & \cos A - \sin B\sin C \\ \sin C & \cos A - \sin B\sin C & 1 - \sin^2 C \end{vmatrix}$$

which can be expanded along R_1 to give

$$a^2[\cos^2 B\cos^2 C - (\cos A - \sin B\sin C)^2]. \quad (1)$$

But since A, B and C are the angles of a triangle, we have

$$\cos A = \cos[\pi - (B + C)] = -\cos(B + C)$$

$$= -(\cos B\cos C - \sin B\sin C)$$

$$\Rightarrow \cos A - \sin B\sin C = -\cos B\cos C.$$

Putting this back in (1), we see that the value of the determinant is 0.

48. Doing $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to the given determinant, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 + \sin A & \sin B - \sin A & \sin C - \sin A \\ \sin A + \sin^2 A & \sin^2 B - \sin^2 A & \sin^2 C - \sin^2 A \end{vmatrix},$$

which, after taking $\sin B - \sin A$ common from C_2 and $\sin C - \sin A$ common from C_3, becomes
$(\sin B - \sin A)(\sin C - \sin A)$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 + \sin A & 1 & 1 \\ \sin A + \sin^2 A & \sin B + \sin A & \sin C + \sin A \end{vmatrix}$$

$= (\sin B - \sin A)(\sin C - \sin A)(\sin C - \sin B)$.
Since the determinant is zero, we must have $\sin B = \sin A$ or $\sin C = \sin A$ or $\sin C = \sin B$, i.e., $B = A$ or $C = A$ or $C = B$. In all three cases we will have an isosceles triangle.

49. On expanding the given determinant, we will get six terms of the form $(1 + x)^m = g(x)$ (say). The coefficient of x in each of these terms is then $m = g'(0)$. Therefore the coefficient of x in the given determinant $\Delta(x)$ is $\Delta'(0)$. Now,

$$\Delta'(x) = \begin{vmatrix} a_1b_1(1+x)^{a_1b_1 - 1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ a_2b_1(1+x)^{a_2b_1 - 1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ a_3b_1(1+x)^{a_3b_1 - 1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$+ \begin{vmatrix} (1+x)^{a_1b_1} & a_1b_2(1+x)^{a_1b_2 - 1} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & a_2b_2(1+x)^{a_2b_2 - 1} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & a_3b_2(1+x)^{a_3b_2 - 1} & (1+x)^{a_3b_3} \end{vmatrix}$$

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R_3, we get

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix}$$

$\Rightarrow f'(x) = 2ax + b$

Integrating, we get

$f(x) = ax^2 + bx + c$

where c is an arbitrary constant. Since f has a maximum at $x = 5/2$,

$f'(5/2) = 0 \Rightarrow 5a + b = 0$ (1)

Also $f(0) = 2 \Rightarrow c = 2$

and $f(1) = 1 \Rightarrow a + b + c = 1$

$\therefore$ $a + b = -1$. (2)

Solving (1) and (2) for a, b we get

$a = 1/4, b = -5/4$

Now, $f'(0) = b = -5/4$

82. Use $f(x) = x^3 + 3x$

83. Use $R_1 \to R_1 + R_3$ followed by $C_1 \to C_1 - C_3$, to get

$f(x) = 4\cot x \operatorname{cosec}^2 x$

$\Rightarrow f(\pi/4) = 8$

$f'(x) = -4\operatorname{cosec}^4 x - 8\cot^2 x \operatorname{cosec}^2 x$

$\Rightarrow f'(\pi/4) = -16$

Finally, $\int_{\pi/6}^{\pi/3} f(x)\,dx = -2\cot^2 x]_{\pi/6}^{\pi/3} = -16/3$

and $f(\pi/2) = 0$

84. Apply $C_1 \to C_1 + C_2 + C_3$ followed by

$R_2 \to R_1 - R_1$, $R_3 \to R_3 - R_1$ to obtain

$\Delta_1 = -2(x^3 + y^3)$.

In Δ_2, apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ to obtain

$\Delta_2 = xy$

Apply $R_2 \to R_2 + R_1$ to obtain

$\Delta_3 = 3x^2(x + y)$

It is easy to show $\Delta_y = (x - y)(y - 1)(1 - x)$

85. (a) Expand along C_1.

(b) Use $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$ and the expand along R_1.

(c) Use $C_2 \to C_2 - x^3 C_1$.

(d) Expand along R_3.

86. (a) Show $f'(x) = 0$.

(b) $f(x) = \begin{vmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix} = 0$

(c) Show f is an odd function.

(d) Use $C_1 \to C_1 + C_2 + C_3$ to obtain

$f(x) = (2 + \sin 2x)\,\Delta_1$ where

$$\Delta_1 = \begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 1 & 1 + \cos^2 x & \sin 2x \\ 1 & \cos^2 x & 1 + \sin 2x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$.

$\therefore$ Maximum value of $f(x)$ is 3.

87. Let A be a skew symmetric matrix of order n, then

$|A'| = (-1)^n |A| \Rightarrow |A| = (-1)^n |A|$

If n is odd, then $|A| = -|A| \Rightarrow 2|A| = 0 \Rightarrow |A| = 0$.

We have

$$\Delta\left(\frac{\pi}{4}\right) = \begin{vmatrix} 0 & \sin(\pi/12) & \cos(\pi/12) \\ -\sin(\pi/12) & 0 & \sec(\pi/12) \\ -\cos(\pi/12) & -\sec(\pi/12) & 0 \end{vmatrix}$$

$= 0$ [using statement-2]

88. Statement-2 is true, See theory page 4.3.

The roots of $x^9 = p$ are $p^{1/9}\,\omega^r$ where

$\omega = \cos\frac{2\pi}{9} + i \sin\frac{2\pi}{9}$.

alue of determinant Δ depends an $\alpha_1, \alpha_2, \ldots \alpha_9$. If we put $\alpha_k = p^{1/9}\,\omega^k$, then

$$\Delta = (p^{1/9})^3 \begin{vmatrix} 1 & \omega^8 & \omega^7 \\ 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \end{vmatrix} = 0$$

However, if $\alpha_1 = p^{1/9}$, $\alpha_2 = p^{1/9}\,\omega^8$,

$a_3 = p^{1/9}\,\omega^7$, $\alpha_4 = p^{1/9}\,\omega$, $\alpha_5 = p^{1/9}\,\omega^5$

$\alpha_6 = p^{1/9}\,\omega^4$, $\alpha_7 = p^{1/9}\,\omega^2$, $\alpha_8 = p^{1/9}\,\omega^3$,

$\alpha_9 = p^{1/9}\,\omega^6$, then

$$\Delta = (p^{1/9})^3 \begin{vmatrix} 1 & \omega^8 & \omega^7 \\ \omega & \omega^5 & \omega^4 \\ \omega^2 & \omega^3 & \omega^6 \end{vmatrix}$$

$= p^{1/3}(2\omega^2 - \omega^7 - \omega^6) \neq 0$.

89. Statement-2 is true.

Using $R_2 \to R_2 - \sqrt{2}\,R_1$ and $R_3 \to R_3 - \sqrt{3}\,R_1$, we get

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9.7 TOTAL PROBABILITY THEOREM

Let $\{H_i\}, i = 1, 2, \cdots, n$, be n events such that $H_i \cap H_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} H_i = S$. Suppose that $P(H_i) > 0$ for $1 \leq i \leq n$. Then for any event A

$$P(A) = \sum_{i=1}^{n} P(H_i)\ P(A \mid H_i)$$

9.8 BAYES' RULE

Let S be a sample space and $H_1, \cdots, H_n$ be n mutually exclusive events such that $\bigcup_{j=1}^{n} H_j = S$ and $P(H_j) > 0$ for $j = 1, 2, \cdots, n$. We can think of the H_j's as the 'causes' (or the hypotheses) that lead to the outcome of an experiment. The probabilities $P(H_j)$, $j = 1, 2, \cdots, n$ are called *prior probabilities*. Suppose the experiment results in an outcome of event A, where $P(A) > 0$. What is the probability that the observed event was due to cause H_j? In other words, we seek the conditional probability $P(H_j \mid A)$. These probabilities are frequently called *posterior probabilities*. The information that A has occurred makes us reassess the probability $P(H_j)$ assigned to H_j.

9.9 THEOREM (BAYES' RULE)

Let $\{H_j\}$ be mutually exclusive events such that $P(H_j) > 0$ for $j = 1, 2, \cdots, n$ and $S = \bigcup_{j=1}^{n} H_j$. Let A be an event with $P(A) > 0$. Then for $j = 1, 2, \cdots, n$

$$P(H_j \mid A) = \frac{P(H_j)\, P(A \mid H_j)}{\sum_{k=1}^{n} P(H_k)\, P(A \mid H_k)}$$

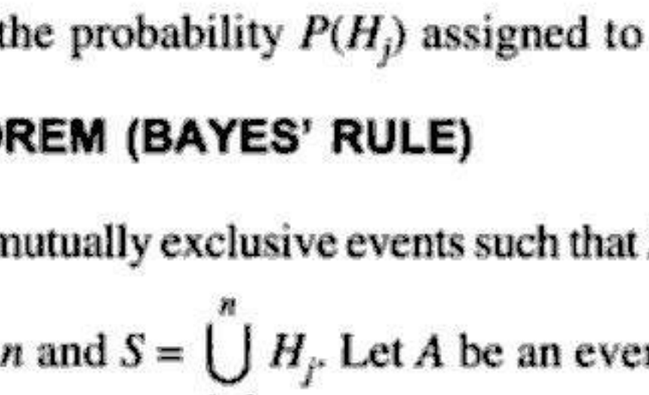

Fig. 9.4

9.10 INDEPENDENT EVENTS

We say that two events A and B are *independent* if the occurrence or non-occurrence of $A(B)$ does not affect the probability of occurrence or non-occurrence of $B(A)$, that is, if $P(B \mid A) = P(B)$ provided $P(A) \neq 0$

$$\Rightarrow \quad \frac{P(B \cap A)}{P(A)} = P(B) \quad \Rightarrow \quad P(A \cap B) = P(A)\ P(B)$$

Thus, two events A and B are independent if and only if $P(A \cap B) = P(A)\ P(B)$.

If two events are not independent, they are said to be *dependent*.

We note that if $P(A) = 0$ then for any event B, $0 \leq P(A \cap B) \leq P(A) = 0$, so that $P(A \cap B) = P(A)\ P(B)$ and A is independent of any other event.

Difference between Mutual Exclusiveness and Independence

We advise the reader to distinguish between independent and mutually exclusive events. Independence is a property of probability, whereas mutual exclusion is a set-theoretic property. If A and B are mutually exclusive events with $P(A) > 0$ and $P(B) > 0$, then $P(A \cap B) = 0 \neq P(A)\ P(B)$, so that A and B cannot be independent. In fact, $P(A \mid B) = 0 = P(B \mid A)$, so that if A occurs, B cannot occur and vice versa. Thus A and B are strongly dependent when $A \cap B = \emptyset$.

We also note the following fact: two events A and B are independent if and only if

(i) A and B' are independent, or
(ii) A' and B are independent, or
(iii) A' and B' are independent.

Three events A, B and C are said to be *pairwise independent* if

$$P(A \cap B) = P(A)\ P(B),\ P(B \cap C) = P(B)\ P(C),$$
$$P(C \cap A) = P(C)P(A)$$

Three events A, B and C are said to be *mutually independent* if

$$P(A \cap B) = P(A)\ P(B),$$
$$P(B \cap C) = P(B)\ P(C),$$
$$P(C \cap A) = P(C)\ P(A)$$

and $P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$

9.11 RANDOM VARIABLES

Let S be a sample space. A *random variable* X is a function from the set S to $\mathbf{R}$, the set of real numbers. If X is a random variable defined on the sample space S and r is a real number, then $\{X = r\} = \{w \in S \mid \times (w) = r\}$ is an event. If the random variable X takes n distinct values $x_1, x_2, \cdots, x_n$, then $\{X = x_1\}, \{X = x_2\}, \cdots, \{X = x_n\}$ are mutually exclusive and exhaustive events (Fig. 9.5).

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(a) 7 (b) 6 (c) 5 (d) 3.

Ans. (d)

Solution Suppose the coin is tossed n times. Let X be the number of heads obtained. Then X follows a binomial distribution with parameters n and $p = 1/2$. We have $P(X \geq 1) \geq 0.8 \Rightarrow 1 - P(X = 0) \geq 0.8$

$$\Rightarrow 1 - {}^nC_0 p^0 (1-p)^n \geq 0.8 \Rightarrow \left(\frac{1}{2}\right)^n \leq 0.2 = \frac{1}{5} \Rightarrow 2^n \geq 5$$

This show that the least value of n is 3.

Example 14 For the three events A, B and C, P (exactly one of the events A or B occurs) = P(exactly one of the events B or C occurs) = P(exactly one of the events C or A occurs) = p and P(all the three events occur simultaneously) = p^2, where $0 < p < 1/2$. Then the probability of at least one of the three events A, B and C occurring is

(a) $\frac{3p + 2p^2}{2}$ (b) $\frac{p + 3p^2}{4}$

(c) $\frac{p + 3p^2}{2}$ (d) $\frac{3p + 2p^2}{4}$

Ans. (a)

Solution We know that

P(exactly one of A or B occurs)

$$= P(A) + P(B) - 2P(A \cap B)$$

Therefore, $P(A) + P(B) - 2P(A \cap B) = p$ (1)

Similarly, $P(B) + P(C) - 2P(B \cap C) = p$ (2)

and $P(C) + P(A) - 2P(C \cap A) = p$ (3)

Adding (1), (2) and (3) we get

$$2[P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)] = 3p$$

$$\Rightarrow P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = 3p/2 \quad (4)$$

We are also given that $P(A \cap B \cap C) = p^2$ (5)

Now, P(at least one of A, B and C)

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{3p}{2} + p^2 = \frac{3p + 2p^2}{2} \text{ [using (4) and (5)]}$$

Example 15 Nine identical balls are numbers 1, 2, ...9. are put in a bag. A draws a ball and gets the number a. The ball is put back the bag. Next B draws a ball gets the number b. The probability that a and b satisfics the inequality $a - 2b + 10 > 0$ is

(a) $\frac{52}{81}$ (b) $\frac{55}{81}$

(c) $\frac{61}{81}$ (d) $\frac{62}{81}$

Ans. (c)

Solution $a - 2b + 10 > 0$

$\Rightarrow 1 \leq b < (a + 10)/2$

Thus, $b \in [1, (a + 10)/2)$

When $a = 1, 2,$ $b \in [1, 5]$

When $a = 3, 4,$ $b \in [1, 6]$

When $a = 5, 6,$ $b \in [1, 7]$

When $a = 7, 8,$ $b \in [1, 8]$

When $a = 9$ $b \in [1, 9]$

Thus, the inequality $a - 2b + 10 > 0$ is satisfied for $2 \times 5 + 2 \times 6 + 2 \times 7 + 2 \times 8 + 9 = 61$ ordered pairs (a, b).

Total number of ways of choosing (a, b) is $9 \times 9 = 81$

$\therefore$ required probability is $\frac{61}{81}$

Example 16 An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5 is then,

(a) 16/81 (b) 1/81 (c) 80/81 (d) 65/81

Ans. (a)

Solution

Let p = probability of getting face value not less than 2 and not more than 5 in a single throw of die = 4/6 = 2/3.

n = number of times die is rolled

X = number of times we get a number not less than 2 and not more than 5.

Then $X \sim B(n, p)$

Required probability

$$= P(X = 4) = {}^4C_4 \, p^4 = (2/3)^4 = 16/81.$$

Example 17 If the papers of 4 students can be checked by any one of the 7 teachers, then the probability that all the 4 papers are checked by exactly 2 teachers is

(a) 2/7 (b) 12/49

(c) 32/343 (d) none of these.

Ans. (b)

Solution The total number of ways in which papers of 4 students can be checked by seven teachers is 7^4.

The number of ways of choosing two teachers out of 7 is ${}^7C_2$. The number of ways in which they can check four papers is 2^4. But this includes two ways in which all the papers will be checked by a single teacher. Therefore, the number of ways in which 4 papers can be checked by exactly two teachers is $2^4 - 2 = 14$.

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both X and Y are binomial variate with parameters $n = 3$ and $p = 1/2$.

Probability that both A and B obtain the same number of heads is

$\therefore \quad P(X = 0)\ P(Y = 0) + P(X = 1)\ P(Y = 1) + P(X = 2)\ P(Y = 2) + P(X = 3)\ P(Y = 3)$

$$= \left[{}^3C_0\left(\frac{1}{2}\right)^3\right]^2 + \left[{}^3C_1\left(\frac{1}{2}\right)^3\right]^2 + \left[{}^3C_2\left(\frac{1}{2}\right)^3\right]^2 + \left[{}^3C_3\left(\frac{1}{2}\right)^3\right]^2$$

$$= \left(\frac{1}{2}\right)^6 [1 + 9 + 9 + 1] = \frac{20}{64} = \frac{5}{16}.$$

Example 32 The probability that a student is not a swimmer is 1/5. The probability that out of 5 students exactly 4 are swimmer is

(a) $\left(\frac{4}{5}\right)^3$ (b) $\left(\frac{4}{5}\right)^4$

(c) $5\left(\frac{4}{5}\right)^4$ (d) $3\left(\frac{4}{5}\right)^3$

Ans. (b)

Solution Let p = probability that a student selected at random is a swimmer = 1 – 1/5 = 4/5.

n = number of students selected = 5

and X = number of swimmers

Then $X \sim B(n, p)$ where $n = 5$ and $p = 4/5$

Probability that exactly 4 students are swimmer is

$$= {}^5C_4\ p^4q = 5\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right) = \left(\frac{4}{5}\right)^4.$$

Example 33 Six distinct numbers are selected from first 150 natural numbers. The probability all the six numbers are divisible both by 3 and 5 is

(a) 1/33 (b) 1/35

(c) 1/49 (d) none of these

Ans. (d)

Solution Total number of ways of selecting 6 numbers out of 1, 2, ..., 150 is ${}^{150}C_6$.

A number is divisible by 3 and 5 both if and only if it is divisible by 15. Thee are 10 numbers viz. 15, 30, ..., 150 that are divisible by 15. Thus, probability of required event

$$= \frac{{}^{10}C_6}{{}^{150}C_6}.$$

Example 34 There are two balls in an urn whose colours are not known (each ball can be either white or black). A white ball is put into the urn. A ball is drawn from the urn. The probability that it is white is

(a) 1/4 (b) 1/3 (c) 2/3 (d) 1/6

Ans. (c)

Solution Let E_i $(0 \le i \le 2)$ denote the event that urn contains i white and $(2 - i)$ black balls. Let A denote the event that a white ball is drawn from the urn.

We have $P(E_i) = 1/3$ for $i = 0, 1, 2$. and

$P(A \mid E_1) = 1/3,\ P(A \mid E_2) = 2/3,\ P(A \mid E_3) = 1.$

By the total probability rule,

$$P(A) = P(E_1)\ P(A \mid E_1) + P(E_2)\ P(A \mid E_2) + P(E_3)\ P(A \mid E_3)$$

$$= \frac{1}{3}\left[\frac{1}{3} + \frac{2}{3} + 1\right] = \frac{2}{3}$$

Example 35 A natural number x is chosen at random from the first one hundred natural numbers. The probability that $\frac{(x-20)(x-40)}{x-30} < 0$ is

(a) 1/50 (b) 3/50 (c) 3/25 (d) 7/25

Ans. (d)

Solution

Let $E = \frac{(x-20)(x-40)}{x-30} = \frac{(x-20)(x-30)(x-40)}{(x-30)^2}$

Sign of E is same as that of sign of

$(x - 20)(x - 30)(x - 40) = F$(say).

Note that $F < 0$ if and only if

$0 < x < 20$ or $30 < x < 40$.

$\therefore \quad E < 0$ in $(0, 20) \cup (30, 40)$

Thus E is negative for $x = 1, 2, \cdots, 19, 31, 32, \cdots, 39$, that is $E, < 0$ for 28 natural numbers.

$\therefore$ Required probability $= \frac{28}{100} = \frac{7}{25}$.

Example 36 An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If a consists of 4 outcomes, the number of outcomes that B must have so that A and B are independent, is

(a) 2, 4 or 8 (b) 3, 6 or 9

(c) 4 or 8 (d) 5 or 10.

Ans. (d)

Solution Let n_B and $n_{A\cap B}$ denote the number outcomes favourable to B and $A \cap B$ respectively.

As A and B are independent.

$$P(A \cap B) = P(A)\ P(B)$$

$$\Rightarrow \quad \frac{n_{A\cap B}}{10} = \frac{4}{10} \times \frac{n_B}{10}$$

$$\Rightarrow \quad 5\ n_{A\cap B} = 2n_B \Rightarrow 5 \mid n_B$$

$$\Rightarrow \quad n_B = 5 \text{ or } 10. \quad [\because\ 1 \le n_B \le 10.]$$

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$\Rightarrow$ $P(B) = 1/6,\ 4/5.$

Example 52 A student appears for tests I, II, and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II, III are p, q and $1/2$, respectively. If the probability that the student is successful is $1/2$, then

(a) $p = 1,\ q = 0$

(b) $p = 2/3,\ q = 1/2$

(c) $p = 3/5,\ q = 2/3$

(d) there are infinitely many values of p and q.

Ans. (a), (b), (c), (d)

Solution Let A, B and C be the events that the student is successful in tests I, II and III, respectively. Then

P(the student is successful)

$$= P[(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A \cap B \cap C)]$$

$$= P(A \cap B \cap C') + P(A \cap B' \cap C) + P(A \cap B \cap C)$$

$$= P(A)\ P(B)\ P(C') + P(A)\ P(B')\ P(C) + P(A)\ P(B)\ P(C)$$

[$\because$ A, B and C are independent]

$$= pq\ (1 - 1/2) + p\,(1 - q)\,(1/2) + (pq)\,(1/2)$$

$$= \frac{1}{2}\ [pq + p(1 - q) + pq] = \frac{1}{2}\ p(1 + q)$$

$$\therefore \quad \frac{1}{2} = \frac{1}{2}\ p(1 + q) \quad \Rightarrow \quad p\ (1 + q) = 1$$

This equation is satisfied for all pairs of values in (a), (b) and (c). Also, it is satisfied for infinitely many values of p and q. For instance, when $p = n/(n + 1)$ and $q = 1/n$, where n is any positive integer.

Example 53 Let X be a set containing n elements. If two subsets A and B of X are picked at random, the probability that A and B have the same number of elements is

(a) $\dfrac{{}^{2n}C_n}{2^{2n}}$ (b) $\dfrac{1}{{}^{2n}C_n}$

(c) $\dfrac{1\cdot3\cdot5\cdots(2n-1)}{2^n\cdot(n!)}$ (d) $\dfrac{3^n}{4^n}$.

Ans. (a), (c)

Solution We know that the number of subsets of a set containing n elements is 2^n. Therefore, the number of ways of choosing A and B is $2^n \cdot 2^n = 2^{2n}$. We also know that the number of subsets (of X) which contain exactly r elements is ${}^nC_r$. Therefore, the number of ways of choosing A and B so that they have the same number of elements is

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + ({}^nC_n)^2$$

$$= {}^{2n}C_n = \frac{1\cdot2\cdot3\cdots(2n-1)\,(2n)}{n!n!}$$

$$= \frac{[1\cdot3\cdot5\cdots(2n-1)]\,[2\cdot4\cdot6\cdots(2n)]}{n!n!}$$

$$= \frac{2^n\,(n!)[1\cdot3\cdot5\cdots(2n-1)]}{n!n!} = \frac{2^n[1\cdot3\cdot5\cdots(2n-1)]}{n!}$$

Thus, the probability of the required event is

$$\frac{{}^{2n}C_n}{2^{2n}} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2^n\ (n!)}.$$

Example 54 Suppose m boys and m girls take their seats randomly around a circle. The probability of their sitting is $({}^{2m-1}C_m)^{-1}$ when

(a) no two boys sit together

(b) no two girls sit together

(c) boys and girls sit alternatively

(d) all the boys sit together.

Ans. (a), (b), (c).

Solution The number of ways in which m boys and m girls can take their seats around a circle is $(2m - 1)!$

(a) We make the girls sit first around the circle. This can be done in $(m - 1)!$ After this boys can take their seats in $m!$.

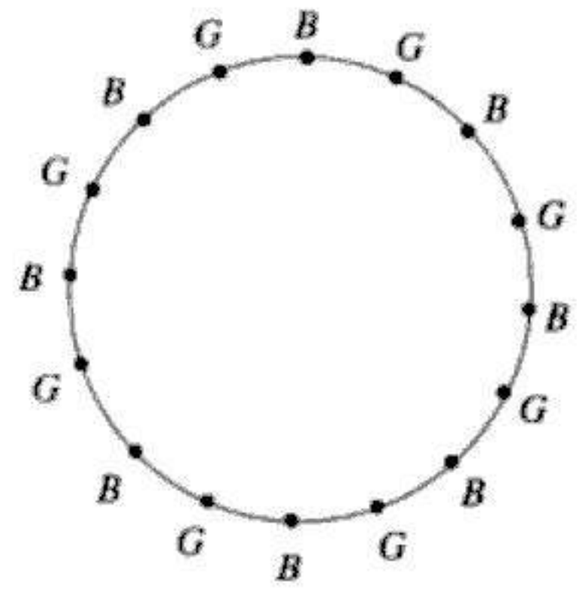

Fig. 9.7

(As in Fig. 9.7, exactly one boy between two girls means m boys are to be adjusted at m places.)

Thus, the number of ways in which no two boys sit together is $m!\ (m - 1)!$

$\therefore$ The probability no two boys sit together

$$= \frac{m!\,(m-1)!}{(2m-1)!} = \frac{1}{{}^{2m-1}C_m}$$

(b) Similar to (a).

(c) Boys and girls sit alternatively if and only if no two boys (girls) sit together. Therefore, the probability of this event is also $({}^{2m-1}C_m)^{-1}$.

(d) We tie the boys together and treat them as a single boy. We can put $(m + 1)$ objects (m girls and one boy) around a circle in $m!$ ways. But the boys can be tied in $m!$ ways. Therefore, boys will sit together in $m!\ m!$ ways. Thus, the probability of

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Example 70 If $P(E_k) \propto k$ for $0 \le k \le n$, then

$\lim_{n\to\infty} \sum_{k=0}^{n} P(E_k|A)$ equals

(a) 0 (b) 1/2
(c) 1/6 (d) 1

Ans. 66.(a), 67.(c), 68.(b), 69.(b), 70.(d),

Solution 66. $P(E_k) = C$ for $0 \le k \le n$ and

$$\sum_{k=0}^{n} P(E_k) = 1 \Rightarrow C = 1/(n+1).$$

Also $$P(A|E_k) = \frac{{}^{n-1}C_{k-1}}{{}^{n}C_{k}} = \frac{k}{n}$$

By the total probability rule

$$P(A) = \sum_{k=0}^{n} P(E_k)\ P(A|E_k)$$

$$= \frac{1}{n(n+1)} \frac{n(n+1)}{2} = \frac{1}{2}.$$

67. By the Bayes' rule

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(A)} = \frac{2}{n(n+1)}.$$

68. Let $P(E_k) = k\alpha$ where α is the constant of proportionality. Since $\sum_{k=0}^{n} P(E_k) = 1$, we get $\alpha = 2/n(n+1)$

Now, $$P(A) = \sum_{k=0}^{n} P(E_k)\,P(A|E_k) = \frac{2}{n^2(n+1)} \sum_{k=0}^{n} k^2$$

$$= \frac{2}{n^2(n+1)} \frac{n(n+1)(2n+1)}{6} = \frac{2n+1}{3n}.$$

69. $$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(A)}$$

$$= \frac{2}{n(n+1)} \frac{1}{n} \times \frac{3n}{2n+1}$$

$$= \frac{6}{n(n+1)(2n+1)}.$$

70. $$P(E_k|A) = \frac{P(E_k)P(A|E_k)}{P(A)}$$

$$= \frac{2k}{n(n+1)} \frac{k}{n} \times \frac{3n}{2n+1}$$

$$= \frac{6k^2}{n(n+1)(2n+1)}$$

$$\Rightarrow \sum_{k=0}^{n} P(E_k|A) = 1.$$

Paragraph for Question Nos. 71 to 75

Suppose E_1, E_2, E_3 be three mutually exclusive events such that $P(E_i) = p_i$ for $i = 1, 2, 3$.

Example 71 If $p_1 = \frac{1}{2}(1-p)$, $p_2 = \frac{1}{3}(1+2p)$ and $p_3 = \frac{1}{5}(2+3p)$, then p belongs to

(a) $(-1, 2)$ (b) $(-1/3, 2/3)$
(c) $(-1, 1/47)$ (d) $(-1/2, -7/23)$

Example 72 If p_1, p_2, p_3 are the roots of

$$27x^3 - 27x^2 + ax - 1 = 0$$

then value of a is

(a) 9 (b) 6
(c) 3 (d) [0, 1/5]

Example 73 If $p_1 = \frac{1}{3}(1+3p)$, $p_2 = \frac{1}{4}(1-p)$ and $p_3 = \frac{1}{4}(1-2p)$, then p belongs to

(a) $(-1/3, 1)$ (b) $(-1/9, 1/4)$
(c) $(-2/3, 2/3)$ (d) $(-1, 0)$

Example 74 P(none of E_1, E_2, E_3) equals

(a) 0
(b) $p_1 + p_2 + p_3$
(c) $(1-p_1)(1-p_2)(1-p_3)$
(d) none of these

Example 75 $P(E_1 \cap E_2') + P(E_2 \cap E_3') + P(E_3 \cap E_1')$ equals

(a) $p_1(1-p_2) + p_2(1-p_3) + p_3(1-p_1)$
(b) $p_1p_2 + p_2p_3 + p_3p_1$
(c) $p_1 + p_2 + p_3$
(d) none of these

Ans. 71. (d), 72. (a), 73. (b), 74. (d), 75. (c)

Solution 71. $p_1 \ge 0, p_2 \ge 0, p_3 \ge 0$

and $$p_1 + p_2 + p_3 \le 1$$

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Example 94 If n positive integers are taken at random and multiplied together, and p_n is the probability that the last digit of the product is 2, 4, 6 or 8 find $125p_3$.

Ans. 56.

Solution The last digit of the product will be 1, 2, 3, 4, 6, 7, 8 or 9 if and only if each of the n positive integers ends in any of these digits. Now, the probability of an integer ending in 1, 2, 3, 4, 6, 7, 8 or 9 is $8/10 = 4/5$. Therefore, the probability that the last digit of the product of n integers is 1, 2, 3, 4, 6, 7, 8 or 9 is $(4/5)^n$. Next, the last digit of the product will be 1, 3, 7 or 9 if and only if each of the n positive integers ends in 1, 3, 7 or 9. The probability for an integer to end in 1, 3, 7 or 9 is $4/10 = 2/5$. Therefore, the probability for the product of n positive integers to end in 1, 3, 7 or 9 is $(2/5)^n$. Hence, the probability of the required event is

$$\left(\frac{4}{5}\right)^n - \left(\frac{2}{5}\right)^n = \frac{4^n - 2^n}{5^n}$$

Put $n = 3$.

Example 95 Two integers r and s are drawn one at a time without replacement from the set 1, 2, ⋯, n. If $p_k = P(r \le k \mid s \le k)$, find $4p_7$ if $n = 25$.

Ans. 1.

Solution We have

$$P(r \le k \mid s \le k) = \frac{P[(r \le k) \text{ and } (s \le k)]}{P(s \le k)}$$

Also, $$P(s \le k) = \frac{k}{n} \text{ and}$$

$$P[(r \le k) \text{ and } (s \le k)] = \frac{{}^kC_2}{{}^nC_2} = \frac{k(k-1)}{n(n-1)}$$

Thus, $$p_k = (k-1)/(n-1)$$

Put $n = 25$ and $k = 7$

Example 96 A biased coin with probability $p(0 < p < 1)$ of heads is tossed until a head appears for the first time. If the probability that the number of tosses required is even is 2/5, find $3p$.

Ans. 1.

Solution Let X denote the number of tosses required. Then $P(X = r) = (1-p)^{r-1}p$ for $r = 1, 2, 3, \cdots$ Let E denote the event that the number of tosses required is even. Then

$$\begin{aligned} P(E) &= P[(X = 2) \cup (X = 4) \cup (X = 6) \cup \cdots] \\ &= P(X = 2) + P(X = 4) + P(X = 6) + \cdots \\ &= (1-p)p + (1-p)^3 p + (1-p)^5 p + \cdots \end{aligned}$$

$$= \frac{p(1-p)}{1-(1-p)^2} = \frac{p(1-p)}{2p-p^2} = \frac{1-p}{2-p}$$

As we are given that $P(E) = 2/5$, we get

$$5(1-p) = 2(2-p) \Rightarrow 1 - 3p = 0 \quad \text{or} \quad p = 1/3$$

Example 97 There are 10 pairs of shoes in a cupboard, from which 4 shoes are picked at random. If p is the probability that there is at least one pair, find $323p$.

Ans. 99.

Solution That total number of ways of choos-ing 4 shoes (in order) out of 10 pairs (or 20 shoes) is $20 \times 19 \times 18 \times 17$. The number of ways in which no pair is selected is $20 \times 18 \times 16 \times 14$. Thus, the probability of not getting a pair is

$$\frac{20 \times 18 \times 16 \times 14}{20 \times 19 \times 18 \times 17} = \frac{224}{323}$$

Hence, the probability of getting at least one pair is

$$1 - 224/323 = 99/323.$$

Example 98 The probability of a bomb hitting a bridge is 1/2 and two direct hits are needed to destroy it. Find the least number of bombs required so that the probability of the bridge being destroyed is greater than 0.9.

Ans. 8.

Solution Let n be the least number of bombs required and X the number of bombs that hit the bridge. Then X follows a binomial distribution with parameters n and $p = 1/2$. Now

$$P(X \ge 2) > 0.9 \Rightarrow 1 - P(X < 2) > 0.9$$

$$\Rightarrow \quad P(X = 0) + P(X = 1) < 0.1$$

$$\Rightarrow \quad {}^nC_0 \left(\frac{1}{2}\right)^n + {}^nC_1 \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{2}\right) < \frac{1}{10}$$

$$\Rightarrow \quad \frac{n+1}{2^n} < \frac{1}{10} \quad \Rightarrow \quad 10(n+1) < 2^n$$

By trial and error, we get $n \ge 8$. Thus, the least value of n is 8.

Example 99 Suppose X follows a binomial distribution with parameters $n = 6$ and p. If $9P(X = 4) = P(X = 2)$, find $4p$.

Ans. 1.

Solution

$$9P(X = 4) = P(X = 2)$$

$$\Rightarrow \quad 9 \cdot {}^6C_4\, p^4 q^2 = {}^6C_2\, p^2 q^4 \qquad [q = 1 - p]$$

$$\Rightarrow \quad 9p^2 = q^2 \qquad [\because {}^6C_4 = {}^6C_2]$$

$$\Rightarrow \quad 3p = q = 1 - p \qquad [\because q > 0]$$

$$\Rightarrow \quad 4p = 1$$

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$\therefore$ probability of the required event

$$= \frac{4!\,3!\,2!}{9!} = \frac{1}{\left({}^7C_3\right)\left({}^9C_2\right)}$$

(b) We can permute *OIL I, N, D, I, A, N* in $\frac{7!}{2!\,2!}$ ways.

$\therefore$ probability of the required event

$$= \frac{2!\,2!\,3!\,2!}{7!\,9!} = \frac{1}{\left({}^5C_2\right)\left({}^7C_2\right)(9!)}$$

(c) Fixing an *I* at the first place and *L* at the last place, we can permute the remaining letters viz.

A, D, I, I, N, N, O in

$\frac{7!}{2!\,2!}$ ways.

$\therefore$ probability of the required event is

$$= \frac{2!\,2!\,3!\,2!}{7!\,9!} = \frac{1}{\left({}^5C_2\right)\left({}^7C_2\right)(9!)}$$

(d) Vowels can be arranged at odd places viz 1^{st}, 3^{rd}, 5^{th}, 7^{th} and 9^{th} in $\frac{5!}{3!}$ ways.

The remaining letters can be arranged at 4 even places in $\frac{4!}{2!}$ ways.

$\therefore$ probability of the required event

$$= \frac{5!\,4!}{3!\,2!} \times \frac{3!\,2!}{9!} = \frac{1}{{}^9C_4} = \frac{1}{{}^9C_5}$$

SECTION VI

REASONING TYPE

Example 108 Statement-1: Let *A* and *B* be two events such that $P(A \cup B) = P(A \cap B)$ then

$P(A \cap B') = P(A' \cap B) = 0$

Statement-2: Let *A* and *B* be two events such that $P(A \cup B) = P(A \cap B)$ then $P(A) + P(B) = 1$.

Ans. (c)

Solution $P(A \cup B) = P(A \cap B)$

$\Leftrightarrow P(A) + P(B) - P(A \cap B) = P(A \cap B)$

$\Leftrightarrow [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] = 0.$ (1)

But $P(A), P(B) \geq P(A \cap B)$.

Thus, (1) is possible if and only if

$P(A) - P(A \cap B) = 0$ and $P(B) - P(A \cap B) = 0$

$\Leftrightarrow P(A) = P(B) = P(A \cap B)$

$\therefore$ $P(A) + P(B) = 1$ need not hold.

Next, $P(A \cap B') = P(A) - P(A \cap B) = 0$

Similarly $P(A' \cap B) = P(B) - P(A \cap B) = 0$

Example 109 Statement-1: If $\frac{1}{5}(1 + 5p)$, $\frac{1}{3}(1 + 2p)$, $\frac{1}{3}(1 - p)$ and $\frac{1}{5}(1 - 3p)$ are probabilities of four mutually exclusive events, then p can take infinite number of values.

Statement-2: If *A*, *B*, *C* and *D* are four mutually exclusive events, then

$P(A), P(B), P(C), P(D) \geq 0$

$P(A) + P(B) + P(C) + P(D) \leq 1.$

Ans. (a)

Solution Statement-2 is true. See theory.

Now, $\frac{1}{5}(1 + 5p), \frac{1}{3}(1 + 2p), \frac{1}{3}(1 - p)\ \frac{1}{5}(1 - 3p) \geq 0$

$\Rightarrow p \geq -1/5, p \geq -1/2, p \leq 1, p \leq 1/3$

$\Rightarrow -1/5 \leq p \leq 1/3$ (1)

and $\frac{1}{5}(1 + 5p) + \frac{1}{3}(1 + 2p) + \frac{1}{3}(1 - p) + \frac{1}{5}(1 - 3p) \leq 1$

$\Rightarrow \frac{1}{5}(2 + 2p) + \frac{1}{3}(2 + p) \leq 1 \Rightarrow 6 + 6p + 10 + 5p \leq 15$

$\Rightarrow 11p \leq -1 \Rightarrow p \leq -1/11.$ (2)

From (1) and (2) we get $-1/5 \leq p \leq -1/11$

$\Rightarrow$ there are infinite values of p.

Example 110 Statement-1: A natural number is chosen at random. The probability that sum of the digits of its square is 93 is 0.

Statement-2: A number is divisible by 31 if and only if sum of its digits is divisible by 31.

Ans. (c)

Solution Let $n \in N$, be such that sum of digits of its square is 93.

Then $3|n^2 \Rightarrow 3|n \Rightarrow 9|n^2$.

A contradiction.

Example 111 Let $m \in N$, and suppose three numbers are chosen at random from the numbers 1, 2, 3, ..., *m*.

Statement-1: If $m = 2n$ for some $n \in N$, then the chosen numbers are in A.P. with probability $\frac{3}{2(2n-1)}$.

Statement-2: If $m = 2n + 1$, then the chosen numbers are in A.P. with probability $\frac{3n}{4n^2 - 1}$

Ans. (b)

Solution We can choose three numbers out of *m* in ${}^mC_3$ ways. Let the numbers be x_1, x_2, x_3. Now,

x_1, x_2, x_3 are in A.P. if and only if

$x_1 + x_3 = 2x_2$, that is, if and only if either both x_1, x_3 are odd or both x_1, x_3 are even.

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22. If X and Y are independent binomial variate B (5, 1/2) and B (7, 1/2), then $P(X + Y = 3)$ is
    (a) 55/1024 (b) 55/4098
    (c) 55/2048 (d) none of these

23. Suppose n (≥ 3) persons are sitting in a row. Two of them are selected at random. The probability that they are not together is
    (a) $1 - \frac{2}{n}$ (b) $\frac{2}{n-1}$
    (c) $1 - \frac{1}{n}$ (d) none of these

24. A bag contains four tickets marked with numbers 112, 121, 211, 222. One ticket is drawn at random from the bag. Let E_i (i = 1, 2, 3) denote the event that ith digit on the ticket is 2. Then which of the following is not true
    (a) E_1 and E_2 are independent
    (b) E_2 and E_3 are independent
    (c) E_3 and E_1 are independent
    (d) E_1, E_2, E_3 are independent

25. If the letters of the word PROBABILITY are written down at random in a row, the probability that two B's are together is
    (a) 2/11 (b) 10/11 (c) 3/11 (d) 6/11

26. If four positive integers are taken at random and multiplied together, then the probability that the last digit is 1, 3, 7 or 9 is
    (a) 1/8 (b) 2/7 (c) 1/625 (d) 16/625

27. Two contestants play a game as follows: each is asked to select a digit from 1 to 9. If the two digits match they both win a prize. The probability that they will win a prize in a single trial is
    (a) 1/81 (b) 7/81 (c) 1/9 (d) 3/11

28. If A and B are two events, then which of the following does not represent the probability that exactly one of A, B occurs is
    (a) $P(A) + P(B) - P(A \cap B)$
    (b) $P(A \cap B') + P(A' \cap B)$
    (c) $P(A') + P(B') - 2P(A' \cap B')$
    (d) $P(A) + P(B) - 2P(A \cap B)$

29. If A and B are two events, then which one of the following is not always true
    (a) $P(A \cap B) \geq P(A) + P(B) - 1$
    (b) $P(A \cap B) \leq P(A)$
    (c) $P(A' \cap B') \geq P(A') + P(B') - 1$
    (d) $P(A \cap B) = P(A)\,P(B)$

30. A sum of money is rounded off to the nearest rupee. The probability that the round off error is at most 10 paise is
    (a) 1/10 (b) 11/100
    (c) 3/25 (d) none of these

31. The probability that an electric bulb will last 150 days or more is 0.7 and that it will last at most 160 days is 0.8. The probability that the bulb will last 150 to 160 days is
    (a) 0.5 (b) 1/3
    (c) 0.56 (d) none of these

32. A bag contains three tickets numbered 1, 2 and 3. A ticket is drawn at random and put back in the bag, and this is done four times. The probability that the sum of the numbers drawn is even is
    (a) 40/81 (b) 41/81 (c) 14/27 (d) 13/81

33. Two numbers x and y are selected at random from the set $\{1, 2, 3, \ldots, 3N\}$. The probability that $x^2 - y^2$ is divisible by 3 is
    (a) $(3N - 1)/3N$ (b) $(N - 1)/N$
    (c) $(5N - 3)/(9N - 3)$ (d) none of these

34. Three persons A, B and C are to speak at a function along with 7 other persons. The probability that A speaks before B and B speaks before C is
    (a) 3/70 (b) 1/6
    (c) 3/7 (d) none of these

35. n men and n women are seated at round table in random order. The probalility that they can be divided into n non-interrecting pairs so that each pair consists of a man and a woman is
    (a) $1/2n$ (b) $2(2^n - 1)/{}^{2n}C_n$
    (c) $2n/{}^{2n}C_n$ (d) $1/({}^nC_n)^2$

36. If a is an integer lying in $[-5, 30]$, then the probability that the graph of $y = x^2 + 2(a + 4)x - 5a + 64$ is strictly above the x – axis is
    (a) 1/6 (b) 7/36
    (c) 2/9 (d) 3/5.

37. Two numbers X and Y are chosen at random (without replacement) from the set $\{1, 2, \cdots 5N\}$. The probability that $X^4 - Y^4$ is divisible by 5 is
    (a) $\frac{N-1}{5N-1}$ (b) $\frac{4(4N-1)}{5(5N-1)}$
    (c) $\frac{17N-5}{5(5N-1)}$ (d) none of these.

38. There is a five-volume dictionary among 50 books arranged on a shelf in random order. The probability that these volumes stand in increasing order from left to right (the volumes are not necessarily kept side-by-side) is

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SECTION IV

INTEGER ANSWER TYPE

81. Let $P(A) = 0.4$ and $P(A \cup B) = P(A \cap B)$, find $5P(B)$.

82. A pair of fair dice is rolled together till a sum of 7 or 11 is obtained. Let p denote the probability that 7 comes before 11. Find the value of $4p$.

83. In a test an examine either gusses or copies or knows the answer to a multiple choice question with m choices out of which exactly one is correct. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct given that he copied it, is 1/8. If the probability that he knew the answer to the question given that he correctly answered it is 120/141, find m.

84. If $\frac{1}{4}(1+4p)$, $\frac{1}{4}(1-p)$ and $\frac{1}{2}(1-2p)$ are the probabilities of three mutually exclusive events, find $2p$.

85. The digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are written in random order to form a nine digit number. Let p be the probability that this number is divisible by 36, find $9p$.

86. Two squares are chosen at random on a chessboard. If p denotes the probability that they have exactly one side in common, find $144p$.

87. In a multiple-choice question, there are five alternative answers, of which one or more than one are correct. A candidate will get marks on the question if he ticks all the correct answers. If he decides to tick answers at random, find the least number of choices should he be allowed so that the probability of his getting marks on the question exceeds 1/8?

88. Two persons each make a single throw with a pair of dice. If p denotes the probability that their throws are unequal, find $648p$.

89. In a sequence of independent trials, the probability of success is 1/4. If p denotes the probability that the second success occurs on the fourth trial or later trial, find $32p$.

90. If X follows a binomial distribution with parameters $n = 100$ and $p = 1/3$, find r for which $P(X = r)$ is maximum.

SECTION V

MATRIX MATCH TYPE

91. Three players A, B and C alternatively throw a die in that order, the first player to throw a 6 being deemed the winner. A's die is fair whereas B and C throw dice with probabilities p_1 and p_2 respectively of throwing a 6.

(a) If $p_1 = 1/5$, $p_2 = 1/4$, the probability that A wins the game.	(p) $\frac{1}{2}$
(b) If $p_1 = 1/5$, $p_2 = 1/4$, the probability that C wins the game	(q) $\frac{1}{3}$
(c) Value of p_1 so that $P(A \text{ wins}) = P(B \text{ wins})$	(r) $\frac{1}{4}$
(d) Value of p_2 so that the game is equiprobable to all the three players.	(s) $\frac{1}{5}$

92. Consider a town with N people. A person spreads a rumor to a second, who in turn repeats it to a third, and so on. Suppose that at each stage, the recipient of the rumor is chosen at random from the remaining $(N - 1)$ people. Suppose rumor is repeated n (≥ 3) times, then probability that it will not be repeated to

(a) originator	(p) $\left(1-\frac{1}{N-1}\right)^{n-1}$
(b) first recipient	(q) $\left(1-\frac{1}{N-1}\right)^{n-2}$
(c) second recipient	(r) $\left(1-\frac{1}{N+1}\right)^{n-2}$
(d) any of the earlier recipient	(s) ${}^{N-1}P_n/(N-1)^n$

93. A prisoner escapes from a jail and is equally likely to choose one of the four roads I, II,III or IV to reach away from the hands of law. If he choose I road, he is successful with probability 1/6 and for II, III and IV this is 1/8, 1/10 and 1/12. If the prisoner is successful, the probability that he chose road

(a) I	(p) 12/57
(b) II	(q) 15/57
(c) III	(r) 20/57
(d) IV	(s) 10/57

94. Sixteen players S_1, S_2, ..., S_{16} play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of

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SECTION III

COMPREHESION TYPE

Paragraph for Questions 31 to 35

Let x and y be two numbers chosen at random from the set $\{1, 2, 3, ..., n\}$ with replacement. Let $Q_n(p)$ denote the probability that $(x^{p-1} - y^{p-1})$ is divisible by p.

31. $Q_{25}(3)$ equals

(a) 0.5648 (b) 0.5678

(c) 0.5698 (d) 0.5628

32. $Q_{50}(5)$ equals

(a) 0.52 (b) 0.64

(c) 0.68 (d) 0.75

33. If $[x]$ denotes the greatest integer $\le x$ then $Q_n(p)$ equals

(a) $1 + 2\left[\frac{n}{p}\right] - \frac{2}{n^2}\left[\frac{n}{p}\right]^2$

(b) $1 - 3\left[\frac{n}{p}\right] - \frac{1}{n^2}\left[\frac{n}{p}\right]^2$

(c) $1 - \left[\frac{n}{p}\right] + \frac{1}{n^2}\left[\frac{n}{p}\right]^2$

(d) none of these.

34. $\lim_{n\to\infty} Q_n(p)$ equals

(a) $1 - \frac{2}{p} + \frac{2}{p^2}$ (b) $1 - \frac{2}{p}$

(c) $1 - \frac{2}{p^2}$ (d) none of these

35. $\lim_{p\to\infty} Q_n(p)$ equals

(a) 1/2 (b) 1

(c) $1/\pi$ (d) 0

Paragraph for Questions 36 to 40

A **permutation** of n objects is a one-to-one function from $A = \{1, 2, ..., n\}$ onto itself. If $a_1, a_2, ... a_k$ are n distinct elements of A such that a function f sends a_1 to a_2, a_2 to a_3, ..., $a_{k-1} \to a_k$ and $a_k \to a_1$, then $a_1, a_2, ..., a_k$ are said to form a cycle of length k.

A permutation is chosen at random from the set of all permutations of the set $A = \{1, 2, ..., n\}$.

36. The probability that chosen permutation is a cycle such that $1 \to 2 \to 3 \to ... \to (n-1) \to 1$, is

(a) $1/n!$ (b) $1/(n-1)!$

(c) $2/n!$ (d) 0

37. Let $a_1, a_2, ..., a_k, b_1, b_2, ..., b_k \in A$ be such that a_i's are distinct and b_j's are distinct. The probability that the chosen permutation sends $a_i \to b_i$ for $i = 1, 2, ..., k$, is

(a) $k!/n!$ (b) $k!\,(n-k)!/n!$

(c) 0 (d) $(n-k)!/n!$

38. The probability that the element i forms a unit cycle i.e. $i \to i$, is

(a) $1/n!$ (b) $1/(n-1)!$

(c) $1/n$ (d) 0

39. If p_n denotes the probability that there is at least one cycle of unit length, then p_n equals

(a) $1 - \sum_{k=2}^{n} (-1)^k \frac{1}{k!}$ (b) $\sum_{k=1}^{n} (-1)^k \frac{1}{k!}$

(c) $1 - \sum_{k=1}^{n} (-1)^k \frac{1}{k!}$ (d) none of these

40. $\lim_{n\to\infty} p_n$ equals

(a) $1/e$ (b) $1 - 1/e$

(c) 0 (d) none of these

SECTION IV

INTEGER ANSWER TYPE

41. Two persons each make a throw with a pair of cubical die and let p denote the probability that their throws are equal. Find the value of $648p$.

42. A bag contains $n + 1$ coins. It is known that one of these coins shows heads on both sides, whereas the other coins are fair. One coin is selected at random and tossed. If the probability that the toss results in heads is 7/12, find n.

43. Two squares are chosen at random on a chess board and let p denote the probability that they have exactly one side in common. Find $36p$.

44. Two integers r and s are chosen one at a without replacement from the numbers 1, 2, 3, ... 100. Let p be the probability that $r \le 25$ given that $s \le 25$. Find the value of $33p$.

45. A child arranges 3 distinct black balls, four green balls, and five red balls in a row randomly. If p is the probability that two red balls are not together, find $99p$.

46. Two numbers x and y are chosen at random from the numbers from the set $\{1, 2, 3, ..., 300\}$. Let p

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B, which option should A choose so that his probability of winning is higher? (No game ends in a draw. [1989]

35. Let A and B be two events such that $P(A) = 0.3$ and $P(A \cup B) = 0.8$. If A and B are independent events, then $P(B) =$ _____. [1990]

36. A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen at random. Find the probability that P and Q have no common elements. [1990]

37. If the mean and variance of a binomial variate X are 2 and 1 respectively, then the probability that X takes a value greater than one is equal to _____.
[1991]

38. For any two events A and B in a sample space

(a) $P(A|B) \geq \dfrac{P(A)+P(B)-1}{P(B)}$, $P(B) \neq 0$, is always true

(b) $P(A \cap B') = P(A) - P(A \cap B)$ does not hold

(c) $P(A \cup B) = 1 - P(A')\,P(B')$, if A and B are independent

(d) $P(A \cup B) = 1 - P(A')\,P(B')$, if A and B are disjoint [1991]

39. In a test, an examine guesses, copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct given that he copied it is 1/8. Find the probability that he knew the answer to the question, given that he answered it correctly. [1991]

40. Three faces of a fair die are yellow, two faces red and one blue. The die is tossed three times. The probability that the colours yellow, red and blue appear in the first, second and the third tosses respectively, is_____. [1992]

41. India plays two matches each with the West Indies and Australia. In any match, the probabilities of India getting 0, 1 and 2 points are 0.45, 0.05 and 0.50, respectively. Assuming that the outcomes are independent, the probability of India getting at least 7 points is

(a) 0.8750 (b) 0.0875
(c) 0.0625 (d) 0.0250 [1992]

42. A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B and C are defined as

A = {the first bulb is defective}
B = {the second bulb is non-defective}
C = {the two bulbs are both defective or both non-defective}

Determine whether (i) A, B and C are pairwise independent and (ii) A, B and C are independent. [1992]

43. An unbiased die with faces marked, 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5 is then,

(a) 16/81 (b) 1/81
(c) 80/81 (d) 65/81 [1993]

44. Let E and F be two independent events. The probability that both E and F happen is 1/2 and the probability that neither E nor F happens is 1/12. Then,

(a) $P(E) = 1/3$, $P(F) = 1/4$
(b) $P(E) = 1/2$, $P(F) = 1/6$
(c) $P(E) = 1/6$, $P(F) = 1/2$
(d) $P(E) = 1/4$, $P(F) = 1/3$ [1993]

45. Numbers are selected at random, from the two digit numbers 00, 01, 02, ..., 99 with replacement. An event E occurs if and only if the product of the two digits of a selected number is 18. If four numbers are selected, find the probability that the event E occurs at least 3 times.
[1993]

46. If two events A and B such that $P(A') = 0.3$, $P(B) = 0.4$ and $P(A \cap B) = 0.5$, then $P(B|(A \cup B)) =$ _____. [1994]

47. A box contains 24 identical balls of which 12 are white and 12 are black. The balls are drawn at random from the box one at a time with replacement. The probability that a white ball is drawn for the 4th time on the 7th draw is

(a) $\dfrac{5}{64}$ (b) $\dfrac{27}{32}$
(c) $\dfrac{5}{32}$ (d) $\dfrac{1}{2}$ [1994]

48. An unbiased die is tossed until a number greater than 4 appears. The probability that an even number of tosses is needed is

(a) 1/2 (b) 2/5
(c) 1/5 (d) 2/3 [1994]

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52. (d) 53. (c), (d) 54. (a), (b)
55. (a), (b), (c), (d) 56. (a), (b), (c), (d)
57. (a), (b), (c), (d) 58. (a), (b), (c)
59. (a), (b), (c) 60. (a), (c), (d)
61. (a), (b), (c), (d) 62. (a), (b), (c)
63. (a), (b), (d) 64. (a), (b)
65. (a), (b), (c), (d)

SECTION C

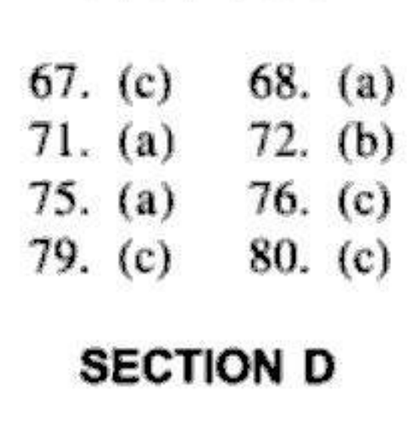
66. (b) 67. (c) 68. (a) 69. (a)
70. (a) 71. (a) 72. (b) 73. (c)
74. (c) 75. (a) 76. (c) 77. (a)
78. (b) 79. (c) 80. (c)

SECTION D

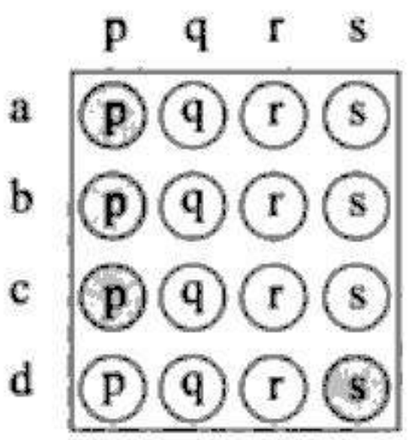
81. 2 82. 3 83. 5 84. 1
85. 2 86. 7 87. 4 88. 73
89. 27 90. 33

SECTION E

91.



92.



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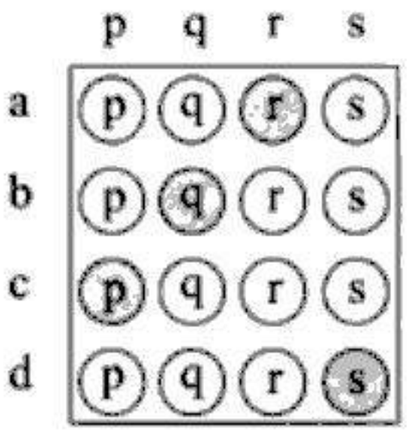

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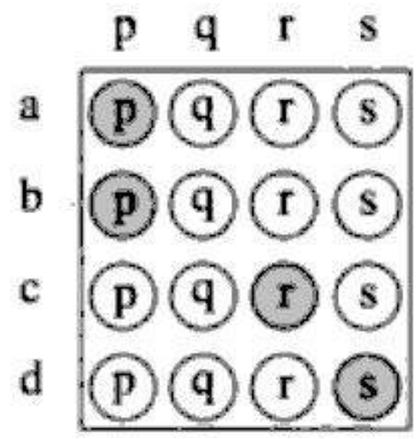
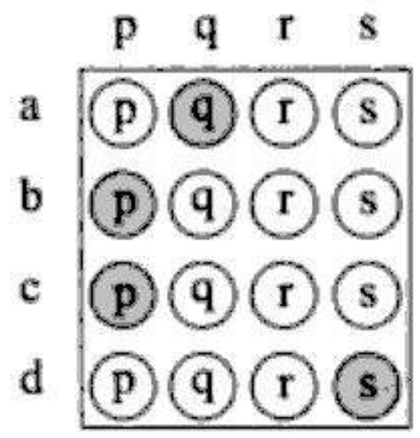


95.



SECTION F

96. (b) 97. (a) 98. (c) 99. (c)
100. (c)

LEVEL 2

SECTION A

1. (b) 2. (a) 3. (c) 4. (d)
5. (b) 6. (d) 7. (c) 8. (d)
9. (d) 10. (b) 11. (c) 12. (d)
13. (c) 14. (a) 15. (a) 16. (b)
17. (b) 18. (d) 19. (a) 20. (d)

SECTION B

21. (a), (c) 22. (a), (b), (c), (d)
23. (a), (b), (d) 24. (a), (b)
25. (a), (d) 26. (a), (b), (c), (d)
27. (a), (b) 28. (a), (b), (c)
29. (a), (b), (c), (d) 30. (b)

SECTION C

31. (a) 32. (c) 33. (d) 34. (a)
35. (b) 36. (a) 37. (d) 38. (c)
39. (a) 40. (b)

SECTION D

41. 73 42. 5 43. 2 44. 8
45. 7 46. 3 47. 9 48. 10

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Now $P(B \mid A) = \dfrac{m-1}{m} \Rightarrow \dfrac{P(A \cap B)}{P(A)} = \dfrac{m-1}{m}$

$\Rightarrow P(A \cap B) = \dfrac{(m-1)^2}{m^2}$

Also $P(B) = P(A)\ P(B \mid A) + P(A')\ P(B \mid A')$

$$= \left(\frac{m-1}{m}\right)\left(\frac{m-1}{m}\right) + \left(\frac{1}{m}\right)(0) = \left(\frac{m-1}{m}\right)^2$$

$\Rightarrow \quad P(B') = 1 - P(B) = 1 - \dfrac{(m-1)^2}{m^2} = \dfrac{2m-1}{m^2}$

$\therefore \quad P(A \mid B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{P(A) - P(A \cap B)}{P(B')}$

$$= \frac{m-1}{2m-1}$$

$$P(A \cup B) = \frac{m-1}{m} + \left(\frac{m-1}{m}\right)^2 - \left(\frac{m-1}{m}\right)^2.$$

58. $P(B) = P(A)\ P(B \mid A) + P(A')\ P(B \mid A')$

$= P(A)\ P(B \mid A) + P(A')\ [1 - P(B' \mid A')]$

$= \alpha(1-\alpha) + (1-\alpha)\ [1 - (1-\alpha)] = 2\alpha(1-\alpha)$

and $P(A' \mid B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)}$

$$= \frac{P(B) - P(A)\ P(B \mid A)}{P(B)}$$

$$= \frac{2\alpha(1-\alpha) - \alpha(1-\alpha)}{2\alpha(1-\alpha)} = \frac{1}{2}$$

$$P(A' \mid B') = \frac{P(A' \cap B')}{P(B')} = \frac{1 - P(A \cup B)}{1 - P(B)}$$

and $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \alpha^2$

59. $P(X \geq n) = \displaystyle\sum_{k=n}^{\infty} P(X = k) = \frac{pq^n}{1-q} = q^n$

$$P(X \geq m+n \mid X \geq m) = \frac{P(X \geq m+n)}{P(X \geq m)} = \frac{q^{m+n}}{q^m} = q^n$$

$$P(X = m+n \mid X \geq m) = \frac{P(X = m+n)}{P(X \geq m)} = pq^n.$$

60. A and B are independent events.

61. $P(E_1) = {}^6C_3/6^3 = 5/54$,
$P(E_2) = 6/6^3 = 1/36$,
and $P(E_1 \cap E_2) = 1/36$.

62. For divisibility with 9, leave one of (1, 8), (2, 7), (3, 6) or (4, 5).
For divisibility with 3, leave one of (1, 2), (1, 5), (1, 8), (2, 4), (2, 7), (3, 6), (3, 9), (4, 5), (4, 8), (5, 7), (6, 9), (7, 8).
For divisibility with 5 unit's digit must be 5.

63. $p_1 = \dfrac{{}^7C_3}{{}^8C_4} = \dfrac{1}{2}$, $p_2 = 0$.

For (d), list all the possible 14 combinations.

64. See Theory.

65. $P(A \cap B) = P(\text{both show odd numbers}) = (3/6)^2$ etc.
$A \cap B \cap C$ means a, b and $a + b$ are odd. Impossible.

66. Use definition.

67. Let E denote the event that the ball lost is white.

$$P(E) = \frac{w}{w+b} \quad P(E') = \frac{b}{w+b}$$

$$P(A \mid E) = \frac{w-1}{w+b-1}, \quad P(A \mid E') = \frac{w}{w+b-1}$$

Now use the total probability rule.

68. Similar to 67.

69. Let E_r $(0 \leq r \leq l)$ denote the event that r white balls are lost.

$$P(E_r) = \frac{{}^wC_r\ {}^bC_{l-r}}{{}^{w+b}C_l}$$

and $P(A \mid E_r) = \dfrac{w-r}{w+b-l}$

Now use the total probability rule and the identity

$$\sum_{r=0}^{l} (w-r)\ ({}^wC_r)\ ({}^bC_{l-r}) = w\ ({}^{w+b-1}C_l).$$

70. $\lim\limits_{w \to \infty} p(l) = \dfrac{1}{1 + b/w} = \dfrac{1}{1+0}$.

71. $a^2 - 1$ is divisible by 10 if and only if last digit of a is 1 or 9.
If $r = 0$, there are $2k$ ways to choose a.

$\therefore \quad p_n = 2k/n$.

72. If $r = 9$, then there are $2(k+1)$ ways to choose a and

$$p_n = 2(k+1)/n.$$

73. If $1 \leq r \leq 8$, a can be chosen in $(2k+1)$ ways.
$\therefore\ p_n = (2k+1)/n$.

74. When $r = 0$, $p_n = 1/5$,

When $r = 9$, $p_n = \dfrac{2(1+1/k)}{10 + r/k} \to \dfrac{1}{5}$

as n and hence $k \to \infty$. Similarly, for $1 \leq r \leq 8$.

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(ii) $a_i \in Q$ and $a_i \notin R$
(iii) $a_i \notin Q$ and $a_i \in R$
(iv) $a_i \notin Q$ and $a_i \notin R$
Thus, total number of choices for picking up Q and R is 4^n.
For a_i not to belong $Q \cap R$, we have 3 choices.
Therefore, $Q \cap R = \phi$ in 3^n ways.
Thus, probability $Q \cap R = \phi$ is $(3/4)^n$.
Next, $a_i \in Q \cup R$ in 3 ways. Therefore, $Q \cup R = P$ in 3^n ways.
Thus, probability $Q \cup R = P$ is $(3/4)^n$.

99. If a and b are both even, then $a + b$ is even, therefore $P(A \cap B \cap C) = 0$.

We have $P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{4}$.

Now, $P[(A \cap B \cap D)|A \cup B]$

$$= \frac{P\left[(A \cap B \cap D) \cap (A \cup B)\right]}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}$$

$$= \frac{1/4}{1/2 + 1/2 - 1/4} = \frac{1}{3}$$

100. Statement-2 is false.

$P(P$ and Q contradict each other)

$= p(1 - 2p) + 2p(1 - p) = 1/2$

$\Rightarrow 8p^2 - 6p + 1 = 0.$

$\Rightarrow (2p - 1)(4p - 1) = 0 \Rightarrow p = 1/2, 1/4.$

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identities, we require the properties of complementary and supplementary angles. For example, if $A + B + C = \pi$, then

1. $\sin (B + C) = \sin A, \quad \cos B = -\cos (C + A)$
2. $\cos (A + B) = -\cos C, \quad \sin C = \sin (A + B)$
3. $\tan (C + A) = -\tan B, \quad \cot A = -\cot (B + C).$
4. $\cos \dfrac{A+B}{2} = \sin \dfrac{C}{2}, \qquad \cos \dfrac{C}{2} = \sin \dfrac{A+B}{2}.$
5. $\sin \dfrac{C+A}{2} = \cos \dfrac{B}{2}, \quad \sin \dfrac{A}{2} = \cos \dfrac{B+C}{2}.$
6. $\tan \dfrac{B+C}{2} = \cot \dfrac{A}{2}, \quad \tan \dfrac{B}{2} = \cot \dfrac{C+A}{2}.$

Some Important Identities If $A + B + C = \pi$, then

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$
2. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$
3. $\tan \dfrac{B}{2} \tan \dfrac{C}{2} + \tan \dfrac{C}{2} \tan \dfrac{A}{2} + \tan \dfrac{A}{2} \tan \dfrac{B}{2} = 1.$
4. $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}.$
5. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$
6. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$
7. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$
8. $\sin A + \sin B + \sin C = 4 \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}.$
9. $\cos A + \cos B + \cos C = 1 + 4 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}.$

SOLVED EXAMPLES

SECTION I

SINGLE CORRECT CHOICE TYPE

Example 1 If $\cot \alpha + \tan \alpha = m$ and $\dfrac{1}{\cos \alpha} - \cos \alpha = n$, then

(a) $m\ (mn^2)^{1/3} - n(nm^2)^{1/3} = 1$
(b) $m(m^2n)^{1/3} - n(mn^2)^{1/3} = 1$
(c) $n(mn^2)^{1/3} - m(nm^2)^{1/3} = 1$
(d) $n(m^2n)^{1/3} - m(mn^2)^{1/3} = 1.$

Ans. (a).

Solution Clearly $\alpha \neq 0$.

$\cot \alpha + \tan \alpha = m \Rightarrow 1 + \tan^2 \alpha = m \tan \alpha$

$\Rightarrow \sec^2 \alpha = m \tan \alpha \qquad (1)$

and $\dfrac{1}{\cos \alpha} - \cos \alpha = n \Rightarrow \sec^2 \alpha - 1 = n \sec \alpha$

$\Rightarrow \tan^2 \alpha = n \sec \alpha \Rightarrow \tan^4 \alpha = n^2 \sec^2 \alpha$

$\Rightarrow \tan^4 \alpha = n^2 m \tan \alpha$ [by (1)]

$\Rightarrow \tan^3 \alpha = n^2 m \Rightarrow \tan \alpha = (n^2 m)^{1/3}$

and $\sec^2 \alpha = m(n^2 m)^{1/3}$ (by (1)]

Now $\sec^2 \alpha - \tan^2 \alpha = 1 \Rightarrow m(n^2m)^{1/3} - (n^2m)^{2/3} = 1$

$\Rightarrow m(mn^2)^{1/3} - n(nm^2)^{1/3} = 1.$

Example 2 If $a \cos^3 \alpha + 3a \cos \alpha \sin^2 \alpha = m$ and $a \sin^3 \alpha + 3a \cos^2 \alpha \sin \alpha = n$, then $(m + n)^{2/3} + (m - n)^{2/3}$ is equal to

(a) $2a^2$ (b) $2a^{1/3}$ (c) $2a^{2/3}$ (d) $2a^3.$

Ans. (c).

Solution From the given relations, we get

$$m + n = a \cos^3 \alpha + 3a \cos \alpha \sin^2 \alpha + 3a \cos^2 \alpha \sin \alpha + a \sin^3 \alpha$$
$$= a(\cos \alpha + \sin \alpha)^3$$

Similarly $m - n = a(\cos \alpha - \sin \alpha)^3$

$\therefore \ (m + n)^{2/3} + (m - n)^{2/3}$

$= a^{2/3}[(\cos \alpha + \sin \alpha)^2 + (\cos \alpha - \sin \alpha)^2]$

$= a^{2/3}\ [2\ (\cos^2 \alpha + \sin^2 \alpha)] = 2a^{2/3}$

Example 3 If $\dfrac{2 \sin \alpha}{1 + \cos \alpha + \sin \alpha} = y$, then $\dfrac{1 - \cos \alpha + \sin \alpha}{1 + \sin \alpha}$ is equal to

(a) $1/y$ (b) y (c) $1 - y$ (d) $1 + y.$

Ans. (b)

Solution $\dfrac{1 - \cos \alpha + \sin \alpha}{1 + \sin \alpha}$

$$= \frac{1 - \cos \alpha + \sin \alpha}{1 + \sin \alpha} \cdot \frac{1 + \cos \alpha + \sin \alpha}{1 + \cos \alpha + \sin \alpha}$$

$$= \frac{(1 + \sin \alpha)^2 - \cos^2 \alpha}{(1 + \sin \alpha)(1 + \cos \alpha + \sin \alpha)}$$

$$= \frac{1 + 2\sin \alpha + \sin^2 \alpha - 1 + \sin^2 \alpha}{(1 + \sin \alpha)(1 + \cos \alpha + \sin \alpha)}$$

$$= \frac{2\sin \alpha(1 + \sin \alpha)}{(1 + \sin \alpha)(1 + \cos \alpha + \sin \alpha)}$$

$$= \frac{2 \sin \alpha}{1 + \cos \alpha + \sin \alpha} = y.$$

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Solution $\sin\beta = \frac{1}{\sqrt{10}} \Rightarrow \cos\beta = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}}$

$\Rightarrow \quad \tan\beta = 1/3$

$\therefore \quad \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\cdot 1/3}{1-1/9} = \frac{3}{4}$

and $\quad \tan(\alpha+2\beta)$

$$= \frac{\tan\alpha+\tan 2\beta}{1-\tan\alpha\tan 2\beta} = \frac{1/7+3/4}{1-(1/7)(3/4)} = \frac{25}{25} = 1$$

Since $0<\beta<\pi/2$ and $\tan 2\beta = 3/4 > 0$, we get $0<2\beta<\pi/2$. Also, $0<\alpha<\pi/2$. Hence, $0<\alpha+2\beta<\pi$ and $\tan(\alpha+2\beta)=1$, so that

$$\alpha+2\beta = \pi/4 \quad \Rightarrow \quad 2\beta = \pi/4 - \alpha.$$

Example 22 If $3\pi/4 < \alpha < \pi$, then

$\sqrt{2\cot\alpha + \frac{1}{\sin^2\alpha}}$ is equal to

(a) $1+\cot\alpha$ (b) $-1-\cot\alpha$

(c) $1-\cot\alpha$ (d) $-1+\cot\alpha$.

Ans. (b).

Solution

$$\sqrt{2\cot\alpha+\frac{1}{\sin^2\alpha}} = \sqrt{2\cot\alpha+\operatorname{cosec}^2\alpha}$$

$$\sqrt{2\cot\alpha+1+\cot^2\alpha} = \sqrt{(1+\cot\alpha)^2} = |1+\cot\alpha|$$

Since $\cot\alpha < -1$ when $3\pi/4 < \alpha < \pi$, we have $|1+\cot\alpha| = -1-\cot\alpha$.

Example 23 $(2\sqrt{3}+4)\sin x + 4\cos x$ lies in the interval

(a) $(-4, 4)$ (b) $(-2\sqrt{5}, 2\sqrt{5})$

(c) $(-2+\sqrt{5}, 2+\sqrt{5})$ (d) $(-2(2+\sqrt{5}), 2(2+\sqrt{5}))$.

Ans. (d).

Solution The given expression is equal to $2[(\sqrt{3}+2)\sin x + 2\cos x]$. Put $\sqrt{3}+2 = r\cos\theta$ and $2 = r\sin\theta$, so that $r^2 = (\sqrt{3}+2)^2 + 2^2 = 11+4\sqrt{3}$. Then the expression can be written as

$$2(r\cos\theta\sin x + r\sin\theta\cos x)$$

$$= 2r\sin(\theta+x) = y \qquad \text{(say)}$$

Since $11+4\sqrt{3} < 9+4\sqrt{5}$, we have

$$\sqrt{11+4\sqrt{3}} < \sqrt{9+4\sqrt{5}} \quad \Rightarrow \quad \sqrt{11+4\sqrt{3}} < 2+\sqrt{5} \quad (1)$$

Also, since $-1 \le \sin(\theta+x) \le 1$

$$-2r \le 2r\sin(\theta+x) \le 2r$$

$$\Rightarrow \quad -2\sqrt{11+4\sqrt{3}} \le y \le 2\sqrt{11+4\sqrt{3}}$$

$$\Rightarrow \quad -2(2+\sqrt{5}) < y < 2(2+\sqrt{5}) \quad \text{[from (1)]}$$

Example 24 If $\cos(\theta-\alpha) = a$ and $\sin(\theta-\beta) = b$ $(0 < \theta-\alpha, \theta-\beta < \pi/2)$, then $\cos^2(\alpha-\beta) + 2ab\sin(\alpha-\beta)$ is equal to

(a) $4a^2b^2$ (b) a^2-b^2

(c) a^2+b^2 (d) $-a^2b^2$.

Ans. (c).

Solution We have

$$\sin(\alpha-\beta) = \sin(\alpha-\theta+\theta-\beta)$$

$$= \sin[(\theta-\beta)-(\theta-\alpha)]$$

$$= \sin(\theta-\beta)\cos(\theta-\alpha) - \cos(\theta-\beta)\sin(\theta-\alpha)$$

$$= ba - \sqrt{1-b^2}\sqrt{1-a^2}$$

and $\cos(\alpha-\beta) = \cos[(\theta-\beta)-(\theta-\alpha)]$

$$= \cos(\theta-\beta)\cos(\theta-\alpha) + \sin(\theta-\beta)\sin(\theta-\alpha)$$

$$= a\sqrt{1-b^2} + b\sqrt{1-a^2}$$

Substituting these values in the given expression, we get

$$\cos^2(\alpha-\beta) + 2ab\sin(\alpha-\beta)$$

$$= \left(a\sqrt{1-b^2}+b\sqrt{1-a^2}\right)^2 + 2ab\left[ab - \sqrt{(1-a^2)}\sqrt{(1-b^2)}\right]$$

$$= a^2(1-b^2) + b^2(1-a^2) + 2ab\sqrt{(1-a^2)}\sqrt{(1-b^2)}$$

$$+ 2a^2b^2 - 2ab\sqrt{(1-a^2)}\sqrt{(1-b^2)} = a^2+b^2.$$

Example 25 The expression

$$\frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{\cos 5x + 5\cos 3x + 10\cos x}$$ is equal to

(a) $\cos 2x$ (b) $2\cos x$

(c) $\cos^2 x$ (d) $1+\cos x$.

Ans. (b).

Solution The given expression can be written as

$$\frac{(\cos 6x+\cos 4x)+5(\cos 4x+\cos 2x)+10(\cos 2x+1)}{\cos 5x+5\cos 3x+10\cos x}$$

$$= \frac{2\cos 5x\cos x + 5\cdot 2\cos 3x\cos x + 10\cdot 2\cos^2 x}{\cos 5x+5\cos 3x+10\cos x}$$

$$= \frac{2\cos x(\cos 5x+5\cos 3x+10\cos x)}{\cos 5x+5\cos 3x+10\cos x} = 2\cos x.$$

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Since $\tan^2 \alpha$ is non-negative, either $x < 1/3$ or $x \geq 3$, so x cannot lie between 1/3 and 3.

Example 44 For a given pair of values x and y satisfying $x = \sin \alpha$, $y = \sin \beta$, there are four different values of $z = \sin(\alpha + \beta)$ whose product is equal to

(a) $x^2 - y^2$ (b) $x^2 + y^2$
(c) $(x^2 + y^2)^2$ (d) $(x^2 - y^2)^2$

Ans. (d)

Solution $z = \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$= \pm\left(x\sqrt{1-y^2}\right) \pm \left(y\sqrt{1-x^2}\right)$$

There are four values of z, given by

$$\pm\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] \text{ and } \pm\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$$

and their product is equal to

$$[x^2(1-y^2) - y^2(1-x^2)]^2 = (x^2 - y^2)^2.$$

Example 45 If $0 < A < \pi/2$ the value of the expression

$$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} - \sec A \operatorname{cosec} A \text{ is equal to}$$

(a) -1 (b) 0
(c) 1 (d) $\sin A + \cos A$

Ans. (c)

Solution The given expression is equal to

$$\frac{\sin^2 A}{\cos A(\sin A - \cos A)} + \frac{\cos^2 A}{\sin A(\cos A - \sin A)} - \sec A \operatorname{cosec} A$$

$$= \frac{\sin^3 A - \cos^3 A}{\sin A \cos A(\sin A - \cos A)} - \sec A \operatorname{cosec} A$$

$$= \frac{\sin^2 A + \cos^2 A + \sin A \cos A}{\sin A \cos A} - \sec A \operatorname{cosec} A$$

$$= \sec A \operatorname{cosec} A + 1 - \sec A \operatorname{cosec} A = 1.$$

Example 46 If $a^2 - 2a \cos x + 1 = 674$ and $\tan(x/2) = 7$ then the integral value of a is

(a) 25 (b) 49 (c) 67 (d) 74

Ans. (a)

Solution
$$674 = a^2 - 2a\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} + 1$$

$$= a^2 - 2a \times \frac{1-49}{1+49} + 1$$

$$= a^2 + 2a \times \frac{48}{50} + 1$$

$$\Rightarrow \quad 25a^2 + 48a - 673 \times 25 = 0$$

$$\Rightarrow \quad (a - 25)(25a + 673) = 0$$

$$\Rightarrow \quad a = 25 \text{ (Taking the integral value of } a\text{).}$$

Example 47 If $\sin 2x = \alpha - 1$ and $\cos 2x = \beta - 1$, then the value of $\dfrac{\sec^2 x[(\cos^2 x - \sin^2 x) - 2\sin x \cos x]}{1 + \sin 2x}$ is equal to

(a) $2(\alpha - \beta)$ (b) $\dfrac{2}{\alpha} - \dfrac{2}{\beta}$
(c) $\dfrac{2}{\beta} - \dfrac{2}{\alpha}$ (d) $2(\alpha + \beta)$

Ans. (b)

Solution The given expression can be written as

$$\frac{2(\cos 2x - \sin 2x)}{2\cos^2 x(1 + \sin 2x)}$$

$$= \frac{2(\cos 2x - \sin 2x)}{(1+\cos 2x)(1+\sin 2x)}$$

$$= 2\left[\frac{1}{1+\sin 2x} - \frac{1}{1+\cos 2x}\right]$$

$$= 2\left[\frac{1}{\alpha} - \frac{1}{\beta}\right]$$

Example 48 The value of the expression

$$\frac{1}{\cos 290^\circ} + \frac{1}{\sqrt{3}\sin 250^\circ} \text{ is equal to}$$

(a) $\sqrt{3}/4$ (b) $4/\sqrt{3}$ (c) $2/\sqrt{3}$ (d) $\sqrt{3}/2$

Ans. (b)

Solution The given expression is equal to

$$= \frac{1}{\cos(270^\circ + 20^\circ)} + \frac{1}{\sqrt{3}\sin(270^\circ - 20^\circ)}$$

$$= \frac{1}{\sin 20^\circ} + \frac{1}{\sqrt{3}(-\cos 20^\circ)}$$

$$= \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sqrt{3}\sin 20^\circ \cos 20^\circ} = \frac{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ}{\frac{\sqrt{3}}{4}\sin 40^\circ}$$

$$= \frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\frac{\sqrt{3}}{4}\sin 40^\circ}$$

$$= \frac{\sin 40^\circ}{\frac{\sqrt{3}}{4}\sin 40^\circ} = \frac{4}{\sqrt{3}}.$$

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Solution $$y = \frac{\sqrt{(\cos 2A - \sin 2A)^2} + 1}{\sqrt{(\cos 2A + \sin 2A)^2} - 1}$$

$$\Rightarrow \quad y = \frac{\pm(\cos 2A - \sin 2A) + 1}{\pm(\cos 2A + \sin 2A) - 1}$$

which gives us four values of y, say y_1, y_2, y_3 and y_4. We have

$$y_1 = \frac{\cos 2A - \sin 2A + 1}{\cos 2A + \sin 2A - 1} = \frac{(1 + \cos 2A) - \sin 2A}{(\cos 2A - 1) + \sin 2A}$$

$$= \frac{2\cos^2 A - 2\sin A \cos A}{-2\sin^2 A + 2\sin A \cos A}$$

$$= \frac{\cos A(\cos A - \sin A)}{\sin A(\cos A - \sin A)} = \cot A$$

$$y_2 = \frac{-(\cos 2A - \sin 2A) + 1}{-(\cos 2A + \sin 2A) - 1} = \frac{(1 - \cos 2A) + \sin 2A}{-(1 + \cos 2A) - \sin 2A}$$

$$= \frac{2\sin^2 A + 2\sin A \cos A}{-2\cos^2 A - 2\sin A \cos A} = -\tan A$$

$$y_3 = \frac{(\cos 2A - \sin 2A) + 1}{-(\cos 2A + \sin 2A) - 1} = \frac{(1 + \cos 2A) - \sin 2A}{-(1 + \cos 2A) - \sin 2A}$$

$$= \frac{2\cos^2 A - 2\sin A \cos A}{-2\cos^2 A - 2\sin A \cos A} = -\frac{\cos A - \sin A}{\cos A + \sin A}$$

$$= -\frac{1 - \tan A}{1 + \tan A} = -\tan\left(\frac{\pi}{4} - A\right) = -\cot\left(\frac{\pi}{4} + A\right)$$

$$y_4 = \frac{-(\cos 2A - \sin 2A) + 1}{(\cos 2A + \sin 2A) - 1} = \frac{(1 - \cos 2A) + \sin 2A}{-(1 - \cos 2A) + \sin 2A}$$

$$= \frac{2\sin^2 A + 2\sin A \cos A}{-2\sin^2 A + 2\sin A \cos A} = \frac{\cos A + \sin A}{\cos A - \sin A}$$

$$= \frac{1 + \tan A}{1 - \tan A} = \tan\left(\frac{\pi}{4} + A\right).$$

Example 64 If $\cos 5\theta = a \cos \theta + b \cos^3 \theta + c \cos^5 \theta + d$, then

(a) $a = 20$ (b) $b = -20$ (c) $c = 16$ (d) $d = 5$

Ans. (b) and (c)

Solution $\cos 5\theta = \cos(4\theta + \theta) =$
$\cos 4\theta \cos \theta - \sin 4\theta \sin \theta$

$= (2\cos^2 2\theta - 1)\cos \theta - 2\sin 2\theta \cos 2\theta \sin \theta$

$= [2(2\cos^2 \theta - 1)^2 - 1]\cos \theta - 2 \cdot 2\cos \theta \sin^2 \theta (2\cos^2 \theta - 1)$

$= [2(4\cos^4 \theta - 4\cos^2 \theta + 1) - 1]\cos \theta - 4\cos \theta (2\cos^2 \theta - 1)(1 - \cos^2 \theta)$

$= \cos \theta (8\cos^4 \theta - 8\cos^2 \theta + 1) - 4\cos \theta (3\cos^2 \theta - 2\cos^4 \theta - 1)$

$= \cos \theta (16\cos^4 \theta - 20\cos^2 \theta + 5)$

$= 16\cos^5 \theta - 20\cos^3 \theta + 5\cos \theta$

clearly, $a = 5$, $b = -20$, $c = 16$ and $d = 0$. Satisfy the given relation.

Example 65 If $(x - a)\cos \theta + y \sin \theta = (x - a)\cos \phi + y \sin \phi = a$ and $\tan(\theta/2) - \tan(\phi/2) = 2b$, then

(a) $y^2 = 2ax - (1 - b^2)x^2$

(b) $\tan \frac{\theta}{2} = \frac{1}{x}(y + bx)$

(c) $y^2 = 2bx - (1 - a^2)x^2$

(d) $\tan \frac{\phi}{2} = \frac{1}{x}(y - bx)$.

Ans. (a), (b) and (d)

Solution Let $\tan(\theta/2) = \alpha$ and $\tan(\phi/2) = \beta$, so that $\alpha - \beta = 2b$.

Also $$\cos \theta = \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{1 - \alpha^2}{1 + \alpha^2}$$

and $$\sin \theta = \frac{2\tan(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{2\alpha}{1 + \alpha^2}$$

Similarly $\cos \phi = \frac{1 - \beta^2}{1 + \beta^2}$ and $\sin \phi = \frac{2\beta}{1 + \beta^2}$

Therefore, we have from the given relations

$$(x - a)\frac{1 - \alpha^2}{1 + \alpha^2} + y\left(\frac{2\alpha}{1 + \alpha^2}\right) = a$$

$$\Rightarrow \quad x\alpha^2 - 2y\alpha + 2a - x = 0$$

Similarly $x\beta^2 - 2y\beta + 2a - x = 0$.

We see that α and β are the roots of the equation $xz^2 - 2yz + 2a - x = 0$, so that $\alpha + \beta = 2y/x$ and $\alpha\beta = (2a - x)/x$. Now, from $(\alpha + \beta)^2 = (\alpha - \beta)^2 + 4\alpha\beta$, we get

$$\left(\frac{2y}{x}\right)^2 = (2b)^2 + \frac{4(2a - x)}{x}$$

$$\Rightarrow \quad y^2 = 2ax - (1 - b^2)x^2$$

Also, from $\alpha + \beta = 2y/x$ and $\alpha - \beta = 2b$, we get

$$\alpha = y/x + b \text{ and } \beta = y/x - b$$

$$\Rightarrow \quad \tan \frac{\theta}{2} = \frac{1}{x}(y + bx) \text{ and } \tan \frac{\phi}{2} = \frac{1}{x}(y - bx)$$

Example 66 For $0 < \phi < \pi/2$, if

$$x = \sum_{n=0}^{\infty} \cos^{2n} \phi, \quad y = \sum_{n=0}^{\infty} \sin^{2n} \phi,$$

$$z = \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi,$$

then

(a) $xyz = xz + y$ (b) $xyz = xy + z$

(c) $xyz = x + y + z$ (d) $xy^2 = y^2 + x$.

Ans. (b) and (c)

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Ans. (a), (c), (d)

Solution We are given that

$$(a+b)\left(\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b}\right)=1$$

$$\Rightarrow \quad \sin^4\alpha+\cos^4\alpha+\frac{b}{a}\sin^4\alpha+\frac{a}{b}\cos^4\alpha=1$$

$$=(\sin^2\alpha+\cos^2\alpha)^2$$

$$\Rightarrow \quad \frac{b}{a}\sin^4\alpha-2\sin^2\alpha\cos^2\alpha+\frac{a}{b}\cos^4\alpha=0$$

$$\Rightarrow \quad \left(\sqrt{\frac{b}{a}}\sin^2\alpha-\sqrt{\frac{a}{b}}\cos^2\alpha\right)^2=0$$

$$\Rightarrow \quad \frac{b}{a}\sin^4\alpha=\frac{a}{b}\cos^4\alpha$$

$$\Rightarrow \quad \frac{\sin^4\alpha}{a^2}=\frac{\cos^4\alpha}{b^2}=k \text{ say}$$

Since $\frac{\sin^4\alpha}{a}+\frac{\cos^4\alpha}{b}=\frac{1}{a+b}$, we get

$$ak+bk=\frac{1}{a+b} \quad \Rightarrow \quad k=\frac{1}{(a+b)^2}$$

$$\therefore \quad \frac{\sin^8\alpha}{a^3}+\frac{\cos^8\alpha}{b^3}=\frac{(a^2k)^2}{a^3}+\frac{(b^2k)^2}{b^3}$$

$$=ak^2+bk^2=(a+b)k^2$$

$$=(a+b)\cdot\frac{1}{(a+b)^4}=\frac{1}{(a+b)^3}.$$

Example 82 If $P_n=\cos^n\theta+\sin^n\theta$, then

(a) $2P_6-3P_4=-1$

(b) $2P_6-3P_4=1$

(c) $6P_{10}-15P_8+10P_6=0$

(d) $6P_{10}-15P_8+10P_6=1$

Ans. (a), (d)

Solution $2P_6-3P_4+1$

$$=2(\cos^6\theta+\sin^6\theta)-3(\cos^4\theta+\sin^4\theta)+1$$

$$=2[(\cos^2\theta+\sin^2\theta)^3-3\sin^2\theta\cos^2\theta(\cos^2\theta+\sin^2\theta)]$$

$$-3[(\cos^2\theta+\sin^2\theta)^2-2\sin^2\theta\cos^2\theta]+1$$

$$=2(1-3\sin^2\theta\cos^2\theta)-3(1-2\sin^2\theta\cos^2\theta)+1=0$$

Again for $n\geq 4$, we have

$$P_n-P_{n-2}$$

$$=\cos^n\theta+\sin^n\theta-(\cos^{n-2}\theta+\sin^{n-2}\theta)$$

$$=\cos^{n-2}\theta(\cos^2\theta-1)+\sin^{n-2}\theta(\sin^2\theta-1)$$

$$=-\sin^2\theta\cos^{n-2}\theta-\cos^2\theta\sin^{n-2}\theta$$

$$=-\sin^2\theta\cos^2\theta(\cos^{n-4}\theta+\sin^{n-4}\theta)$$

$$=-\sin^2\theta\cos^2\theta P_{n-4}$$

$\therefore$ $6P_{10}-15P_8+10P_6-1$

$$=6(P_{10}-P_8)-9(P_8-P_6)+(P_6-P_4)+P_4-P_2$$

$$=-\sin^2\theta\cos^2\theta(6P_6-9P_4+P_2+P_0)$$

$$=-3\sin^2\theta\cos^2\theta(2P_6-3P_4)-\sin^2\theta\cos^2\theta(1+2)$$

$$[\because \quad P_2=1,\ P_0=2]$$

$$=-3\sin^2\theta\cos^2\theta(-1)-3\sin^2\theta\cos^2\theta=0.$$

$$[\because\ 2P_6-3P_4+1=0 \text{ (as proved)}]$$

Example 83 The equation $\sin^4 x+\cos^4 x=a$ has a real solution for

(a) all values of a (b) $a=1/2$

(c) $a=7/10$ (d) $a=1$

Ans. (b), (c), (d)

Solution We have $\sin^4 x+\cos^4 x\leq \sin^2 x+\cos^2 x$, as $|\sin x|\leq 1$ and $|\cos x|\leq 1$

$$\Rightarrow \quad a\leq 1 \qquad (1)$$

Next, $\sin^4 x+\cos^4 x=a$

$$\Rightarrow \quad (\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x=a$$

$$\Rightarrow \quad \frac{1}{2}\sin^2 2x=1-a$$

$$\Rightarrow \quad 1-a\leq 1/2 \qquad [\because \quad \sin^2 2x\leq 1]$$

$$\Rightarrow \quad 1/2\leq a \qquad (2)$$

From (1) and (2) we get $1/2\leq a\leq 1$. Note that $a=1/2$ for $x=\pi/4$ and $a=1$ for $x=\pi/2$.

Example 84 If $\sin\theta(1+\sin\theta)+\cos\theta(1+\cos\theta)=x$ and $\sin\theta(1-\sin\theta)+\cos\theta(1-\cos\theta)=y$ then

(a) $x^2-2x=\sin 2\theta$ (b) $y^2+2y=\sin 2\theta$

(c) $xy=\sin 2\theta$ (d) $x-y=2$

Ans. (a), (b), (c), (d)

Solution $\sin\theta+\cos\theta=x-1=y+1$

$$\sin 2\theta=(\sin\theta+\cos\theta)^2-1=x^2-2x=y^2+2y$$

$$xy=(\sin\theta+\cos\theta)^2-1=\sin 2\theta$$

$$x-y=2$$

Example 85 For a positive integer n, let $f_n(\theta)=\tan(\theta/2)(1+\sec\theta)(1+\sec 2\theta)\ldots(1+\sec 2^n\theta)$ then

(a) $f_2(\pi/16)=1$ (b) $f_3(\pi/32)=1$

(c) $f_4(\pi/64)=1$ (d) $f_5(\pi/128)=1$

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Example 123 Given $\frac{\sin x}{a} = \frac{\cos x}{b} = \frac{\tan x}{c} = k$

(a) $a^2 + b^2$	(p) $1/b^2k^4$
(b) $a^2 + b^2 + c^2$	(q) $1/k^2$
(c) bc	(r) $1/ak$
(d) $\frac{1}{ck} + \frac{ak}{1+bk}$	(s) a/k

Ans.

	p	q	r	s
a	p	**q**	r	s
b	**p**	q	r	s
c	p	q	r	**s**
d	p	q	**r**	s

Solution $a^2 + b^2 = \frac{1}{k^2}(\sin^2 x + \cos^2 x) = \frac{1}{k^2}$

$$a^2 + b^2 + c^2 = \frac{1}{k^2} + \frac{1}{k^2}\tan^2 x = \frac{\sec^2 x}{k^2} = \frac{1}{b^2k^4}$$

$$bc = \frac{1}{k^2}\sin x = a/k$$

$$\frac{1}{ck} + \frac{ak}{1+bk} = \cot x + \frac{\sin x}{1+\cos x}$$

$$= \frac{\cos x(1+\cos x) + \sin^2 x}{\sin x(1+\cos x)} = \frac{1}{\sin x} = \frac{1}{ak}.$$

Example 124 If $\sin A = \sin B$ and $\cos A = \cos B$, then

(a) $\sin (1/2)(A - B)$	(p) 0
(b) $\sin 2A + \sin 2B$	(q) $\cos 2A$
(c) $\cos (A + B)$	(r) 1
(d) $\tan ((A + B)/2)$	(s) $2 \sin (A + B)$

Ans

	p	q	r	s
a	**p**	q	r	s
b	p	q	r	**s**
c	p	**q**	r	s
d	p	q	**r**	s

Solution $\sin A - \sin B = \cos B - \cos A = 0$

$$\Rightarrow 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} = 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\Rightarrow \sin\frac{A-B}{2}\left[\cos\frac{A+B}{2} - \sin\frac{A+B}{2}\right] = 0$$

$$\Rightarrow \text{Either } \sin\frac{A-B}{2} = 0 \text{ or } \tan\frac{A+B}{2} = 1$$

$$\sin 2A + \sin 2B = 2\sin A\cos A + 2\sin B\cos B$$
$$= 2\sin A\cos B + 2\cos A\sin B$$
$$= 2\sin(A+B)$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$= \cos^2 A - \sin^2 A = \cos 2A.$$

Example 125 $\frac{\sin 3\alpha}{\cos 2\alpha}$ is

(a) positive	(p) $\alpha \in \left(\frac{13\pi}{48}, \frac{14\pi}{48}\right)$
(b) negative	(q) $\alpha \in \left(\frac{14\pi}{48}, \frac{18\pi}{48}\right)$
(c) positive or negative	(r) $\alpha \in \left(\frac{18\pi}{48}, \frac{23\pi}{48}\right)$
(d) not defined	(s) $\alpha = 23\pi/4$

Ans.

	p	q	r	s
a	p	q	**r**	s
b	**p**	q	r	s
c	p	**q**	r	s
d	p	q	r	**s**

Solution $\alpha \in \left(\frac{13\pi}{48}, \frac{14\pi}{48}\right) \Rightarrow 3\alpha \in \left(\frac{13\pi}{16}, \frac{14\pi}{16}\right)$

$= (\pi - 3\pi/16, \pi - 2\pi/16)$

i.e. $3\alpha \in$ II quadrant $\Rightarrow \sin 3\alpha > 0$

$$2\alpha \in \left(\frac{13\pi}{24}, \frac{14\pi}{24}\right) = \left(\pi - \frac{11\pi}{24}, \pi - \frac{10\pi}{24}\right)$$

i.e. $2\alpha \in$ II quadrant $\Rightarrow \cos 2\alpha < 0$

$\therefore \frac{\sin 3\alpha}{\cos 2\alpha}$ is negative.

Similarly $\alpha \in \left(\frac{14\pi}{48}, \frac{18\pi}{48}\right) \Rightarrow 3\alpha \in \left(\pi - \frac{2\pi}{16}, \pi + \frac{2\pi}{16}\right)$

$\Rightarrow$ $\sin 3\alpha$ can be positive or negative.

$2\alpha \in \left(\pi - \frac{10\pi}{24}, \pi - \frac{61\pi}{24}\right)$ $\cos 2\alpha$ is negative.

so $\frac{\sin 3\alpha}{\cos 2\alpha}$ can be positive or negative.

If $\alpha \in \left(\frac{18\pi}{48}, \frac{23\pi}{48}\right)$, $\sin 3\alpha$ is – ve

and $\cos 2\alpha$ is – ve so $\frac{\sin 3\alpha}{\cos 2\alpha}$ is positive

and if $\alpha = 23\pi/4$ then $\cos 2\alpha = 0$ so $\frac{\sin 3\alpha}{\cos 2\alpha}$ is not defined.

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17. The number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2 \cos(2x) + 9_3 \sin^2 x = 0$ for all x is
(a) zero (b) one
(c) three (d) infinite
(e) none of these [1987]

18. The sides of a triangle inscribed in a circle subtend angels α, β, γ at the centre. The minimum value of the arithmetic mean of [1987]
$\cos(\alpha + \pi/2)$, $\cos(\beta + \pi/2)$, $\cos(\gamma + \pi/2)$ is equal to ... [1987]

19. Prove that $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$ [1988]

20. The value of the expression
$\sqrt{3}$ cosec 20° – sec 20° is equal to
(a) 2 (b) $2\sin 20^\circ/\sin 40^\circ$
(c) 4 (d) $4\sin 20^\circ/\sin 40^\circ$
[1988]

21. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}$
$\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$ is equal to ... [1991]

22. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$ where $[x]$ stands for the greatest integer function then
(a) $f(\pi/2) = -1$ (b) $f(\pi) = 1$
(c) $f(-\pi) = 0$ (d) $f(\pi/4) = 1$
[1991]

23. Show that the value of $\tan x/\tan 3x$, where ever defined, never lies between 1/3 and 3. [1992]

24. For $0 < \theta < \pi/2$, if
$x = \sum_{i=0}^{\infty}\cos^{2n}\phi$, $y = \sum_{n=0}^{\infty}\sin^{2n}\phi$, $z = \sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi$, then
(a) $xyz = xz + y$ (b) $xyz = xy + z$
(c) $xyz = x + y + z$ (d) $xyz = yz + x$
[1993]

25. If $k = \sin(\pi/18)\sin(5\pi/18)\sin(7\pi/18)$, then the numerical value of k is ... [1993]

26. If $A > 0, B > 0$ and $A + B = \pi/3$, then the maximum value of $\tan A \tan B$ is ... [1993]

27. For all values of A, B, C and P, Q, R show that
$$\begin{vmatrix}\cos(A-P) & \cos(A-Q) & \cos(A-R)\\ \cos(B-P) & \cos(B-Q) & \cos(B-R)\\ \cos(C-P) & \cos(C-Q) & \cos(C-R)\end{vmatrix} = 0$$
[1994]

28. The number of points of intersection of two curves $y = 2\sin x$ and $y = 5x^2 + 2x + 3$ is
(a) 0 (b) 1
(c) 2 (d) ∞ [1994]

29. Let $0 < x < \pi/4$, then $(\sec 2x - \tan 2x)$ equals
(a) $\tan(x - \pi/4)$ (b) $\tan(\pi/4 - x)$
(c) $\tan(x + \pi/4)$ (d) $\tan^2(x + \pi/4)$
[1994]

30. Let n be a positive integer such that
$\sin\frac{\pi}{2^n} + \cos\frac{\pi}{2^n} = \frac{\sqrt{n}}{2}$. Then
(a) $6 \le n \le 8$ (b) $4 < n \le 8$
(c) $4 \le n < 8$ (d) $4 < n < 8$
[1994]

31. The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$ where α, β, γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$ is
(a) positive (b) zero
(c) negative (d) – 3 [1995]

32. $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if
(a) $x + y \ne 0$ (b) $x = y, x \ne 0$
(c) $x = y$ (d) $x \ne 0, y \ne 0$
[1996]

33. If $\cos(x - y)$, $\cos x$, $\cos(x + y)$ are in H.P., then $\cos x \sec(y/2) = \ldots$ [1997]

34. The graph of the function
$\cos x \cos(x + 2) - \cos^2(x + 1)$ is
(a) a straight line passing through $(0, -\sin^2 1)$ with slope 2
(b) a straight line passing through (0, 0)
(c) a parabola with vertex $(1, -\sin^2 1)$
(d) a straight line passing through the point $(\pi/2, -\sin^2 1)$ and parallel to x-axis.
[1997]

35. Prove that the value of the function
$\frac{\sin x \cos 3x}{\sin 3x \cos x}$ does not lie between $\frac{1}{3}$ and 3 for any real value of x. [1997]

36. For each natural number k, let C_k denote the circle with radius k centimeter and centre at the origin. On the circle C_k, a particle moves k centimetres in the counter clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1}, in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n then $n = \ldots$
[1997]

37. The number of values of x where the function $f(x) = \cos x + \cos\left(\sqrt{2}\,x\right)$ attains its maximum is

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63. $x^2 + y^2 = 27 = \left(3\sqrt{3}\right)^2 \Rightarrow x = 3\sqrt{3}\cos\theta$,

$y = 3\sqrt{3}\sin\theta$

$x - y = 3\sqrt{3}(\cos\theta - \sin\theta)$

$= 3\sqrt{3} \times \sqrt{2}\cos(\theta + \pi/4) \leq 3\sqrt{6}$ from statement 2 which is true.

64. $\left(\sqrt{p} + \sqrt{q} + \sqrt{r}\right)^2 = p + q + r +$

$2\left(\sqrt{pq} + \sqrt{qr} + \sqrt{rp}\right)$

$\leq p + q + r + 2(p + q + r) = 3.16$

because $p + q + r = 16$

as statement 2 is true $\Rightarrow$ Statement 1 is false.

65. $x\sin\theta.\sin^2\theta + y\cos\theta.\cos^2\theta = \sin\theta\cos\theta$

$\Rightarrow x\sin\theta = \sin\theta\cos\theta \Rightarrow x = \cos\theta \Rightarrow y = \sin\theta$

$\Rightarrow x^2 + y^2 = 1$, so statement 1 is true. Also statement 2 is true but does not imply statement-1.

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Since $-\sqrt{2} \le \cos x \pm \sin x \le \sqrt{2}$

$$-\sqrt{2} \le (1-4n)\frac{\pi}{2p} \le \sqrt{2}$$

$$\Rightarrow \quad |p| \ge |(1-4n)|\frac{\pi}{2\sqrt{2}}$$

The smallest positive value of p is $\frac{\pi}{2\sqrt{2}} = \frac{\pi\sqrt{2}}{4}$

Example 13 Number of solutions of pair x, y of the equation $\sin x \sin y = \min\{-1, \alpha^2 - 4\alpha + 5\}$, $\alpha \in R$ (where $0 < x < \pi, -\pi < y < 0$) is

(a) 1 (b) 2 (c) 3 (d) infinite

Ans. (a)

Solution $\alpha^2 - 4\alpha + 5 = (\alpha - 2)^2 + 1 \ge 1$

So $\min\{-1, \alpha^2 - 4\alpha + 5\} = -1$

and $\sin x \sin y = -1$

$\Rightarrow \quad \sin x = 1, \sin y = -1$ or $\sin x = -1, \sin y = 1$

$\Rightarrow \quad x = \pi/2, y = -\pi/2$ or $x = -\pi/2, y = \pi/2$

So there is only one pair $-\pi/2$ & $\pi/2$ satisfying the given equation.

Example 14 The number of solutions of the equation $\tan^2 x - \sec^{10} x + 1 = 0$ in $(0, 10)$ is

(a) 3 (b) 6 (c) 10 (d) 0

Ans. (a)

Solution $\sec^2 x - \sec^{10} x = 0$

$\Rightarrow \quad \sec^2 x (1 - \sec^8 x) = 0$

$\Rightarrow \quad 1 - \sec^8 x = 0$ as $\sec^2 x \ge 1$

$\Rightarrow \quad \sec^8 x = 1 \Rightarrow \cos^8 x = 1 \Rightarrow \cos x = \pm 1$

$\Rightarrow \quad x = n\pi, n \in \mathbf{I}$

$\Rightarrow \quad x = \{\pi, 2\pi, 3\pi\} \in (0, 10)$

So the required number is 3.

Example 15 If $\left|1 - \frac{|\sin x|}{1+|\sin x|}\right| \ge \frac{2}{3}$, then $\sin x$ lies in

(a) $\left(-\infty, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$ (b) $\left(-\frac{1}{2}, \frac{1}{2}\right)$

(c) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (d) none of these

Ans. (c)

Solution. $|\sin x| \ge 0 \Rightarrow \frac{|\sin x|}{1+|\sin x|} < 1$

$$\Rightarrow \quad 1 - \frac{|\sin x|}{1+|\sin x|} > 0$$

So the given in equation becomes

$$1 - \frac{|\sin x|}{1+|\sin x|} \ge \frac{2}{3}$$

$$\Rightarrow \quad \frac{1}{3} \ge \frac{|\sin x|}{1+|\sin x|} \Rightarrow |\sin x| \le \frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{2} \le \sin x \le \frac{1}{2} \Rightarrow \sin x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Example 16 Number of solutions of the equation $\cos(\pi\sqrt{x}) \cdot \cos(\pi\sqrt{x-4}) = 1$ are

(a) 1 (b) 2 (c) 3 (d) 0

Ans. (a)

Solution $\cos\left(\pi\sqrt{x-4}\right)$ is defined if $x \ge 4$

$$\cos\left(\pi\sqrt{x}\right) \cos\left(\pi\sqrt{x-4}\right) = 1$$

$$\Rightarrow \quad \cos\left(\pi\sqrt{x}\right) = 1, \cos\left(\pi\sqrt{x-4}\right) = 1$$

$$\Rightarrow \quad \pi\sqrt{x} = 2m\pi, \ \pi\sqrt{x-4} = 2n\pi \ (m, n \in \mathbf{I})$$

$$\Rightarrow \quad x = 4m^2, \ x - 4 = 4n^2$$

$$\Rightarrow \quad m^2 - n^2 = 1 \Rightarrow m = 1, n = 0 \Rightarrow x = 4$$

So $x = 4$ is the only solution of the equation

Note

If we take $\cos\left(\pi\sqrt{x}\right) = -1, \cos\left(\pi\sqrt{x-4}\right) = -1$

We will get $(2m+1)^2 - (2n+1)^2 = 4$

$\Rightarrow \quad (m + n + 1)(m - n) = 1$

which does not give any integral solution of m & n.

Example 17 Let $\alpha \ \beta \in [-\pi \ \pi]$ be such that $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/e$, The number of pairs of α, β satisfying the above system of equations is

(a) 0 (b) 1 (c) 2 (d) 4

Ans. (d)

Solution $-\pi \le \alpha, \beta \le \pi \Rightarrow -2\pi \le \alpha - \beta \le 2\pi$

So $\cos(\alpha - \beta) = 1 \Rightarrow \alpha - \beta = -2\pi, 0, 2\pi$

$\Rightarrow \quad \beta = \alpha - 2\pi, \ \alpha, \ \alpha + 2\pi$

As $\cos(\alpha + \beta) = 1/e, \cos 2\alpha = 1/e$

As $-2\pi \le 2\alpha \le 2\pi$ or $-\pi \le \alpha \le \pi$

And $\cos 2\alpha = 1/e$, there are four values α and hence at least four pairs (α, β).

Example 18 The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 2 (d) 4

Ans. (c)

Solution $2\sin^2\theta - \cos 2\theta = 0$

$\Rightarrow \quad 2\sin^2\theta - (1 - 2\sin^2\theta) = 0$

$\Rightarrow \quad 4\sin^2\theta = 1 \Rightarrow \sin\theta = \pm 1/2$

and $\quad 2\cos^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad 2(1 - \sin^2\theta) - 3\sin\theta = 0$

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Solution We can write

$r \sin(x + \alpha) = 2y^2 - 8y + 21$ where $r = \sqrt{144 + 25} = 13$

and $\tan \alpha = 5/12$

L.H.S ≤ 13 and R.H.S $= 2(y-2)^2 + 13 \ge 13$

Roots of the equation exist if L.H.S. = R.H.S = 13

$\Rightarrow \quad y = 2$ and $\sin(x + \tan^{-1}(5/12)) = 1$

$$\Rightarrow \quad x = \frac{\pi}{2} - \tan^{-1}\frac{5}{12} \Rightarrow \cot x = \frac{5}{12}$$

$$\therefore \quad 3864 \cot\left(\frac{xy}{2}\right) = 3864 \times \frac{5}{12}$$

$$= 322 \times 5 = 1610.$$

SECTION V

MATRIX MATCH TYPE

Example 36

(a) $\cos x = -1/2$	(p) $x = 7\pi/3$
(b) $\sin x = \sqrt{3}/2$	(q) $x = 19\pi/6$
(c) $\tan x = 1/\sqrt{3}$	(r) $x = 8\pi/3$
(d) $\cot x = -1$	(s) $x = 11\pi/4$

Ans.

	p	q	r	s
a			●	
b	●			
c		●		
d				●

Solution $\cos x = -1/2 = \cos(2\pi/3) \Rightarrow x = 2n\pi \pm (2\pi/3)$, $n \in \mathbf{I}$

$\Rightarrow \quad x = 8\pi/3$ for $n = 1$

$\sin x = \sqrt{3}/2 = \sin(\pi/3) \Rightarrow x = n\pi + (-1)^n \pi/3$, $n \in \mathbf{I}$.

$\Rightarrow \quad x = 7\pi/3$ for $n = 2$

$\tan x = 1/\sqrt{3} = \tan(\pi/6) \Rightarrow x = n\pi + \pi/6$, $n \in \mathbf{I}$

$\Rightarrow \quad x = 19\pi/6$ for $n = 3$.

$\cot x = -1 = \cot(3\pi/4)$

$\Rightarrow \quad x = n\pi + 3\pi/4 \; n \in I$

$\Rightarrow \quad x = 11\,\pi/4$ for $n = 2$

Example 37

(a) $(2 \sin x - \cos x)(1 + \cos x) = \sin^2 x$	(p) $\sin x = 1/2$
(b) $1 + \sin 2x = \cos x + \sin x$	(q) $\tan x = -1$
(c) $4x^4 + x^6 + \sin^2 5x = 0$	(r) $x = 0$
(d) $\sin x + \sin 2x = 0$	(s) $\cos x = -1/2$

Ans.

	p	q	r	s
a	●			
b		●		
c			●	
d			●	●

Solution $(2 \sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\Rightarrow \; 2 \sin x (1 + \cos x) - \cos x - \cos^2 x - \sin^2 x = 0$

$\Rightarrow \quad (1 + \cos x)(2 \sin x - 1) = 0$

which holds if $\sin x = 1/2$

$1 + \sin 2x = \cos x + \sin x$

$\Rightarrow \quad (\cos x + \sin x)^2 = \cos x + \sin x$

which is satisfied if $\cos x + \sin x = 0$ i.e. $\tan x = -1$

$4x^4 + x^6 + \sin^2 5x = 0$ if $x = 0$

$\sin x + \sin 2x = \sin x (1 + 2\cos x) = 0$

Example 38

(a) $4 \sin^2 x + \sin^2 2x = 3$	(p) $x = \tan^{-1} 1$
(b) $4 \cos^2 2x + 8 \cos^2 x = 7$	(q) $x = \sin^{-1}(-1)$
(c) $3(1 + \sin x) = 1 + \cos 2x$	(r) $x = \tan^{-1}\left(1/\sqrt{3}\right)$
(d) $4 \sin^3 x - 2 \sin^2 x - 2 \sin x + 1 = 0$	(s) $x = \sin^{-1}(1/2)$

Ans.

	p	q	r	s
a	●			
b			●	
c		●		
d	●			●

Solution $4 \sin^2 x + 4 \sin^2 x \cos^2 x = 3(\sin^2 x + \cos^2 x)$

$\Rightarrow \quad \sin^2 x + 4 \sin^2 x \cos^2 x - 3 \cos^2 x = 0$

$\Rightarrow \quad \tan^2 x (1 + \tan^2 x) + 4 \tan^2 x - 3(1 + \tan^2 x) = 0$

$\Rightarrow \quad \tan^4 x + 2 \tan^2 x - 3 = 0$

$\Rightarrow \quad (\tan^2 x - 1)(\tan^2 x + 3) = 0$

which is satisfied by $x = \tan^{-1} 1$,

Next, $4 \cos^2 2x + 8 \cos^2 x = 7$

$\Rightarrow \quad 4(\cos^2 x - \sin^2 x)^2 + 8 \cos^2 x = 7$

$\Rightarrow \quad 4(1 - \tan^2 x)^2 + 8(1 + \tan^2 x) = 7(1 + \tan^2 x)^2$

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We have $\tan(45° \pm \tan 30°) = \dfrac{1 \pm \frac{1}{\sqrt{3}}}{1 \mp \frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3} \pm 1}{\sqrt{3} \mp 1} = \dfrac{4 \pm \sqrt{3}}{2}$

$= 2 \pm \sqrt{3}$

So From (i) $\tan\theta = \tan(45° \pm 30°)$

$\Rightarrow \theta = 15°$ or $75°$ or $\theta = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$

Showing that statement-1 and 2 both are true and statement-2 leads to statement-1.

EXERCISE

LEVEL 1

SECTION I

SINGLE CORRECT CHOICE TYPE

1. $\cos 2x + a \sin x = 2a - 7$ possesses a solution for
   (a) all value of a (b) $a > 6$
   (c) $a < 2$ (d) $2 \le a \le 6$
2. A solution (x, y) of the system of equations $x - y = 1/3$ and $\cos^2(\pi x) - \sin^2(\pi y) = 1/2$ is given by
   (a) (2/3, 1/3) (b) (7/6, 1/6)
   (c) (13/6, 11/6) (d) (1/6, 5/6)
3. If $\sin x = \cos y$, $\sqrt{6} \sin y = \tan z$ and $2 \sin z = \sqrt{3} \cos x$; u, v, w denote respectively $\sin^2 x$, $\sin^2 y$, $\sin^2 z$, then the value of the triplet (u, v, w) is
   (a) (1, 0, 0) (b) (0, 1, 0)
   (c) (1/2, 1/2, 3/4) (d) (1/2, 3/4, 1/2)
4. The least difference between the roots of the equation
   $4 \cos x (2 - 3 \sin^2 x) + (\cos 2x + 1) = 0 \; (0 \le x \le \pi/2)$ is
   (a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$
5. The sum of the roots of the equation
   $4 \cos^3(\pi + x) - 4 \cos^2(\pi - x) + \cos(\pi + x) - 1 = 0$ in the interval [0, 320] is $p\pi$ where p is equal to
   (a) 2500 (b) 2601
   (c) 2600 (d) 2651

SECTION II

MULTIPLE CORRECT CHOICE TYPE

6. $\cos^2 x - 2 \cos x = 4 \sin x - \sin 2x$ if
   (a) $\tan(x/2) = -1/2$ (b) $\tan x = -1/2$
   (c) $\tan(x/2) = 2 + \sqrt{5}$ (d) $\tan(x/2) = 2 - \sqrt{5}$
7. $\sin x + \cos x = 1 + \sin x \cos x$, if
   (a) $\sin(x + \pi/4) = 1/\sqrt{2}$
   (b) $\sin(x - \pi/4) = 1/\sqrt{2}$
   (c) $\cos(x + \pi/4) = 1/\sqrt{2}$
   (d) $\cos(x - \pi/4) = 1/\sqrt{2}$
8. If $x + y + z = \pi$, $\tan x \tan y = 2$, $\tan x + \tan y + \tan z = 6$ then
   (a) $x = m\pi + \pi/4$
   (b) $y = n\pi + \tan^{-1} 2$
   (c) $z = l\pi + \tan^{-1} 3$
   (d) all are correct $(l, m, n \in \mathbf{I})$

SECTION III

LINKED COMPREHENSION TYPE

Paragraph for Question Nos. 9 to 11

Consider the equations

$5\sin^2 x + 3 \sin x \cos x - 3 \cos^2 x = 2$ (1)

$\sin^2 x - \cos 2x = 2 - \sin 2x$ (2)

9. If α is a root of (1) and β is a root of (2) then $\tan\alpha + \tan\beta$ can be equal to
   (a) $1 + \sqrt{69}/6$ (b) $1 - \sqrt{69}/6$
   (c) $\dfrac{-3 + \sqrt{69}}{6}$ (d) $\dfrac{-3 - \sqrt{69}}{3}$
10. If $\tan\alpha$, $\tan\beta$ satisfy (1) and $\cos\gamma$, $\cos\delta$ satisfy (2) then $\tan\alpha \tan\beta + \cos\gamma + \cos\delta$ can be equal to
    (a) -1 (b) $-\dfrac{5}{3} + \dfrac{2}{\sqrt{13}}$
    (c) $\dfrac{5}{3} - \dfrac{2}{\sqrt{13}}$ (d) $-\dfrac{5}{3} - \dfrac{2}{\sqrt{13}}$
11. The number of solutions common to (1) and (2) is
    (a) 0 (b) 1
    (c) finite (d) infinite

SECTION IV

INTEGER ANSWER TYPE

12. $\cos 5\theta = 0$ if $64 \cos^4\theta - 80 \cos^2\theta + 1625$ is equal to
13. If $3 \tan 2x - 4 \tan 3x = \tan^2 3x \tan 2x$ then $\tan 3x + 3 \tan x + 2736$ is equal to

SECTION V

MATRIX MATCH TYPE

14. (a) $2 \cos^2 x + 4 \cos x = 3 \sin^2 x$ (p) $\sin x = 1/2$
    (b) $\tan x + \sec x = 2 \cos x$ (q) $\cos x = -3/5$

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ANSWERS

LEVEL 1

SECTION I

1. (d) 2. (c). 3. (a)
4. (a) 5. (b)

SECTION II

6. (a), (c), (d), 7. (a), (d)
8. (a), (b), (c), (d)

SECTION III

9. (a), (b) 10. (b), (d) 11. (a)

SECTION IV

12. 1605 13. 2736

SECTION V

14.

	p	q	r	s
a	p	q	**r**	s
b	**p**	q	r	s
c	p	**q**	r	**s**
d	**p**	q	r	**s**

15.

	p	q	r	s
a	p	q	**r**	s
b	**p**	q	r	s
c	p	**q**	r	**s**
d	p	q	r	**s**

SECTION VI

16. (c) 17. (a)

LEVEL 2

SECTION I

1. (a) 2. (b) 3. (b)
4. (c) 5. (b)

SECTION II

6. (a), (d) 7. (b), (d) 8. (a), (b)

SECTION III

9. (a), (b) 10. (a), (d) 11. (a), (b)

SECTION IV

12. 8829 13. 5050

SECTION V

14.

	p	q	r	s
a	p	q	**r**	s
b	**p**	q	r	s
c	p	**q**	r	s
d	p	q	r	**s**

15.

	p	q	r	s
a	p	**q**	r	s
b	**p**	q	r	s
c	p	q	r	**s**
d	p	q	**r**	s

SECTION VI

16. (d) 17. (b)

PAST YEARS' IIT QUESTIONS

1. $\pm\pi, \pm\pi/3, -\pi/2$ 2. (a)
3. $(\pi/4, 1/\sqrt{2})$
5. $x = n\pi$ or $x = m\pi \pm (-1)^m \sin^{-1}\left(\frac{-1\pm\sqrt{5}}{4}\right)$, $n, m \in \mathbf{I}$.
6. False 7. $x = \pm\pi/3, \pm 2\pi/3$
8. ϕ (empty)
9. $\theta = n\pi$ or $\theta = m\pi + (-1)^m \pi/6$, $n, m \in \mathbf{I}$
10. (c) 11. $[0, \pi/6] \cup \{\pi/2\} \cup [5\pi/6, \pi]$
12. (c) 13. (b) 14. (d)
15. (c) 16. (a)
17. $(\sqrt{3}-1)/2$ 18. (i) $\leftrightarrow$ (c), (ii) $\leftrightarrow$ (a)
19. (c)
20. $\lambda = \sin 2\alpha + \cos 2\alpha$; $\alpha = m\pi$ or $\alpha = k\pi + \pi/4$ $(m, k \in \mathbf{I})$
21. 30° 22. (d) 23. $\sqrt{2}\pi/4$
24. $m\pi, n\pi \pm \pi/3$, $m, n \in \mathbf{I}$
25. $\pm\pi/3$ 26. $0, \pi/2, -\pi/2$
27. (c) 28. (c) 29. (b)
30. (d)
31. $[-\pi/2, -\pi/10] \cup [3\pi/10, \pi/2]$ 32. (a)
33. (c) 34. (c), (d)

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the given elements and hence evaluated. This process is called the solution of the triangle.

Let ABC be a *right-angled triangle*, right-angled at C (Fig. 12.2).

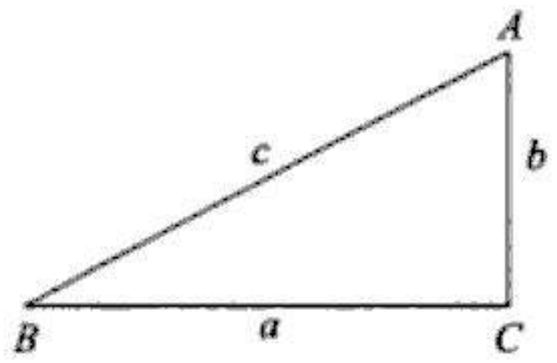

Fig. 12.2

The triangle can be solved as follows:

1. If the hypotenuse c and one side, say b, are given, then $\sin B = b/c$ gives the value of B, $A = 90° - B$ and $a = \sqrt{c^2 - b^2}$.
2. If two sides a and b are given, then $\tan B = b/a$ gives the value of B, $A = 90° - B$ and $c = \sqrt{a^2 + b^2}$.
3. If an angle B and a side a are given, then $A = 90° - B$, side b is obtained from the relation $\tan B = b/a$ and side c from the relation $\cos B = a/c$.
4. If an angle B and the hypotenuse c are given, then $A = 90° - B$, $a = c \cos B$ and $b = c \sin B$.

If the triangle is not *right-angled*, the different cases to be considered are as follows:

1. If the three sides a, b and c are given, angle A can be found from the formulae

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Angles B and C can be found using the corresponding formulae.

2. If two sides b and c and the included angle A are given, and if $b > c$ then

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$$

gives the value of $(B - C)/2$, while

$$\frac{B+C}{2} = 90° - \frac{A}{2}$$

so that B and C can be evaluated. The third side a is given by

$$a = b \frac{\sin A}{\sin B} \quad \text{or} \quad a^2 = b^2 + c^2 - 2bc \cos A$$

3. If two sides b and c and the angle B opposite one of them are given, then angle C is given by $\sin C = (c/b) \sin B$, $A = 180° - (B + C)$ and the third side a is given by $a = b (\sin A/\sin B)$. If $b < c$ and B is an acute angle, then there are two values of C and hence two values of A, so that there are two triangles having the given elements. This is called an *ambiguous case* in the solution of triangles.

4. If one side a and two angles B and C are given, then $A = 180° - (B + C)$ and the sides b, c are obtained from

$$b = \frac{a \sin B}{\sin A} \quad \text{and} \quad c = \frac{a \sin C}{\sin A}.$$

5. If the three angles A, B and C are given, we can only find the ratios of the sides a, b and c, using

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

12.3 GEOMETRICAL PROPERTIES AND FORMULAE FOR A TRIANGLE

(i) In a tiangle the internal bisector of an angle divdes the opposite side in the ratio of the sides containing the angle.

(ii) In an isosceles traiangle the median is perpendicular to the base.

(iii) In similar triangles the corresponding sides are proportional.

(iv) The exterior angle of a traiangle is equal to the sum of interior opppsite angles.

12.4 TRIANGLES, QUADRILATERALS, POLYGONS

Escribed circles The circle touching BC and the two sides AB and AC produced of a triangle ABC is called the *escribed circle* opposite angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite angles B and C, respectively. r_1, r_2, r_3, are known as the ex-radii of triangle ABC.

Some important formulae connecting the sides, angles and radii of a triangle ABC are given as follows.

1. $r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$

$$= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

2. $r_1 = \frac{\Delta}{s-a} = s \tan \frac{A}{2} = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$

$$= 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

Similar formulae for r_2 and r_3 exist.

3. $R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}$

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(a) $\sin(A+B)$ (b) $\cos(A+B)$
(c) $\cos(A-B)$ (d) $\sin A+\sin B$

Ans. (a)

Solution Since $C=90^\circ \Rightarrow A+B=90^\circ$, we have

$$\frac{a^2+b^2}{a^2-b^2}\sin(A-B)$$

$$=\frac{\sin^2 A+\sin^2 B}{\sin^2 A-\sin^2 B}\sin(A-B)$$

$$=\frac{\sin^2 A+\sin^2(90^\circ-A)}{\sin(A+B)\sin(A-B)}\sin(A-B)$$

$$=\frac{\sin^2 A+\cos^2 A}{\sin(A+B)}=\frac{1}{\sin 90^\circ}=1=\sin(A+B).$$

Example 11 If in a triangle ABC

$$\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$$

then

(a) $A=90^\circ$ (b) $B=90^\circ$
(b) $C=90^\circ$ (c) none of these

Ans. (a)

Solution The given relation can be written as

$$\frac{2(b^2+c^2-a^2)+c^2+a^2-b^2+2(a^2+b^2-c^2)}{2abc}=\frac{a^2+b^2}{abc}$$

$$\Rightarrow \quad 3b^2+c^2+a^2=2(a^2+b^2)$$

$$\Rightarrow \quad b^2+c^2-a^2=0$$

$$\Rightarrow \quad \cos A=0 \quad \Rightarrow \quad A=90^\circ.$$

Example 12 In a triangle ABC, AD is the altitude form A (Fig. 12.9). Given $b>c$, $\angle C=23^\circ$ and $AD=\dfrac{abc}{b^2-c^2}$ then $\angle B$. is equal to

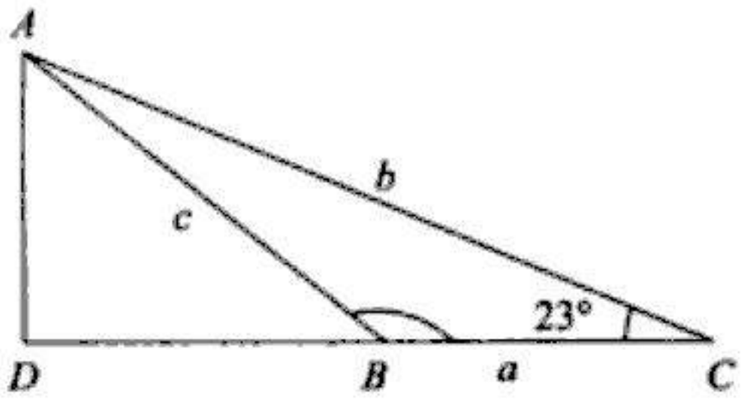

Fig. 12.9

(a) 113° (b) 123° (c) 147° (d) 157°

Ans. (a)

Solution We have $AD=b\sin C$

$$\Rightarrow \quad \frac{abc}{b^2-c^2}=b\sin C$$

$$\Rightarrow \quad \frac{\sin C}{c}=\frac{a}{b^2-c^2}$$

$$\Rightarrow \quad \frac{\sin A}{a}=\frac{a}{b^2-c^2}$$

$$\Rightarrow \quad \sin A=\frac{a^2}{b^2-c^2} \qquad \left[\because \frac{\sin C}{c}=\frac{\sin A}{a}\right]$$

$$=\frac{\sin^2 A}{\sin^2 B-\sin^2 C}$$

$$=\frac{\sin^2 A}{\sin(B+C)\sin(B-C)}$$

$$\Rightarrow \quad \sin(B-C)=1 \qquad [\because \sin(B+C)=\sin A]$$

$$\Rightarrow \quad B-C=90^\circ$$

$$\Rightarrow \quad B=90^\circ+C=90^\circ+23^\circ=113^\circ.$$

Example 13 In a triangle ABC, $a:b:c=4:5:6$. The ratio of the radius of the circumcircle to that of the incircle is

(a) 15/4 (b) 11/5 (c) 16/7 (d) 16/3

Ans. (c)

Solution Let $\dfrac{a}{4}=\dfrac{b}{5}=\dfrac{c}{6}=k$

$$R=\frac{abc}{4\Delta} \quad \text{and} \quad r=\frac{\Delta}{s}$$

$$\Rightarrow \quad \frac{R}{r}=\frac{abc}{4\Delta}\times\frac{s}{\Delta}=\frac{s\,abc}{4\Delta^2}$$

$$=\frac{s\,abc}{4s(s-a)(s-b)(s-c)}$$

$$=\frac{4k\times 5k\times 6k}{4\left(\frac{15}{2}k-4k\right)\left(\frac{15}{2}k-5k\right)\left(\frac{15}{2}k-6k\right)}$$

$$=\frac{5\times 6\times 8}{7\times 5\times 3}=\frac{16}{7}.$$

Example 14 If for a triangle ABC,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$$

then $\sin^3 A+\sin^3 B+\sin^3 C=$

(a) $\sin A+\sin B+\sin C$
(b) $3\sin A\sin B\sin C$
(c) $\sin 3A+\sin 3B+\sin 3C$
(d) $\sin^3 A\sin^3 B\sin^3 C$

Ans. (b)

Solution $$\Delta=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

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Similarly $b > [(b + c - a)(a + b - c)]^{1/2}$

and $a > [(c + a - b)(a + b - c)]^{1/2}$

Multiplying the above inequalities we get

$abc > (b + c - a)(c + a - b)(a + b - c)$

Note

It a, b, c are any distinct positive numbers at most one of $b+c-a, c+a-b, a+b-c$ can be non-positive and in that case also the given expression is negative)

Example 27 The perimeter of a triangle is 6 times the arithmetic mean of the sines of its angles. If the side a is 1, then A is equal to

(a) 30° (b) 60°

(c) 90° (d) 120°

Ans. (a)

Solution $a + b + c = 6\,\dfrac{(\sin A + \sin B + \sin C)}{3}$

$\Rightarrow \quad a + b + c = 2(\sin A + \sin B + \sin C)$

$\Rightarrow \quad 1 + \dfrac{\sin B}{\sin A} + \dfrac{\sin C}{\sin A} = 2(\sin A + \sin B + \sin C)$

$$\left(\text{using } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right)$$

$\Rightarrow \quad \sin A + \sin B + \sin C = 2\sin A\,(\sin A + \sin B + \sin C)$

$\Rightarrow \quad \sin A = \dfrac{1}{2} \quad \Rightarrow A = 30°$

Example 28 If in a triangle ABC, $\cos A + 2\cos B + \cos C = 2$ then a, b, c are in

(a) A.P (b) G.P

(c) H.P (d) none of these

Ans. (a)

Solution $\cos A + \cos C = 2(1 - \cos B)$

$\Rightarrow 2\cos\dfrac{A+C}{2}\cos\dfrac{A-C}{2} = 2 \times 2\sin^2\dfrac{B}{2}$

$\Rightarrow 2\sin\dfrac{B}{2}\cos\dfrac{A-C}{2} = 2 \times 2\sin\dfrac{B}{2}\cos\dfrac{A+C}{2}$

$[\because A + B + C = \pi]$

$\Rightarrow \quad \cos\dfrac{A-C}{2} = 2\cos\dfrac{A+C}{2}$

$\Rightarrow \quad \cos\dfrac{A}{2}\cos\dfrac{C}{2} = 3\sin\dfrac{A}{2}\sin\dfrac{C}{2}$

$\Rightarrow \quad \tan\dfrac{A}{2}\tan\dfrac{C}{2} = \dfrac{1}{3}$

$\Rightarrow \quad \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}} = \dfrac{1}{3}$

$\Rightarrow \quad 3(s - b) = s \Rightarrow 2s = 3b$

$\Rightarrow \quad a + b + c = 3b \Rightarrow b = \dfrac{a+c}{2}$

$\Rightarrow$ a, b, c are in A.P.

Example 29 In a triangle ABC, bisector of angle C meets the side AB at D and circumcentre at E. The maximum value of $CD.\,DE$ is equal to

(a) $a^2/4$ (b) $b^2/4$

(c) $c^2/4$ (d) $(a + b)^2/4$

Ans. (c)

Solution $CD.\,DE = AD.DB$

$AD + DB = c$

Since $A.M. \geq G.M.$

$$\frac{AD + DB}{2} \geq \sqrt{AD \cdot DB}$$

or $\dfrac{c^2}{4} \geq AD.DB$

$\Rightarrow \quad AD.DB \leq c^2/4$

and the required value is $c^2/4$

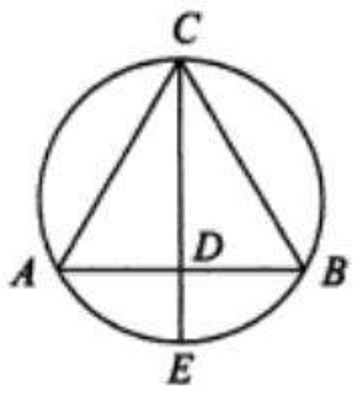

Fig. 12.13

Example 30 In a triangle ABC, $a \geq b \geq c$. If $\dfrac{a^3 + b^3 + c^3}{\sin^3 A + \sin^3 B + \sin^3 C} = 8$, then the maximum value of a is

(a) 1/2 (b) 2

(c) 8 (d) 64

Ans. (b)

Solution we have $\dfrac{(2R)^3(\sin^3 A + \sin^3 B + \sin^3 C)}{\sin^3 A + \sin^3 B + \sin^3 C} = 8$

$\Rightarrow \quad R = 1$, the radius of the circumcircle is 1.

Greatest length of a side of a triangle inscribed in a circle can be equal to the diameter of the circle and hence the maximum value of the greatest side a is equal to 2.

Example 31 If p denotes the perimeter of the triangle ABC, then $b\cos^2\dfrac{C}{2} + c\cos^2\dfrac{B}{2}$ is equal to

(a) p (b) $2p$

(c) $p/2$ (d) p^2

Ans. (c)

Solution $b\cos^2\dfrac{C}{2} + c\cos^2\dfrac{B}{2}$

$= b \times \dfrac{s(s-c)}{ab} + c \times \dfrac{s(s-b)}{ac}$

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Solution Let ABC be the isosceles triangle with $AB = AC = b$ and $\angle B = \angle C = \alpha$ (Fig. 12.19). Let AD be the perpendicular bisector of the side BC. Since ΔABC is isosceles, AD is also the bisector of angle A, so that O and I both lie on AD. We have $OB = R$ and $ID = r$. Also, since O is the circumcentre, we get $OA = OB = R$. Therefore, from isosceles triangle OAB

$$\frac{OB}{\sin(90^\circ - \alpha)} = \frac{AB}{\sin 2\alpha}$$

$$\Rightarrow \qquad R = \frac{b\cos\alpha}{2\sin\alpha\cos\alpha} = \frac{1}{2}b\,\text{cosec}\,\alpha$$

so that (a) is correct.

$$\text{Again } \Delta = BD\cdot AD = b\cos\alpha\cdot b\sin\alpha = \frac{1}{2}b^2\sin 2\alpha$$

so that (b) is not correct.

$$\text{Also, } r = \frac{\Delta}{s} = \frac{\frac{1}{2}b^2\sin 2\alpha}{\frac{1}{2}(b+b+2b\cos\alpha)} = \frac{b\sin 2\alpha}{2(1+\cos\alpha)}$$

so that (c) is correct.

Further $OI = |OD + DI| = |OD + r|$

because $\alpha < \pi/4$, $A > \pi/2$ and O lies on AD produced. Now, from right-angled triangle ODB, we get

$$OD^2 = OB^2 - BD^2 = R^2 - (b\cos\alpha)^2$$

$$= \frac{1}{4}\frac{b^2}{\sin^2\alpha} - b^2\cos^2\alpha$$

$$= \frac{b^2(1 - 4\sin^2\alpha\cos^2\alpha)}{4\sin^2\alpha} \quad \text{[from (a)]}$$

$$= \frac{b^2(\cos^2\alpha - \sin^2\alpha)^2}{4\sin^2\alpha}$$

$$= \frac{b^2\cos^2 2\alpha}{(2\sin\alpha)^2}$$

$$\Rightarrow \qquad OD = \frac{b\cos 2\alpha}{2\sin\alpha}$$

$$\therefore \qquad OI = \left|\frac{b\sin 2\alpha}{2(1+\cos\alpha)} + \frac{b\cos 2\alpha}{2\sin\alpha}\right|$$

$$= \left|\frac{b\sin 2\alpha}{4\cos^2(\alpha/2)} + \frac{b\cos 2\alpha}{4\sin(\alpha/2)\cos(\alpha/2)}\right|$$

$$= \left|\frac{b}{4\cos(\alpha/2)}\cdot\frac{\sin 2\alpha\sin(\alpha/2) + \cos 2\alpha\cos(\alpha/2)}{\sin(\alpha/2)\cos(\alpha/2)}\right|$$

$$= \left|\frac{b\cos(3\alpha/2)}{2\sin\alpha\cos(\alpha/2)}\right|$$

Thus, (d) is also correct.

Example 41 If H is the orthocentre of triangle ABC, then AH is equal to

(a) $2R\cos A$ (b) $2R\sin A$

(c) $a\cot A$ (d) $\frac{2abc}{\Delta}\cos A$.

Ans. (a) and (c).

Solution Referring to ΔAHB (Fig. 12.20), we have

$$\frac{AH}{\sin(90^\circ - A)} = \frac{c}{\sin(A+B)}$$

$$\Rightarrow \qquad AH = \frac{c\cos A}{\sin(180^\circ - C)}$$

$$= \frac{c\cos A}{\sin C} = 2R\cos A \quad \left[\because \frac{c}{\sin C} = 2R\right]$$

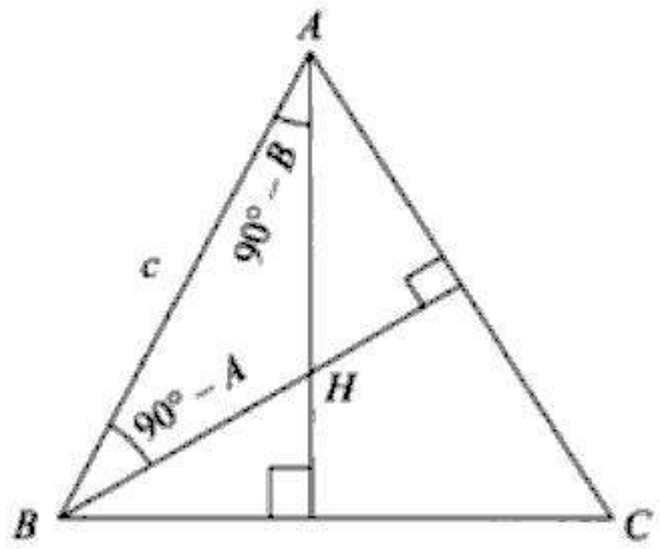

Fig. 12.19

so that (a) is correct while (b) is not correct. Also,

$$AH = 2R\cos A = 2\cdot\frac{abc}{4\Delta}\cos A = \frac{abc}{2\Delta}\cos A$$

and $AH = 2R\cos A = \frac{a}{\sin A}\cdot\cos A = a\cot A$

so that (c) is correct, while (d) is not correct.

Example 42 If the tangents of the angles A and B of a triangle ABC satisfy the equation $abx^2 - c^2x + ab = 0$, then

(a) $\tan A = a/b$

(b) $\tan B = b/a$

(c) $\cos C = 0$

(d) $\sin^2 A + \sin^2 B + \sin^2 C = 2$.

Ans. (a), (b), (c) and (d).

Solution From the given equation, we get

$$\tan A + \tan B = c^2/ab$$

and $\tan A\tan B = 1$. Since

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Example 52 A and B are two points 30 m apart in a line on the horizontal plane through the foot of a tower lying on opposite sides of the tower. The distances of the top of the tower from A and B are 20 m and 15 m respectively, if the angle of elevation of the top of the tower at A is θ and the height of the tower is h, then

(a) $\cos\ \theta = 29/36$ (b) $\cos\ \theta = 43/48$

(c) $h = 5\sqrt{5}/\sqrt{3}$ (d) $h = 5\sqrt{455}/12$

Ans. (b) and (d)

Solution Let OP be the tower of height h, then $AB = 30$, $AP = 20$ and $BP = 15$ (Fig. 12.25)

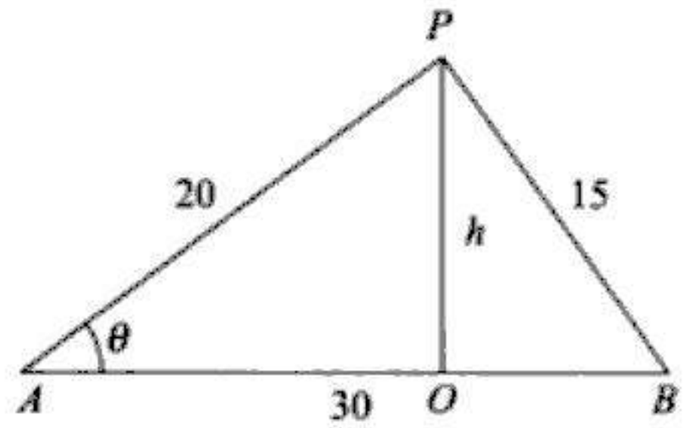

Fig. 12.23

so from ΔABP

$$\cos\ \theta = \frac{20^2 + 30^2 - 15^2}{2 \times 20 \times 30}$$

$$= \frac{1075}{1200} = \frac{43}{48}$$

$$\Rightarrow \quad \sin\ \theta = \sqrt{1 - \left(\frac{43}{48}\right)^2}$$

$$= \frac{\sqrt{2304 - 1849}}{48}$$

$$= \frac{\sqrt{455}}{48}$$

and $$h = 20 \sin\ \theta = 20 \times \frac{\sqrt{455}}{48} = \frac{5\sqrt{455}}{12}$$

Example 53 A pole 50 m high stands on a building 250 m high. To an observer at a height of 300 m, the building and the pole subtend equal angle θ. If the horizontal distance of the observer from the pole is x, then

(a) $x = 25\sqrt{6}$ m (b) $x = 25\sqrt{3}$ m

(c) $\tan\ \theta = \sqrt{2/3}$ (d) $\tan\ \theta = 2\sqrt{6}$

Ans. (a) and (c)

Solution Let PQ be the pole on the building QR and O be the observer.

Then $PQ = 50$, $QR = 250$ (Fig. 12.27)

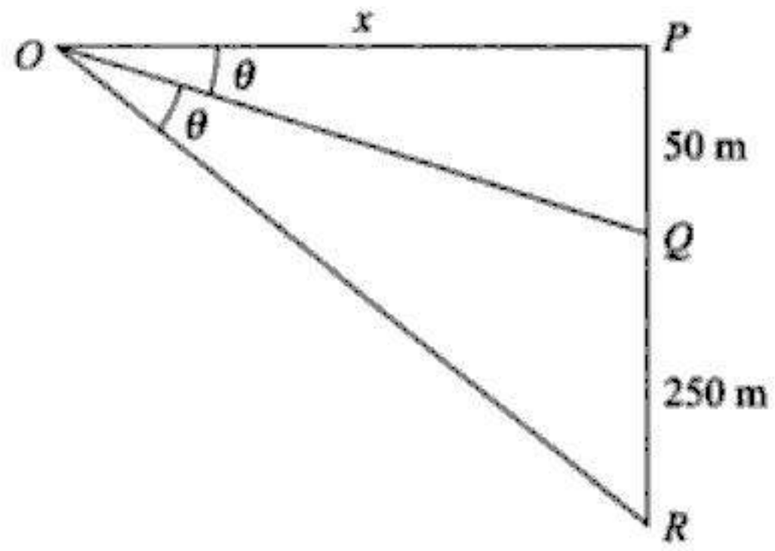

Fig. 12.24

$\Rightarrow PR = 300$ so the observer is at the same height as the top P of the pole. So $OP = x$. Then from right angled triangles OPQ and OPR,

$$\tan\ \theta = \frac{50}{x} \text{ and } \tan\ 2\theta = \frac{300}{x}.$$

so that $$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{300}{x}$$

$$\Rightarrow \quad \frac{2 \times \frac{50}{x}}{1 - \left(\frac{50}{x}\right)^2} = \frac{300}{x} \Rightarrow 3\left\{1 - \left(\frac{50}{x}\right)^2\right\} = 1$$

$$\Rightarrow \left(\frac{50}{x}\right)^2 = \frac{2}{3} \Rightarrow x = \frac{50\sqrt{3}}{\sqrt{2}} = 25\sqrt{6}$$

and $$\tan\ \theta = \frac{50}{25\sqrt{6}} = \sqrt{\frac{2}{3}}.$$

Example 54 A monument $ABCD$ stands at a point A on a level ground. At a point P on the ground the portions AB, AC, AD subtend angles α, β, γ respectively. If $AB = a$, $AC = b$, $AD = c$, $AP = x$ and $\alpha + \beta + \gamma = 180°$ then

(a) $x^2 = (a + b + c)/abc$

(b) $x^2 = abc/(a + b + c)$

(c) $\tan\ \alpha + \tan\ \beta + \tan\ \gamma = (a + b + c)^{3/2}/\sqrt{abc}$

(d) $\tan\ \alpha \tan\ \beta \tan\ \gamma = (a + b + c)^{3/2}/\sqrt{abc}$

Ans. (b), (c) and (d)

Solution We have

$a = AB = AP \tan\ \alpha = x \tan\ \alpha$ (Fig. 12.28)

$b = AC = x \tan\ \beta$

and $c = AD = x \tan\ \gamma$

so that $a + b + c = x\ (\tan\ \alpha + \tan\ \beta + \tan\ \gamma)$

$= x \tan\ \alpha \tan\ \beta \tan\ \gamma$ [$\because\ \alpha + \beta + \gamma = 180°$]

and $abc = x^3 \tan\ \alpha \tan\ \beta \tan\ \gamma$

$$\Rightarrow \quad x^2 = \frac{abc}{a+b+c}$$

so that $\tan\ \alpha + \tan\ \beta + \tan\ \gamma = \tan\ \alpha \tan\ \beta \tan\ \gamma = (a + b + c)/x$

$$= (a + b + c)^{3/2}/\sqrt{abc}.$$

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$$\Rightarrow \qquad h = 106 \text{ ft}$$

$$\Rightarrow \qquad x = 4h - 24 = 400 \text{ ft}$$

$$\Rightarrow \qquad OA' = \sqrt{400^2 + 300^2} = 500 \text{ ft}$$

Paragraph for Question Nos. 64 and 65

A diving stand (vertical) is situated at a point on the edge of a circular swimming pool of radius r. At another point P on the circular edge the angle of elevation of the top of the diving stand in 30°. After moving 10 metres along the circular edge, the angle of elevation is again 30°, and further going along the edge 5 metres, the angle of elevation is 45°.

Example 64 Height of the diving stand is

(a) r (b) $2r$

(c) $(2/3)r$ (d) $3r$

Example 65 The perimeter of the swimming pool is

(a) 15 m (b) 20 m

(c) 30 m (d) 35 m

Ans. 64. (a), 65. (c)

Solution Let OR be the diving stand of height h and C the centre of the swimming pool of radius r.

Let P and Q be the two points on the edge of the pool where the angle of elevation of R the top of the stand is 30° and R' be the middle point of PQ (Fig. 12.34).

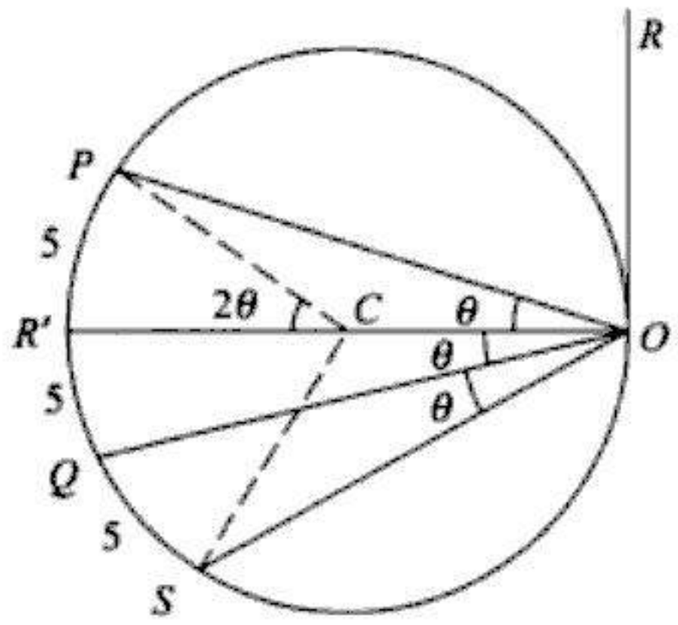

Fig. 12.31

Then arc PR' = arc $R'Q$ = 5 m, as arc PQ = 10 m.

Let S be the point where the angle of elevation of R is 45°.

Then arc QS = 5 m.

We have $OP = OQ = h \cot 30° = h\sqrt{3}$

and $OS = OR \cot 45° = OR = h.$

Let $\angle COP = \angle COQ = \underline{QOS} = \theta.$

From isosceles triangle COP

$$\frac{OP}{\sin(180° - 2\theta)} = \frac{CP}{\sin\theta} \Rightarrow \frac{\sqrt{3}\,h}{2\sin\theta\cos\theta} = \frac{r}{\sin\theta}$$

$$\Rightarrow \qquad \cos\theta = \frac{\sqrt{3}\,h}{2r} \qquad (1)$$

From isosceles triangle COS.

$$\frac{OS}{\sin(180° - 4\theta)} = \frac{CS}{\sin 2\theta}$$

$$\Rightarrow \qquad \frac{h}{2\sin 2\theta\cos 2\theta} = \frac{r}{\sin 2\theta}$$

$$\Rightarrow \qquad \cos 2\theta = \frac{h}{2r} \qquad \Rightarrow \qquad 2\cos^2\theta - 1 = \frac{h}{2r}$$

$$\Rightarrow \qquad 2 \times \frac{3h^2}{4r^2} - \frac{h}{2r} - 1 = 0$$

$$\Rightarrow \qquad 3\left(\frac{h}{r}\right)^2 - \frac{h}{r} - 2 = 0$$

$$\Rightarrow (3h + 2r)(h - r) = 0$$

$$\Rightarrow h = r$$

From (1) $\cos\theta = \sqrt{3}/2 \Rightarrow \theta = \pi/6$

$$2\theta r = PR' = 5$$

$$\Rightarrow r = 15/\pi$$

So the perimeter of the pool is $2\pi \times \frac{15}{\pi} = 30\,m$.

SECTION IV

INTEGER ANSWER TYPE

Example 66 In a triangle ABC; $A = \pi/3$, $b = 50$, $c = 30$, AD is a median through A, then AD^2 is equal to

Ans. 1225.

Solution

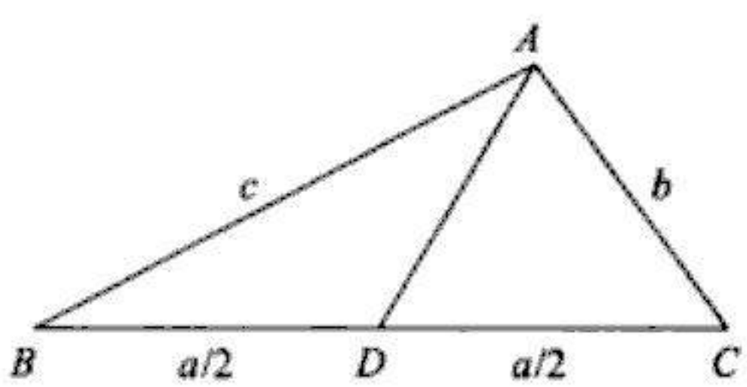

Fig. 12.32

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

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(c) ED (r) $\sqrt{45}$
(d) AE (s) 6

Ans.

	p	q	r	s
a	p	**q**	r	s
b	p	q	**r**	s
c	**p**	q	r	s
d	p	q	r	**s**

Solution $AD = 4$

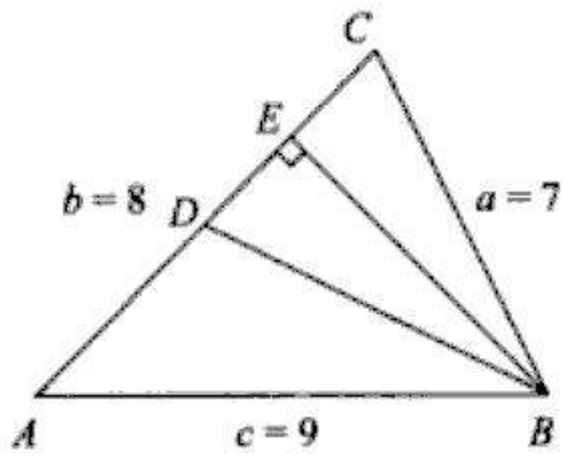

Fig. 12.40

From ΔABD

$$\cos A = \frac{4^2 + 9^2 - BD^2}{2 \times 4 \times 9}$$

$$= \frac{8^2 + 9^2 - 7^2}{2 \times 8 \times 7} \text{ (From } \Delta ABC)$$

$\Rightarrow BD^2 = 49 \Rightarrow BD = 7 \Rightarrow \Delta BCD$ is isosceles

$\Rightarrow ED = 1/2\ CD = 2$

and $BE = \sqrt{7^2 - 2^2} = \sqrt{45}$, $AE = 6$

Example 79 In a triangle ABC, $\sin A \cos B = 1/4$, $3 \tan A = \tan B$,

(a) $B - A$ (p) $60°$
(b) $A + B/2$ (q) $90°$
(c) $A + B$ (r) $30°$
(d) $A + C/2$ (s) $75°$

Ans.

	p	q	r	s
a	p	q	**r**	s
b	**p**	q	r	s
c	p	**q**	r	s
d	p	q	r	**s**

Solution $3 \tan A = \tan B \Rightarrow 3 \sin A \cos B = \cos A \sin B$

$\Rightarrow \cos A \sin B = 3/4 \Rightarrow \sin (A + B) = 1/4 + 3/4 = 1 \Rightarrow C = 90°$

and $\tan A = \tan (90° - B) = \cot B$

$$\Rightarrow \tan^2 A = 1/3 \Rightarrow \tan A = \frac{1}{\sqrt{3}} \Rightarrow A = 30° \Rightarrow, B = 60°$$

Example 80 The angles of a triangle are in the ratio 2 : 3 : 7

(a) smallest side = 2 (p) largest side = $\sqrt{3} + 1$
(b) smallest side = $\sqrt{2}$ (q) largest side = $\sqrt{6} + \sqrt{2}$
(c) $s = 3 + \sqrt{3} + \sqrt{2}$ (r) largest side = 1
(d) $\Delta = \left(\sqrt{3} - 1\right)/4$ (s) largest side = $2 + \sqrt{3}$

Ans.

	p	q	r	s
a	p	**q**	r	s
b	**p**	q	r	s
c	p	q	r	**s**
d	p	q	**r**	s

Solution Angles of the triangle are 30°, 45°, 105°.

then $\frac{a}{\sin 30°} = \frac{b}{\sin 45°} = \frac{c}{\sin 105°}$

$$\Rightarrow \frac{a}{1/2} = \frac{b}{1/\sqrt{2}} = \frac{c}{\left(\sqrt{3} + 1\right)/2\sqrt{2}}$$

$$\Rightarrow \frac{a}{\sqrt{2}} = \frac{b}{2} = \frac{c}{\sqrt{3} + 1} = k \text{ (say)}$$

The smallest side $a = \sqrt{2}\ k$ and largest side $c = \left(\sqrt{3} + 1\right) k$

If $a = 2, k = \sqrt{2}, c = \sqrt{6} + \sqrt{2}$

If $a = \sqrt{2}, k = 1, c = \sqrt{3} + 1$

If $s = 3 + \sqrt{2} + \sqrt{3}, k = 2, c = 2 + 2\sqrt{3}$

If $\Delta = \frac{\sqrt{3} - 1}{4}$, then from $\Delta = \frac{1}{2} bc \sin A$ we have

$$\frac{\sqrt{3} - 1}{4} = \frac{1}{2} \times 2k \times \left(\sqrt{3} + 1\right) k \sin 30°$$

$$\Rightarrow k^2 = \frac{\sqrt{3} - 1}{2\left(\sqrt{3} + 1\right)} = \frac{\left(\sqrt{3} - 1\right)^2}{4} \Rightarrow k = \frac{\sqrt{3} - 1}{2}$$

and hence $c = 1$

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Ans. (a)

Solution Statement-2 can be proved (by using $A + B + C = \pi$) to be true. (conditional identities) From which we get

$3 - 2(\sin^2 A + \sin^2 B + \sin^2 C) = -1 - 4 \cos A \cos B \cos C$

$\Rightarrow \quad 3 - 2(2) = -1 - 4 \cos A \cos B \cos C$

$\Rightarrow \quad \cos A \cos B \cos C = 0 \Rightarrow$ one of the angles A, B, C is equal to $90°$.

Example 90 **Statement-1:** In any triangle ABC

$a \cos A + b \cos B + c \cos C \leq s$.

Statement-2: In any triangle ABC

$$\sin (A/2) \sin (B/2) \sin (C/2) \leq 1/8$$

Ans. (a)

Solution Since L.H.S of the inequality in statement-2 is a symmetric function of sines of the angles of the triangle, its maximum values is attained. When $A = B = C = 60°$ and hence the statement-2 is true. Statement-1 is true if

$2R (\sin A \cos A + \sin B \cos B + \sin C \cos C)$

$\leq R (\sin A + \sin B + \sin C)$

or if $4 \sin A \sin B \sin C \leq 4 \cos (A/2) \cos (B/2) \cos (C/2)$

(From conditional identities)

or if $\sin (A/2) \sin (B/2) \sin (C/2) \leq 1/8$

which is true by statement-2

So statement-1 is correct.

Example 91 **Statement-1:** In any triangle ABC, if $a : b : c = 4 : 5 : 6$ then $R : r = 16 : 7$

Statement-2: In any triangle ABC

$$\frac{R}{r} = \frac{abc}{4s}$$

Ans. (c)

Solution Statement-2 is false as $R = \dfrac{abc}{4\Delta}$, $r = \dfrac{\Delta}{s}$

$$\text{so } \frac{R}{r} = \frac{sabc}{4\Delta^2} = \frac{sabc}{4s(s-a)(s-b)(s-c)}$$

$$= \frac{4 \times 5 \times 6}{4\left(\frac{15}{2}-4\right)\left(\frac{15}{2}-5\right)\left(\frac{15}{2}-6\right)} = \frac{5 \times 6 \times 8}{7 \times 5 \times 3} = \frac{16}{7}$$

and the statement-1 is true.

Example 92 **Statement-1:** Manan observes that the angle of elevation of the top P of the tower OP at a point A on the ground is α, he then walks a distance AB towards the foot O of the tower and finds the angle of elevation as β, he again walks a distance BC in the same direction and observes the angle of elevation now is γ. He notices that $\alpha + \beta + \gamma = 180°$; α, β, γ are in A.P. and the distance of C from the foot of the tower is half the distance of B from the foot O of the tower. If the height of tower is h, the distance of A from the foot O is $5h/3\sqrt{3}$.

Statement-2: If the angles of elevation of the top of a tower at three points on the ground are in A.P. then the distance of the points from the foot of the tower are also in A.P.

Ans. (c)

Solution Statement-1

$\alpha = 60° - \theta, \beta = 60°$

$\gamma = 60° + \theta$

$OC = (1/2)\ OB$

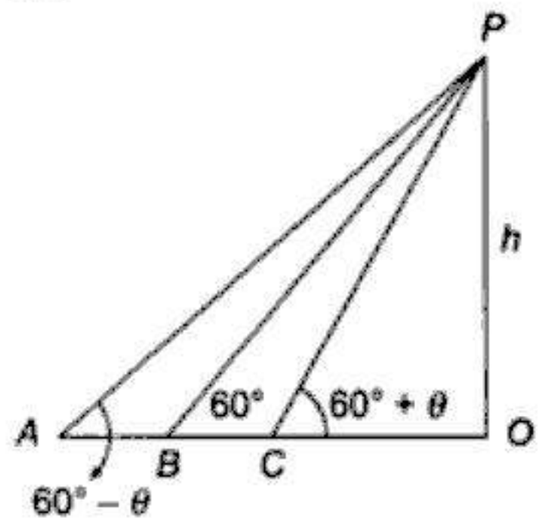

Fig. 12.45

$\Rightarrow \quad 2h \cot (60° + \theta) = h \cot 60°$

$\Rightarrow \quad 2\sqrt{3}(1 - \sqrt{3} \tan \theta) = \sqrt{3} + \tan \theta$

$\Rightarrow \quad \tan \theta = \sqrt{3}/7$

and $OA = h \cot (60° - \theta)$

$$= \frac{h\left(1+\sqrt{3}\tan\theta\right)}{\sqrt{3}-\tan\theta} = \frac{5h}{3\sqrt{3}}$$

Showing that statement-1 is True. But statement-2 is not True. Because if the angles of elevation are $30°$, $45°$ and $60°$ and the height of the tower is h, distances are $h/\sqrt{3}$, h, $h\sqrt{3}$ which are not in A.P.

EXERCISE

LEVEL 1

SECTION I

SINGLE CORRECT CHOICE TYPE

1. The expression

$$\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$$

is equal to

(a) $\cos^2 A$ (b) $\sin^2 A$

(c) $\cos A \cos B \cos C$ (d) none of these

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LEVEL 2

SECTION I

SINGLE CORRECT CHOICE TYPE

1. In a triangle ABC, if a is the arithmetic mean and b, c $(b \neq c)$ are two geometric means between, any two positive real numbers then $\frac{\sin^3 B + \sin^3 C}{\sin A \sin B \sin C}$ is equal to

   (a) 0 (b) 1 (c) 2 (d) 4

2. In a triangle ABC, if $\cot A = (x^3 + x^2 + x)^{1/2}$, $\cot B = (x + x^{-1} + 1)^{1/2}$ and $\cot C = (x^{-3} + x^{-2} + x^{-1})^{-1/2}$, then the triangle is

   (a) isosceles (b) obtuse angled
   (c) right angled (d) equilateral

3. In a triangle ABC, $\frac{r_1 + r_2}{1 + \cos C}$ is equal to

   (a) $\frac{2abc}{\Delta}$ (b) $\frac{a+b}{c\Delta}$ (c) $\frac{abc}{2\Delta}$ (d) $\frac{abc}{\Delta^2}$

4. In a triangle ABC, if $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and $a = 1/\sqrt{6}$ then the area of the triangle in square units is equal to

   (a) 1/24 (b) 1/8
   (c) $1/8\sqrt{3}$ (d) $1/24\sqrt{3}$

5. In a triangle ABC, if $5 \cos C + 6 \cos B = 4$ and $6 \cos A + 4 \cos C = 5$, then $\tan (A/2) \tan (B/2)$ is equal to

   (a) 2/3 (b) 3/2 (c) 1/5 (d) 5

6. In a triangle ABC, $\frac{a(a+c-b)}{b(b+c-a)}$ is equal to

   (a) $\frac{1-\cos A}{1-\cos B}$ (b) $\frac{1+\cos A}{1+\cos B}$
   (c) $\frac{\cos^2 (A/2)}{\sin^2 (B/2)}$ (d) $\frac{\sin^2 A}{\sin^2 B}$

7. If r, R are respectively the radii of the inscribed and circumscribed circles of a regular polygon of n sides such that $\frac{R}{r} = \sqrt{5} - 1$, then n is equal to

   (a) 5 (b) 6
   (c) 10 (d) none of these

8. In a triangle ABC, if $\cos A + 2 \cos B + \cos C = 1$, then a, b, c are in

   (a) A.P. (b) G.P.
   (c) H.P. (d) none of these

9. In a triangle ABC, if $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then $\sin A \sin B + \sin B \sin C + \sin C \sin A$ is equal to

   (a) 0 (b) 9/4
   (c) 1 (d) none of these

10. In a triangle ABC, if $A = 18°$, $b - a = 2$, $ab = 4$, then the triangle is

    (a) acute angled (b) right angled
    (c) obtuse angled (d) isosceles

11. If for a triangle ABC, a, b, A are given, then which of the following gives us two such triangles

    (a) $a < b \sin A$ (b) $a = b \sin A$
    (c) $a > b \sin A$ and $a < b$ (d) $a > b \sin A$ and $a > b$

12. The distance of the incentre of the triangle ABC from A is

    (a) $4R \sin (A/2)$ (b) $4R \sin (B/2) \sin (C/2)$
    (c) $4R \cos (A/2)$ (d) $4R \cos (B/2) \cos (C/2)$

13. If d_1, d_2, d_3 are the diameters of the three escribed circles of a triangle ABC, then $d_1d_2 + d_2d_3 + d_3d_1$ is equal to

    (a) $ab + bc + ca$ (b) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$
    (c) $(a + b + c)^2$ (d) $a^2 + b^2 + c^2$

14. In a triangle ABC, if $\frac{s-a}{11} = \frac{s-b}{12} = \frac{s-c}{13}$, then $\tan^2 (A/2)$ is equal to

    (a) 143/432 (b) 13/33
    (c) 11/39 (d) 12/37

15. A point P is inside an equilateral triangle. If the distances of P from the sides are 8, 10, 12 units, the length of a side of the triangle is

    (a) 10 (b) $10\sqrt{3}$
    (c) $20\sqrt{3}$ (d) $30\sqrt{3}$

SECTION II

MULTIPLE CORRECT CHOICE TYPE

16. If ABC is not a right angled triangle, then which of the following is (are) not possible.

    (a) $\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \sin C$
    (b) $\cos 2A + \cos 2B + \cos 2C = -3/2$
    (c) $\sin A + \sin B + \sin C = 2\sqrt{2} \cos (A/2) \cos (B/2)$
    (d) $\cos A + \cos B + \cos C = 1$

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13. AB is a diameter of a circle and C is any point on the circumference of the circle. Then
    (a) The area of the ΔABC is maximum when it is isosceles.
    (b) The area of the ΔABC is minimum when it is isosceles.
    (c) The perimeter of ΔABC is minimum when it is isosceles.
    (d) none of these. [1983]
14. In a triangle ABC, if $\cot A$, $\cot B$, $\cot C$ are in A.P. then a^2, b^2, c^2 are in ... progression. [1985]
15. In a triangle ABC, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides the angle A into angles of 30° and 45°. Find the length of the side BC. [1985]
16. There exists a triangle ABC satisfying the conditions:
    (a) $b \sin A = a$, $A < \pi/2$
    (b) $b \sin A > a$, $A > \pi/2$
    (c) $b \sin A > a$, $A < \pi/2$
    (d) $b \sin A < a$, $A < \pi/2$, $b > a$
    (e) $b \sin A < a$, $A > \pi/2$, $b = a$ [1986]
17. If in a triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$. Show that $a : b : c = 1 : 1 : \sqrt{2}$ [1986]
18. If a, b and c are distinct positive numbers then the expression
    $(b + c - a)(c + a - b)(a + b - c) - abc$ is
    (a) positive (b) negative
    (c) non-positive (d) non-negative
    (e) none of these [1986]
19. In a triangle, the lengths of the two larger sides are 10 and 9 respectively. If the angles are in A.P. then the length of the third side can be
    (a) $5 - \sqrt{6}$ (b) $3\sqrt{3}$
    (c) 5 (d) $5 + \sqrt{6}$
    (e) none of these [1987]
20. If the angles of a triangle are 30° and 45° and included side is $(\sqrt{3} + 1)$ cm, then the area of the triangle is ... [1988]
21. ABC is a isosceles triangle inscribed in a circle of radius r. If $AB = AC$ and h is the altitude from A to BC, then the triangle ABC has perimeter $P = 2\left(\sqrt{2hr - h^2} + \sqrt{2hr}\right)$ and area $A =$ _____
    also $\lim_{x \to 0} \frac{A}{P^3} =$ _____. [1989]
22. ABC is a triangle such that
    $\sin(2A + B) = \sin(C - A) = -\sin(B + 2C) = 1/2$
    If A, B and C are in arithmetic progression, determine the values of A, B and C. [1990]
23. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides the triangle. [1991]
24. In a triangle of base a, the ratio of the other two sides is r (< 1) show that the altitude of the triangle is less than or equal to $ar/(1 - r^2)$. [1991]
25. If in a triangle ABC
    $$\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$$
    then the value of the angle A is ... degrees. [1993]
26. In a triangle ABC, AD is the altitude from A.
    Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{b^2 + c^2}$
    then $\angle B =$ ... [1994]
27. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is ... [1994]
28. Let $A_1, A_2, \ldots A_n$ be the vertices of an n-sided regular polygon such that
    $$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$$, find the value of n.
    [1994]
29. Consider the following statements concerning a triangle ABC.
    (i) the sides a, b, c and area Δ are rational
    (ii) a, $\tan(B/2)$, $\tan(C/2)$ are rational
    (iii) a, $\sin A$, $\sin B$, $\sin C$ are rational.
    Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) [1994]
30. If the lengths of the sides of triangle are 3, 5, 7, then the largest angle of the triangle is
    (a) $\pi/2$ (b) $5\pi/6$
    (c) $2\pi/3$ (d) $3\pi/4$ [1994]
31. In a triangle ABC, $\angle B = \pi/3$ and $\angle C = \pi/4$. Let D divide BC internally in the ratio 1 : 3 then
    $\frac{\sin \angle BAD}{\sin \angle CAD}$ equals
    (a) $1/\sqrt{6}$ (b) 1/3
    (c) $1/\sqrt{3}$ (d) $\sqrt{2/3}$ [1995]
32. In a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is ... [1996]

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8. (ii) $\frac{BD}{BA} = \frac{AB}{DB} - \frac{AD^2}{AB \times DB}$

8. (iv) B. 9. $75°$ 13. (a)
14. A.P. 15. 2 16. (a), (d)
18. (b) 19. (a), (d)
20. $\left(\sqrt{3}+1\right)/2$ *sq.* cm
21. $h\sqrt{2hr-h^2}$, r/128
22. $A = 45°$, $B = 60°$, $C = 75°$
23. 4, 5, 6 25. $90°$ 26. $113°$
27. $a^2/6$ units 28. 7 30. (c)
31. (a) 32. 16/7
33. $p \in (-\infty, 0) \cup (3 + 2\sqrt{2}, \infty)$
34. (b) 36. (a) 38. (b)
39. (a) 42. (a) 44. (b)
45. (c) 47. (c) 48. (d)
49. (b) 50. (c) 51. (a)
52. (a), (b), (c), (d) 53. (b), (c)

HINTS AND SOLUTIONS

LEVEL I

1. $$\frac{\left[(b+c)^2-a^2\right]\left[a^2-(b-c)^2\right]}{4b^2c^2}$$

$$= \frac{\left[2bc+\left(b^2+c^2-a^2\right)\right]\left[2bc-\left(b^2+c^2-a^2\right)\right]}{4b^2c^2}$$

$$= \frac{2bc(1+\cos A)\,2bc(1-\cos A)}{4b^2c^2} = 1-\cos^2 A$$

$$= \sin^2 A$$

2.

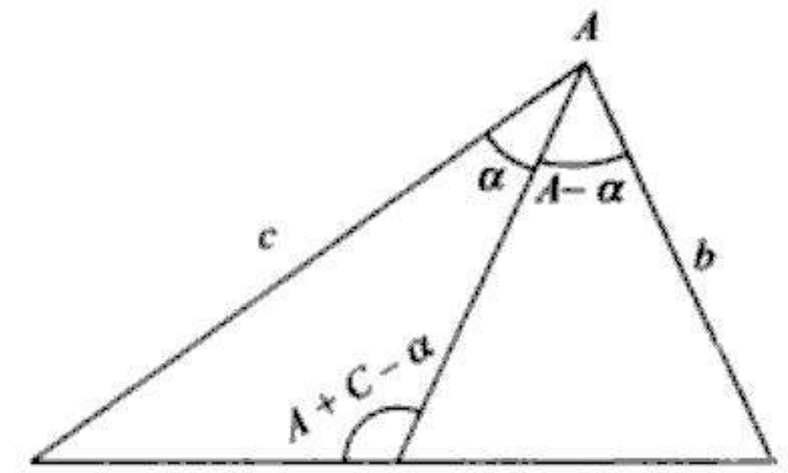

Fig. 12.46

$$\frac{b}{\sin(A+C-\alpha)} = \frac{a/2}{\sin(A-\alpha)}$$ Fig. 12.48

$$\frac{c}{\sin(A+C-\alpha)} = \frac{a/2}{\sin\alpha}$$

$\therefore b \sin(A-\alpha) = c \sin a$

3. $BD = \left(\frac{c}{b+c}\right)a$, $DC = \left(\frac{b}{b+c}\right)a$,

$$\frac{\sin\theta}{b} = \frac{\sin(A/2)}{\frac{ba}{b+c}}$$

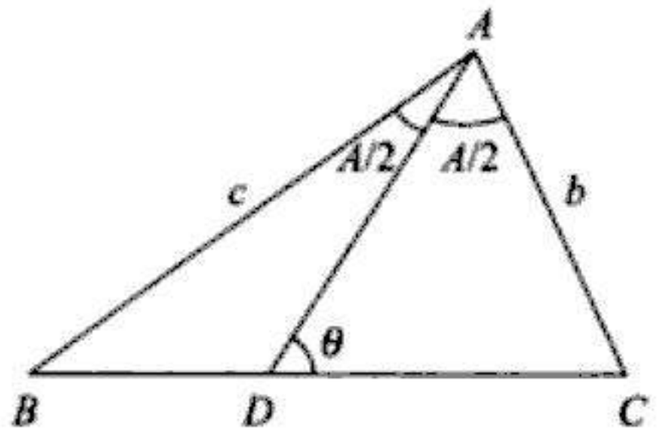

Fig. 12.47

$$\Rightarrow \sin\theta = \frac{b+c}{a}\sin\frac{A}{2}$$

$$= \frac{\sin B+\sin C}{\sin A}.\sin\frac{A}{2} = \cos\frac{B-C}{2}$$

4. $$\frac{a}{\sin 45°} = \frac{b}{\sin 22\frac{1}{2}°} = \frac{c}{\sin 112\frac{1}{2}°}$$

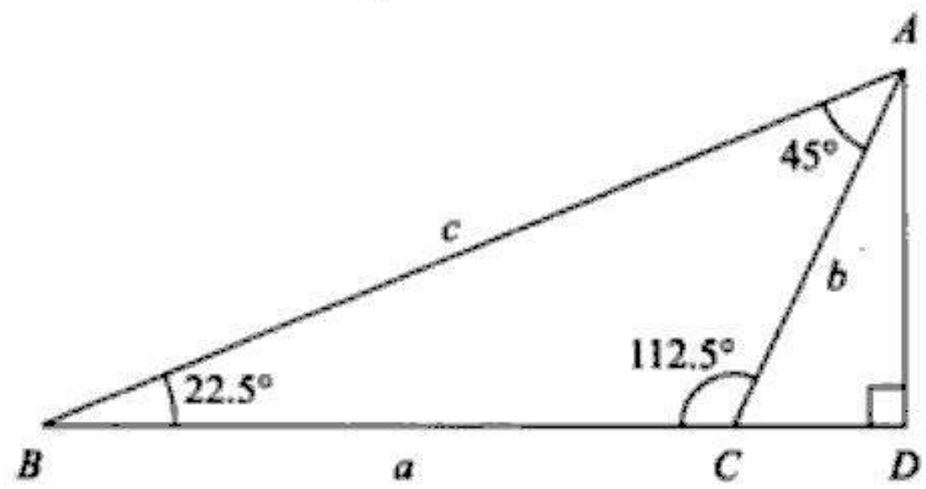

Fig. 12.48

$$a = \frac{c\sin 45°}{\sin 112\frac{1}{2}°} = \frac{c\sin 45°}{\cos 22\frac{1}{2}°}$$

$$= 2c \sin 22\tfrac{1}{2} = 2 \text{ AD}$$

5. $a(1+\cos C)+c(1+\cos A) = 3b$

$\Rightarrow a+c+a\cos C+c\cos A = 3b$

$\Rightarrow a+c+b = 3b \Rightarrow a+c = 2b$

$\Rightarrow a, b, c$ are in A.P.

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38. $\frac{BD}{AD} = \frac{\sin(A/3)}{\sin B}$ (From ΔADE)

$$DE = \frac{\sin(A/3)\,AD}{\sin(C+(A/3))}$$

$$AE = \frac{\sin(B+(A/3))\,AD}{\sin(C+(A/3))}$$

$$EC = \frac{AE\sin(A/3)}{\sin C}$$

$$= \frac{\sin(B+(A/3))\sin(A/3)\,AD}{\sin(C+(A/3))\,\sin C}$$

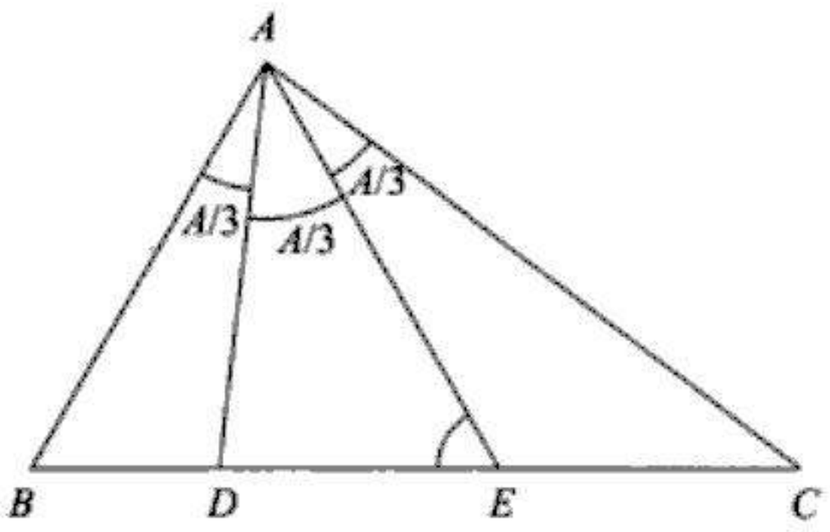

Fig. 12.54

39. $B = 60°$, $\sin C = \frac{c}{b}\sin B = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$

$\Rightarrow C = 45°$

and $A = 75° \Rightarrow \cos(A-C) = \cos 30° = \sin 60° = \sin B$

$\sin(A-C) = \sin(B/2)$, $\sin(A+C) = \sin 2B$ and $\sin(2C-A) = \sin 15° = \sin(B/4)$

40. $AD = h$, $\tan\alpha = 45/28$

$AC = h - 17 = h\cot\alpha = \frac{28}{45}h$

$\Rightarrow 45(h-17) = 28h$

$\Rightarrow h = 45 = AB$

$AC = 45 - 17 = 28$

$BC = \sqrt{(45)^2 + (28)^2} = 53.$

$BD = 45\sqrt{2}$

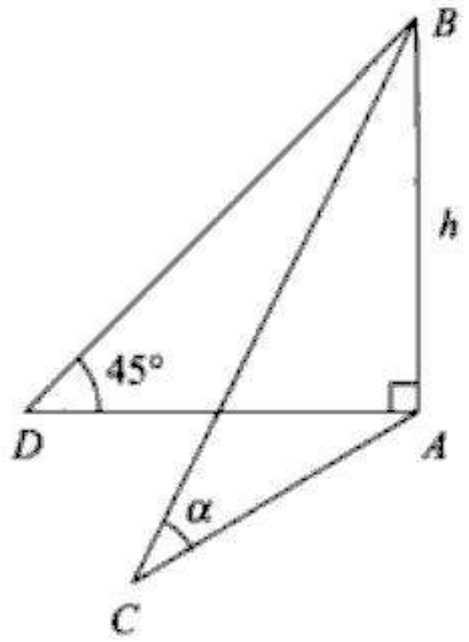

Fig. 12.55

41. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \Delta = \frac{abc}{4R} = rs = \frac{ca\sin B}{2}$

42. A is acute if $b^2 + c^2 > a^2$ and obtuse if $b^2 + c^2 < a^2$.

43. $a + b + \sqrt{a^2+b^2} = (1/2)\,ab$

$\Rightarrow (2a + 2b - ab)^2 = 4(a^2 + b^2)$

$\Rightarrow ab + 8 - 4a - 4b = 0$

$\Rightarrow (a-4)(b-4) = 8$

$\Rightarrow a - 4 = 1, b - 4 = 8$, or $a - 4 = 2, b - 4 = 4$

ordered pairs are (5, 12) or (6, 8)

So Statement 1 is True

Perimeter is maximum when $(a, b) = (5, 12)$

and is equal to $5 + 12 + \sqrt{5^2 + 12^2} = 30$

so statement 2 is also true. But statement-1 does not follow from statement-2.

44. $a = 5, b = 7, \cos C = \frac{a^2+b^2-c^2}{2ab}$ (true) < 0

$\Rightarrow 25 + 49 - c^2 < 0$ or $c^2 > 74$

45. Statement-2 is False

as $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow \tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1} < 0$ [$\because A$ is obtuse]

$\Rightarrow \tan B \tan C < 1 \Rightarrow$ Statement-1 is True.

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Ans. (c)

Solution $\cot(A+B)$

$$= \frac{\cot A \cot B - 1}{\cot A + \cot B} = \frac{1+x+x^2-1}{\sqrt{1+x+x^2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)}$$

$$= \frac{x(1+x)\sqrt{x}}{(1+x)(1+x+x^2)^{1/2}} = x^{3/2}(1+x+x^2)^{-1/2} = \cot C$$

$\Rightarrow$ $A + B = C = \pi/2$ Triangle ABC is right angled.

Example 8 The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is equal to

(a) 3/4 (b) $-3/4$ (c) 1/16 (d) 1/4

Ans. (a)

Solution Let $\cos^{-1}(1/8) = \theta$, where $0 < \theta < \pi$. Then

$$\frac{1}{2}\cos^{-1}\frac{1}{8} = \frac{1}{2}\theta \Rightarrow \cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right) = \cos\frac{1}{2}\theta$$

Now $\cos^{-1}\frac{1}{8} = \theta \Rightarrow \cos\theta = \frac{1}{8} \Rightarrow 2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8}$

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{9}{16} \Rightarrow \cos\frac{\theta}{2} = \frac{3}{4}$$

$$\left[\because 0 < \frac{\theta}{2} < \frac{\pi}{2}, \text{ so } \cos\frac{\theta}{2} \neq -\frac{3}{4}\right]$$

Example 9 The inequality $\sin^{-1}(\sin 5) > x^2 - 4x$ holds if

(a) $x = 2 - \sqrt{9-2\pi}$

(b) $x = 2 + \sqrt{9-2\pi}$

(c) $x \in (2 - \sqrt{9-2\pi}, 2 + \sqrt{9-2\pi})$

(d) $x > 2 + \sqrt{9-2\pi}$

Ans. (c)

Solution Since $3\pi/2 < 5 < 2\pi$, we have $\sin 5 < 0$, so $\sin^{-1}(\sin 5) = 2\pi - 5$. Therefore, the given inequality can be written as

$$2\pi - 5 > x^2 - 4x \quad \text{or} \quad x^2 - 4x - (2\pi - 5) < 0$$

$$\Rightarrow \left[x - \frac{4 - \sqrt{16 - 4(2\pi-5)}}{2}\right]\left[x - \frac{4 + \sqrt{16-4(2\pi-5)}}{2}\right] < 0$$

$$\Rightarrow [x - (2 - \sqrt{9-2\pi})]\,[x - (2 + \sqrt{9-2\pi})] < 0$$

$$\Rightarrow x \in (2 - \sqrt{9-2\pi}, 2 + \sqrt{9-2\pi})$$

Example 10 If $1 < x < \sqrt{2}$, the number of solutions of the equation

$\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$ is

(a) 0 (b) 1 (c) 2 (d) 3

Ans. (a)

Solution The given equation can be written as

$$\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\frac{x-1+x+1}{1-(x-1)(x+1)} = \tan^{-1}\frac{3x-x}{1+3x^2}$$

$$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2} \Rightarrow x + 3x^3 = 2x - x^3$$

$$\Rightarrow 4x^3 - x = 0 \Rightarrow x(4x^2 - 1) = 0$$

$$\Rightarrow x = 0, x = \pm 1/2$$

none of which satisfies $1 < x < \sqrt{2}$

Example 11 If $a\sin^{-1}x - b\cos^{-1}x = c$, then $a\sin^{-1}x + b\cos^{-1}x$ is equal to

(a) 0 (b) $\frac{\pi ab + c(b-a)}{a+b}$

(c) $\pi/2$ (d) $\frac{\pi ab + c(a-b)}{a+b}$

Ans. (d)

Solution We have $b\sin^{-1}x + b\cos^{-1}x = \frac{b\pi}{2}$

and $a\sin^{-1}x - b\cos^{-1}x = c$ (given)

$$\Rightarrow (a+b)\sin^{-1}x = \frac{b\pi}{2} + c$$

$$\Rightarrow \sin^{-1}x = \frac{(b\pi)/2 + c}{a+b}$$

Similarly $$\cos^{-1}x = \frac{(a\pi)/2 - c}{a+b}$$

so that $$a\sin^{-1}x + b\cos^{-1}x = \frac{\pi ab + c(a-b)}{a+b}$$

Example 12 If $\frac{1}{2}\sin^{-1}\left[\frac{3\sin 2\theta}{5+4\cos 2\theta}\right] = \tan^{-1}x$, then $x =$

(a) $\tan 3\theta$ (b) $3\tan\theta$

(c) $(1/3)\tan\theta$ (d) $3\cot\theta$

Ans. (c)

Solution $$\frac{3\sin 2\theta}{5+4\cos 2\theta} = \frac{6\tan\theta}{9+\tan^2\theta} = \frac{2\tan\phi}{1+\tan^2\phi} = \sin 2\phi$$

where $\tan\theta = 3\tan\phi$

$$\Rightarrow \left(\frac{1}{2}\right)\sin^{-1}\left[\frac{3\sin 2\theta}{5+4\cos 2\theta}\right] = \phi = \tan^{-1}\left[\left(\frac{1}{3}\right)\tan\theta\right]$$

so that $x = (1/3)\tan\theta$.

Example 13 If $x > 0$, $y > 0$ and $x > y$, then $\tan^{-1}(x/y) + \tan^{-1}[(x+y)/(x-y)]$ is equal to

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so that $a + b = -\frac{\pi}{3} - \frac{\pi}{4} = \frac{-7\pi}{12}$

$$a - b = -\frac{\pi}{4} + \frac{2\pi}{3} + \frac{\pi}{3} + \frac{2\pi}{3} = \frac{17\pi}{12}$$

Example 29 If α, β are the roots of the equation $(\tan^{-1}(x/5))^2 + (\sqrt{3} - 1)\tan^{-1}(x/5) - \sqrt{3} = 0$, $|\alpha| > |\beta|$ then

(a) $\alpha + \beta = -5\pi/12$ (b) $|\alpha - \beta| = 35\ \pi/12$

(c) $\alpha\beta = -25\pi^2/12$ (d) $3\alpha + 4\beta = 0$

Ans. (a), (b), (c), (d)

Solution $(\tan^{-1}(x/5) + \sqrt{3})(\tan^{-1}(x/5) - 1) = 0$

$$\Rightarrow \quad \tan^{-1}(x/5) = -\sqrt{3} \Rightarrow \frac{x}{5} = -\frac{\pi}{3} \Rightarrow x = -\frac{5\pi}{3}$$

and $\tan^{-1}(x/5) = 1 \Rightarrow \frac{x}{5} = \frac{\pi}{4} \Rightarrow x = \frac{5\pi}{4}$

Let $\alpha = -\frac{5\pi}{3}, \beta = \frac{5\pi}{4}$

$\Rightarrow \quad \alpha + \beta = -5\pi/12, |\alpha - \beta| = 35\pi/12$

$\alpha\beta = -25\pi^2/12$ and $3\alpha + 4\beta = 0$.

Example 30 $\theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}((1/3)\tan\theta)$ if $\tan\theta$ is equal to

(a) -2 (b) 1 (c) 2/3 (d) 2

Ans. (a), (b)

Solution $\theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}((1/3)\tan\theta)$

$$\Rightarrow \quad \tan\theta = \frac{2\tan^2\theta - (1/3)\tan\theta}{1 + (2/3)\tan^3\theta}$$

$$\Rightarrow \quad \tan\theta\left[\frac{2\tan\theta - (1/3)}{1 + (2/3)\tan^3\theta} - 1\right] = 0$$

which is true if $\tan\theta = 0$

or $\frac{2\tan\theta - (1/3)}{1 + (2/3)\tan^3\theta} = 1 \Rightarrow \tan^3\theta - 3\tan\theta + 2 = 0$

$\Rightarrow \quad (\tan\theta - 1)^2(\tan\theta + 2) = 0$ which holds if $\tan\theta = 1$ or $\tan\theta = -2$.

Example 31 $\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}\,x = -\pi/2$ if x is equal to

(a) $-1/12$ (b) 1/6

(c) 1/12 (d) $-1/6$

Ans. (a)

Solution The given equation can be written as

$$\sin^{-1} 6x = -\pi/2 - \sin^{-1} 6\sqrt{3}\,x$$

$$\Rightarrow \quad \sin(\sin^{-1} 6x) = \sin\left(-\pi/2 - \sin^{-1} 6\sqrt{3}\,x\right)$$

$$\Rightarrow \quad 6x = -\cos(\sin^{-1} 6\sqrt{3}\,x)$$

$$\Rightarrow \quad 6x = -\sqrt{1 - 108x^2} \qquad (1)$$

Squaring both sides, we get

$$36x^2 = 1 - 108x^2 \Rightarrow 144x^2 = 1$$

$$\Rightarrow \quad x = \pm 1/12$$

Substituting $x = -1/12$ in the given equation, we get

$$\text{LHS} = \sin^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{2} = \text{RHS}$$

Therefore, $x = -1/12$ is a root of the given equation. Again, substituting $x = 1/12$ in the given equation, we get LHS $= \pi/2$, so that $x = 1/12$ is an extraneous root of the given equation. Hence $x = -1/12$ is the only root.

Example 32 If $A = \tan^{-1}(1/7)$ and $B = \tan^{-1}(1/3)$, then

(a) $\cos 2A = 24/25$ (b) $\cos 2B = 4/5$

(c) $\cos 2A = \sin 4B$ (d) $\tan 2B = 3/4$

Ans. (a), (b), (c), (d)

Solution We have $A = \tan^{-1}(1/7)$, or $\tan A = 1/7$.

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{1 - 1/49}{1 + 1/49} = \frac{48}{50} = \frac{24}{25}$$

Also, from $B = \tan^{-1}(1/3)$, or $\tan B = 1/3$, we get

$$\sin 2B = \frac{2\tan B}{1 + \tan^2 B} = \frac{2(1/3)}{1 + (1/9)} = \frac{2}{3}\cdot\frac{9}{10} = \frac{3}{5}$$

and $\cos 2B = \frac{1 - 1/9}{1 + 1/9} = \frac{8}{10} = \frac{4}{5}$

$\therefore \quad \sin 4B = 2\sin 2B\cos 2B = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$

Example 33 $\cos^{-1} x$ is equal to

(a) $2\sin^{-1}\sqrt{\frac{1-x}{2}}$ (b) $2\cos^{-1}\sqrt{\frac{1-x}{2}}$

(c) $2\cos^{-1}\sqrt{\frac{1+x}{2}}$ (d) $2\sin^{-1}\sqrt{\frac{1+x}{2}}$

Ans. (a), (c)

Solution Let $\cos^{-1} x = y$, so that $x = \cos y$. Then

$$\Rightarrow \quad \sqrt{\frac{1-x}{2}} = \sqrt{\frac{1-\cos y}{2}} = \sqrt{\frac{2\sin^2(y/2)}{2}} = \sin\frac{y}{2}$$

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Example 52 If $\theta = \cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18$, then $\cot^8 \theta$ is equal to

Ans. 6561

Solution $\cot\theta = \dfrac{7\times8\times18-(7+8+18)}{56+144+126-1} = \dfrac{975}{325} = 3$

$\Rightarrow \quad \cot^8\theta = 3^8 = 6561$

Example 53 If $\tan^{-1}\left(x+\dfrac{2}{x}\right) - \tan^{-1}\left(x-\dfrac{2}{x}\right) = \tan^{-1}\dfrac{4}{x}$ then the value and $250x^4 + 320x^2 + 137$ is equal to

Ans. 1777

Solution We have

$$\frac{x+\frac{2}{x}-\left(x-\frac{2}{x}\right)}{1+\left(x^2-\frac{4}{x^2}\right)} = \frac{4}{x} \quad \Rightarrow \quad x^2 - \frac{4}{x^2} = 0$$

$\Rightarrow \qquad x^2 = 2$

So $250x^4 + 320x^2 + 137 = 250\times4 + 320\times2 + 137 = 1777$

Example 54 $\tan^{-1} x = 2\tan^{-1}\dfrac{1}{5} - \dfrac{\pi}{4}$, then $289x^2 + 238x + 9876$ is equal to

Ans. 9827

Solution $2\tan^{-1}(1/5) = \tan^{-1} 5/12$

$$x = \tan(\tan^{-1}(5/12) - \pi/4) = \frac{\frac{5}{12}-1}{1+\frac{5}{12}} = \frac{-7}{17}$$

$\Rightarrow \qquad 17x + 7 = 0 \Rightarrow 289x^2 + 238x + 49 = 0$

So the required value = 9876 – 49 = 9827

Example 55 Maximum value of $2997\sin x + 3996\cos x$ is equal to

Ans. 4995

Solution $2997\sin x + 3996\cos x$

$= 999(3\sin x + 4\cos x)$

$= 999\,r\sin(x+\alpha)$ where $r\sin\alpha = 4$, $r\cos\alpha = 3$ and $r = 5$

$\le 999\times5 = 4995$

SECTION E

MATRIX MATCH TYPE

Example 56

(a) $\sin(\tan^{-1} x)$		(p) $\sqrt{1-x^2}$
(b) $\cos(\tan^{-1} x)$		(q) $\dfrac{x}{\sqrt{1+x^2}}$
(c) $\cos(\sin^{-1} x)$		(r) $\dfrac{1}{\sqrt{1+x^2}}$
(d) $\cos(2\cos^{-1} x)$		(s) $2x^2 - 1$

Ans.

	p	q	r	s
a	p	**q**	r	s
b	p	q	**r**	s
c	**p**	q	r	s
d	p	q	r	**s**

Solution $\sin(\tan^{-1} x) = \sin\theta$ where $x = \tan\theta$

$$= \frac{\tan\theta}{1+\tan^2\theta} = \frac{x}{\sqrt{1+x^2}}$$

and $\cos(\tan^{-1} x) = \cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}} = \dfrac{1}{\sqrt{1+x^2}}$

$\cos(\sin^{-1} x) = \cos\alpha$ where $x = \sin\alpha = \sqrt{1-x^2}$

$\cos(2\cos^{-1} x) = \cos(2\varphi) = 2\cos^2\varphi - 1 = 2x^2 - 1$ $(x = \cos\phi)$

Example 57

(a) $\tan^{-1} 3 + \tan^{-1} 4$		(p) $\pi/2$
(b) $\tan^{-1}(1/3) + \tan^{-1}(1/4)$		(q) $\pi - \tan^{-1}(7/11)$
(c) $\sin^{-1}(1/3) + \cos^{-1}(1/3)$		(r) $\tan^{-1}(7/11)$
(d) $\tan^{-1}(3) + \cot^{-1}(4)$		(s) $\tan^{-1}(13)$

Ans.

	p	q	r	s
a	p	**q**	r	s
b	p	q	**r**	s
c	**p**	q	r	s
d	p	q	r	**s**

Solution $\tan^{-1} 3 + \tan^{-1} 4 = \pi + \tan^{-1}\dfrac{3+4}{1-3\times4}$

$[\because\ 3>0, 4>0, 1-3\times4<0]$

$= \pi - \tan^{-1}(7/11)$

$\tan^{-1}(1/3) + \tan^{-1}(1/4) = \tan^{-1}\dfrac{1/3+1/4}{1-(1/3)(1/4)}$

$= \tan^{-1}(7/11)$, and $\sin^{-1}(1/3) + \cos^{-1}(1/3) = \pi/2$

$= \tan^{-1}(3) + \cot^{-1}(4) = \tan^{-1} 3 + \tan^{-1}(1/4) = \tan^{-1} 13$

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is equal to 29/3.

Statement-2: $\{x \cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 = \frac{51}{50}$ if $x = 1/5\sqrt{2}$.

Ans. (d)

Solution Expression in statement-1 is $\tan(\tan^{-1} 9 - \tan^{-1} 5) = 2/23$ showing that the statement is False.

In statement-2, let $\cot^{-1} x = y \Rightarrow x = \cot y$, so

$$\text{L.H.S} = (\cot y \cos y + \sin y)^2 = \frac{(\cos^2 y + \sin^2 y)^2}{\sin^2 y}$$

$$= 1 + \cot^2 y = 1 + x^2 = 1 + \frac{1}{50} = \frac{51}{50}$$

Showing that statement-2 is True.

Example 69 Statement-1: The value of the determinant

$$\begin{vmatrix} \tan^{-1} x & \cot^{-1} x & \pi/2 \\ \sin^{-1}(4/5) & \sin^{-1}(3/5) & \sin^{-1} 1 \\ \cos^{-1}(3/5) & \cos^{-1}(4/5) & 1 \end{vmatrix}$$

is equal to zero for all values of x.

Statement-2: $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$ if $-1 \le x \le 1$.

Ans. (c)

Solution **Statement-2** is true if $0 \le x \le 1$

For $-1 \le x < 0$, $2\cos^{-1} x = \pi - \cos^{-1}(2x^2 - 1)$

So statement-2 is false.

If Δ denotes the given determinant

Then Δ

$$= \begin{vmatrix} \tan^{-1} x + \cot^{-1} x & \cot^{-1} x & \pi/2 \\ \sin^{-1}(4/5) + \sin^{-1}(3/5) & \sin^{-1}(3/5) & \sin^{-1} 1 \\ \cos^{-1}(3/5) + \cos^{-1}(4/5) & \cos^{-1}(4/5) & 1 \end{vmatrix}$$

(Applying $C_1 \to C_1 + C_2$)

$$= \begin{vmatrix} \pi/2 & \cot^{-1} x & \pi/2 \\ \sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{9}{25}} + \frac{3}{5}\sqrt{1-\frac{16}{25}}\right] & \sin^{-1}(3/5) & \sin^{-1} 1 \\ \cos^{-1}\left[\frac{3}{5}\times\frac{4}{5} - \sqrt{1-\frac{9}{25}}\sqrt{1-\frac{16}{25}}\right] & \cos^{-1}(4/5) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \pi/2 & \cot^{-1} x & \pi/2 \\ \sin^{-1} 1 & \sin^{-1}(3/5) & \sin^{-1} 1 \\ \cos^{-1} 0 & \cos^{-1}(4/5) & 1 \end{vmatrix}$$

$= 0$ ($\because \cos^{-1} 0 = 1$)

(C_1 and C_2 are identical)

So statement-1 is true.

Example 70 Statement-1: $3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right) < 2\pi/3$ and $\tan^{-1}(2\sqrt{2} - 1) > \pi/3$

Satement-2: If $3\sin^{-1} x = \pi/6$, then $6x - 8x^3 - 1 = 0$

Ans. (b)

Solution $3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$$= \sin^{-1}\left(3\times\frac{1}{3} - 4(1/3)^3\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\left(1 - \frac{4}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\frac{23}{27} + \sin^{-1}\frac{3}{5}$$

$$< \sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{\sqrt{3}}{2}$$

$$\left[\because \frac{23}{27} = 0.85, \frac{3}{5} = 0.6 \text{ and } \frac{\sqrt{3}}{2} = 0.86\right]$$

$$= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

and $\tan^{-1}(2\sqrt{2} - 1) > \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

[$\because 2\sqrt{2} - 1 = 1.8$, $\sqrt{3} = 1.7$]

So statement-1 is true

In statement-2, $3\sin^{-1} x = \frac{\pi}{6}$

$$\Rightarrow \quad \sin(3\sin^{-1} x) = \frac{1}{2}$$

$$\Rightarrow \quad 3x - 4x^3 = 1/2$$

$$\Rightarrow \quad 6x - 8x^3 - 1 = 0$$

Showing that statement-2 is also true but does not lead to statement-1.

EXERCISE

LEVEL 1

SECTION I

SINGLE CORRECT CHOICE TYPE

1. The equation $\sin^{-1} x = 2\sin^{-1} a$ has a solution for
   (a) all real values of a (b) $a < 1$

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SECTION IV

INTEGER ANSWER TYPE

21. If $\alpha = \sin^{-1}(4/5)$, $\beta = \cos^{-1}(5/13)$ and $x = \sin(2\alpha + \beta)$ then the value of $\frac{540}{x}$ is equal to

22. If $y = \tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}$ and $\pi = 22/7$, then the value of $(2x + 14y)^3 - 7^3$ is equal to

23. $x = \sin[\sin^{-1}(3/5) + \sin^{-1}(8/17)]$
$y = \cos[\cos^{-1}(4/5) + \cos^{-1}(12/13)]$
then the value of $\frac{2431}{xy}$ is equal to

24. If $2\tan^{-1} x = \cos^{-1}(\sqrt{5}/3)$ then the value of $60x^4 - 540x^2 + 360x + 9261$ is equal to

25. If $x = \sin(2\tan^{-1}(1/3)) + \cos(\tan^{-1}(2\sqrt{2}))$ then $\frac{1}{(1-x)^3} + \frac{1}{(1-x)^2} + \frac{1}{1-x}$ is equal to

SECTION V

MATRIX MATCH TYPE

26.

(a) $\sin^{-1}(2/3) + \cos^{-1}(2/3)$	(p)	$\sin^{-1}\sin(\pi/6)$
(b) $\sin^{-1}(\sin(7\pi/6))$	(q)	$\sin^{-1}\sin(11\pi/6)$
(c) $\cos^{-1}(\cos(\pi/6))$	(r)	$\tan^{-1}2 + \cot^{-1}2$
(d) $\tan^{-1}3 + \tan^{-1}(1/3)$	(s)	$\pi/2$

27.

(a) $\sin^{-1} x + \sin^{-1} 2x = \pi/3$	(p)	4/3,
(b) $\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$	(q)	$\frac{1}{2}\sqrt{\frac{3}{7}}$
(c) $\tan^{-1}\cos x = \sin(\tan^{-1}2)$	(r)	$\sqrt{5}/3$
(d) $\sin\tan^{-1} x = -3/\sqrt{73}$	(s)	$-3/8$

28.

(a) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$	(p)	x
(b) $\tan^{-1}\frac{\cos x \pm \sin x}{\cos x \pm \sin x}$	(q)	$\frac{\pi}{4} - x$
(c) $2\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$	(r)	$x/2$
(d) $\sin^{-1}\frac{\cos 2x}{\sqrt{2}(\cos x \pm \sin x)}$	(s)	$\pi/4 + x$

SECTION VI

REASONING TYPE

29. **Statement-1:**
$$\cot^{-1}\left[\frac{\log(e/x^2)}{\log(ex^2)}\right] + \cot^{-1}\left[\frac{\log(ex^4)}{\log(e^2/x^2)}\right] = \pi - \tan^{-1}3$$
Statement-2
$$\tan^{-1}\left[\frac{1+\log x^2}{1-\log x^2}\right] = \tan^{-1}1 + \tan^{-1}(\log x^2)$$

30. **Statement-1:** Let $f(x) = \cot^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$, $g(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
then $f(x) = g(x)$ if $x = \frac{\sqrt{25 - 10\sqrt{5}}}{5}$.
Statement-2: $\sin 18° = \frac{\sqrt{5}-1}{4}$.

PAST YEARS' IIT QUESTIONS

1. Find the value of $\cos(2\cos^{-1} x + \sin^{-1} x)$ at $x = 1/5$ where $0 \le \cos^{-1} x \le \pi$ and $-\pi/2 \le \sin^{-1} x \le \pi/2$ [1981]

2. The numerical value of $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$ is equal to [1984]

3. The value of $\tan\left\{\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right\}$ is
(a) 6/17 (b) 7/16
(c) 16/7 (d) none of these [1983]

4. The principal value of $\sin^{-1}\left[\sin\frac{2\pi}{3}\right]$ is
(a) $-2\pi/3$ (b) $2\pi/3$
(c) $4\pi/3$ (d) $5\pi/3$
(e) none of these. [1986]

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$$\beta = 2\tan^{-1} 9 = \sin^{-1}\frac{2\times 9}{1+81} = \sin^{-1}\frac{9}{41}$$

$$\Rightarrow \quad \sin\beta = \frac{9}{41} \Rightarrow 81\ \text{cosec}^2\beta = 1681$$

$$80\ \text{cosec}^2\alpha + 81\ \text{cosec}^2\beta = 1681 + 81 = 1762$$

24. $\tan\theta = \dfrac{1/70 - 1/99}{1+(1/70)(1/99)} = \dfrac{29}{6931} = \dfrac{1}{239}$

$$9\cot\theta + 890 = 239\times 9 + 890 = 2151 + 890 = 3041$$

25. $\tan^{-1}\dfrac{2x}{1-x^2} + \tan^{-1}\dfrac{2x}{1-x^2} = \pi/3$

$$\Rightarrow \quad 2\tan^{-1}\frac{2x}{1-x^2} = \pi/3$$

$$\Rightarrow \quad \frac{2x}{1-x^2} = \tan(\pi/6) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad x^4 - 14x^2 + 1 = 0 \quad \Rightarrow \quad x^4 + 1 = 14x^2$$

$$\Rightarrow \quad x^8 + 1 = 194x^4 \quad \Rightarrow \quad x^4 + \frac{1}{x^4} = 194$$

$$\Rightarrow \quad 25\left(x^4 + \frac{1}{x^4}\right) + 8 = 4858$$

26. $\cos(3\cos^{-1}x) = \cos 3\theta = 4\cos^3\theta - 3\cos\theta$
$= 4x^3 - 3x\ [x = \cos\theta]$

$\sin(2\sin^{-1}x) = \sin 2\theta = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$
$[x = \sin\theta]$

$\tan(3\tan^{-1}x) = \tan 3\theta = \dfrac{3x - x^3}{1-3x^2}$ $(x = \tan\theta)$

$\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$

27. $x = \sin\theta \Rightarrow \cos 2\theta = 1/9 \Rightarrow 1 - 2\sin^2\theta = 1/9$
$\Rightarrow \sin\theta = \pm 2/3$

$x = 1$ satisfies (ii) and $x = -1$ satisfies (iii).

$\tan^{-1}x + \sin^{-1}x + \cot^{-1}x = \pi$

$$\Rightarrow \frac{\pi}{2} + \frac{\pi}{2} - \cos^{-1}x = \pi \quad \Rightarrow \quad \cos^{-1}x = 0$$

$\Rightarrow x = 1$

28. Statement-2 is false as $\tan(\sin^{-1}x + \cos^{-1}x) = \tan(\pi/2)$

In statement-1, $p = 2\tan\dfrac{\sin^{-1}x + \cos^{-1}x}{2} = 2$,

$q = \sqrt{\tan^{-1}x\cot^{-1}x} = 1$ and we get $x^2 - 2x + 1 = 0$
$\Rightarrow \quad x = 1$, so it is True.

29. In statement-1, $(\tan^{-1}x)^2 - \left(\dfrac{\pi}{6} + \dfrac{\pi}{3}\right)\tan^{-1}x +$

$$\frac{\pi}{6}\times\frac{\pi}{3} = 1$$

$\Rightarrow \tan^{-1}x = \pi/6$ or $\tan^{-1}x = \pi/3 \Rightarrow x = 1/\sqrt{3};\ \sqrt{3}$.

$\Rightarrow \alpha + \beta = \left(1/\sqrt{3}\right) + \sqrt{3} = 4/\sqrt{3} \Rightarrow$ statement-1 is True

L.H.S of Statement-2 = $\sec^2(\sec^{-1}4) + \text{cosec}^2(\text{cosec}^{-1}5)$

$= 16 + 25 = 41$ and the statement-2 is True.

30. $\sin^{-1}x\tan(\tan^{-1}x + \tan^{-1}(1-x)) = \pi/2$

$$\Rightarrow \quad \frac{x+1-x}{1-x(1-x)} = \sin\pi/2 = 1 \Rightarrow 1 - x(1-x) = 1$$

$\Rightarrow x = 1$ is a non zero solution and thus the statement is false.

Next, $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = (\sin^{-1}x + \cos^{-1}x)^3 - 3\sin^{-1}x\cos^{-1}x(\sin^{-1}x + \cos^{-1}x)$

$= (\pi/2)^3 - (3\pi/2)\sin^{-1}x(\pi/2 - \sin^{-1}x)$

$= (\pi^3/8) - (3\pi^2/4)\sin^{-1}x + (3\pi/2)(\sin^{-1}x)^2$

$= (\pi^3/8) + (3\pi/2)[\sin^{-1}x - (\pi/4)]^2 - (3\pi^3/32)$

$= (\pi^3/32) + (3\pi/2)[\sin^{-1}x - (\pi/4)]^2$

Greatest and least value of $\sin^{-1}x - \pi/4$ is $(3\pi/4)$ and 0 respectively.

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Example 10 *OPQR* is a square and *M*, *N* are the middle points of the sides *PQ* and *QR* respectively then the ratio of the areas of the square and the triangle *OMN* is

(a) 4 : 1 (b) 2 : 1 (c) 8 : 3 (d) 4 : 3

Ans. (c)

Solution Taking the coordinates of vertices *O*, *P*, *Q*, *R* as (0, 0), (*a*, 0), (*a*, *a*), (0, *a*) respectively we get the coordinates of *M* as (*a*, *a*/2) and of *N* as (*a*/2, *a*) (Fig. 14.1)

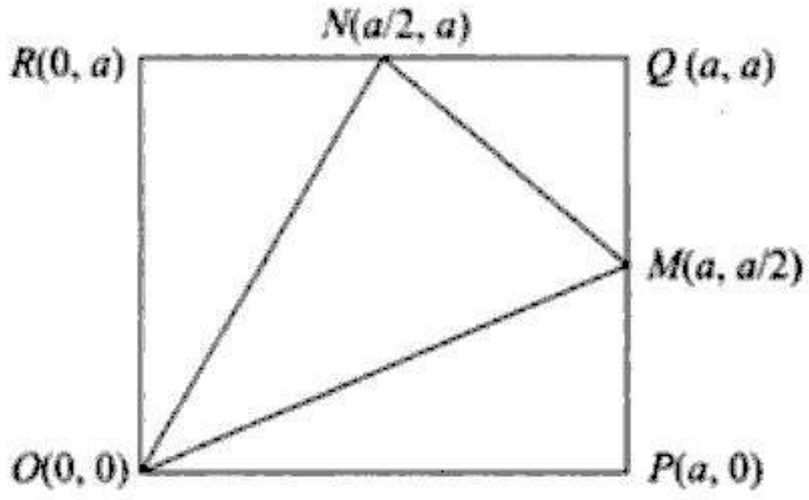

Fig. 14.1

$$\therefore \quad \text{Area of the } \Delta OMN = \frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ a & a/2 & 1 \\ a/2 & a & 1 \end{vmatrix}\right| = \frac{3a^2}{8}$$

Area of the square = a^2

$\therefore$ the required ratio is 8 : 3.

Example 11 If $p, x_1, x_2 \ldots x_i, \ldots$ and $q, y_1, y_2 \ldots, y_i, \ldots$ are in A.P., with common difference a and b respectively, then locus of the centre of mean position of the points A_i (x_i, y_i), $i = 1, 2 \ldots n$ is

(a) $ax - by = aq - bp$

(b) $bx - ay = ap - bq$

(c) $bx - ay = bp - aq$

(d) $ax - by = bq - ap$

Ans. (c) [**Note.** Centre of Mean Position is $\left(\frac{\Sigma xi}{n}, \frac{\Sigma yi}{n}\right)$]

Solution Let the coordinates of the centre of mean position of the points

A_i, $i = 1, 2, \ldots n$ be (x, y), then

$$x = \frac{x_1 + x_2 + \cdots + x_n}{n}, \; y = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

$$\Rightarrow \quad x = \frac{np + a(1 + 2 + \cdots + n)}{n},$$

$$y = \frac{nq + b(1 + 2 + \cdots + n)}{n}$$

$$\Rightarrow \quad x = p + \frac{n(n+1)}{2n} a \;, \; y = q + \frac{n(n+1)}{2n} b$$

$$\Rightarrow \quad x = p + \frac{n+1}{2} a, \; y = q + \frac{n+1}{2} b$$

$$\Rightarrow \quad 2\frac{(x-p)}{a} = 2\frac{(y-q)}{b} \Rightarrow bx - ay = bp - aq.$$

Example 12 If α, β, γ are the real roots of the equation $x^3 - 3px^2 + 3qx - 1 = 0$, then the centroid of the triangle with vertices $\left(\alpha, \frac{1}{\alpha}\right), \left(\beta, \frac{1}{\beta}\right)$ and $\left(\gamma, \frac{1}{\gamma}\right)$ is at the point

(a) (p, q) (b) $(p/3, q/3)$

(c) $(p + q, p - q)$ (d) $(3p, 3q)$

Ans. (a)

Solution The centroid of the given triangle is the point

$$\left(\frac{\alpha+\beta+\gamma}{3}, \frac{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}{3}\right) = \left(\frac{3p}{3}, \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{3\alpha\beta\gamma}\right)$$

$$= (p, q)$$

$[\because \alpha+\beta+\gamma = 3p, \; \alpha\beta+\beta\gamma+\gamma\alpha = 3q, \; \alpha\beta\gamma = 1]$

Example 13 The number of points (p, q) such that $p, q \in \{1, 2, 3, 4\}$ and the equation $px^2 + qx + 1 = 0$ has real roots is

(a) 7 (b) 8

(c) 9 (d) none of these

Ans. (a)

Solution $px^2 + qx + 1 = 0$ has real roots

if $q^2 - 4p \geq 0$ or $q^2 \geq 4p$

Since $p, q \in \{1, 2, 3, 4\}$

The required points are

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (4, 4)

So the required number is 7.

Example 14 If *G* is the centroid and *I* the incentre of the triangle with vertices *A*(–36, 7), *B*(20, 7) and *C*(0, –8), then *GI* is equal to.

(a) $\sqrt{173}/3$ (b) $\sqrt{397}/3$

(c) $\sqrt{205}/3$ (d) None of these

Ans. (c)

Solution We have $a = BC = \sqrt{(20)^2 + (7+8)^2} = 25$,

$b = CA = \sqrt{(36)^2 + (8+7)^2} = 39$ and

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In Ex. 28.

$$P_1\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

$$P_2\left(\frac{x_2+x_3+x_4}{3}, \frac{y_2+y_3+y_4}{3}\right)$$

$$P_3\left(\frac{x_3+x_4+x_5}{3}, \frac{y_3+y_4+y_5}{3}\right)$$

.
.
.
.

$$P_{n-2}\left(\frac{x_{n-2}+x_{n-1}+x_n}{3}, \frac{y_{n-2}+y_{n-1}+y_n}{3}\right)$$

So the coordinates of the centre of mean position of $P_1, P_2 \ldots P_{n-2}$ is $(\bar{x}, \bar{y})$
where

$$\bar{x} = \frac{x_1+2x_2+3x_3+\ldots+3x_{n-2}+2x_{n-1}+x_n}{n-2}$$

$$\bar{y} = \frac{y_1+2y_2+3y_3+\ldots+3y_{n-2}+2y_{n-1}+y_n}{n-2}$$

Paragraph for Question Nos. 29 to 32

Given two points $A(-2, 0)$ and $B(0, 4)$, M is a point with coordinates (x, x), $x \geq 0$. P divides the joint A and B in the ratio 2 : 1. C and D are the mid-points of BM and AM respectively.

Example 29 Area of the ΔAMB is minimum, if the coordinates of M are

(a) (1, 1) (b) (0, 0)
(c) (2, 2) (d) (3, 3)

Example 30 Ratio of the areas of the triangles APM and BPM is

(a) 2 : 1 (b) 1 : 2
(c) 2 : 3 (d) 1 : 3

Example 31 Perimeter of the quadrilateral $ABCD$ is

(a) $2\sqrt{5}$ (b) $3+3\sqrt{5}$
(c) $2+\sqrt{5}$ (d) none of these

Example 32 Area of the quadrilateral $ABCD$ in units is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. 29. (b) 30. (a) 31. (b)
32. (c)

Solution Area of the $\Delta AMB = \frac{1}{2}\begin{vmatrix} x & x & 1 \\ -2 & 0 & 1 \\ 0 & 4 & 1 \end{vmatrix}$

$$= \left|\frac{1}{2}(-4x+2x-8)\right| = |-(x+4)|$$

which is minimum for $x = 0$
and thus the coordinates of M are (0, 0).
In example 30,
as P divides AB in the ratio 2:1.
The base of the triangles APM and BPM are in the ratio 2:1 and the length of the perpendicular from the vertex M on the base is same. So, the ratio of the areas of the triangles APM and BPM is also 2 : 1.

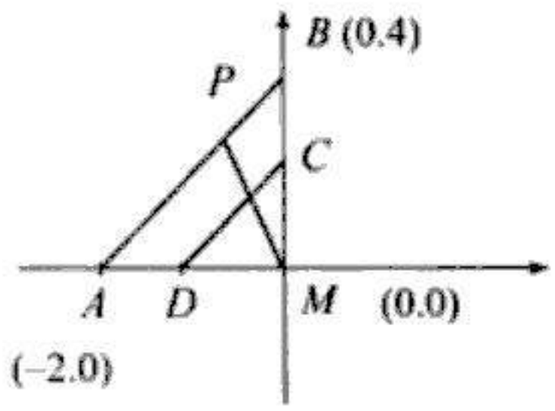

Fig. 14.2

In example 31,
$ABCD$ is a trapezium with $AD = 1$, $BC = 2$

$$DC = \frac{1}{2}AB = \frac{1}{2}\sqrt{2^2+4^2} = \sqrt{5}$$

So, the required perimeter is

$$1+2+\sqrt{5}+2\sqrt{5} = 3+3\sqrt{5}$$

In example 32,
Area of the trapezium $ABCD$
= Area of the ΔAMB – Area of the ΔCMD.

$$= \frac{1}{2}(8-2) = 3$$

Paragraph for Question Nos. 33 to 35

$A(p, 0)$, $B(4, 0)$, $C(5, 6)$ and $D(1, 4)$ are the vertices of a quadrilateral $ABCD$.

Example 33 If $\angle ADC$ is obtuse, the maximum integral value of p is

(a) 1 (b) 2
(c) 3 (d) 0

Example 34 The value of p for which $\angle ADC$ is obtuse and $\angle BAD$ is a right angle is

(a) 1 (b) 2
(c) –3 (d) –4

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$(PQ)^2 = 29,\ (QR)^2 = 205,\ (PR)^2 = 320$

$(PS)^2 = [(2/5)\ (PQ)]^2 = \dfrac{116}{25}.$

$(QS)^2 = [(3/5)\ (PQ)]^2 = \dfrac{261}{25}.$

$(QT)^2 = [(3/5)\ (QR)]^2 = \dfrac{369}{5}.$

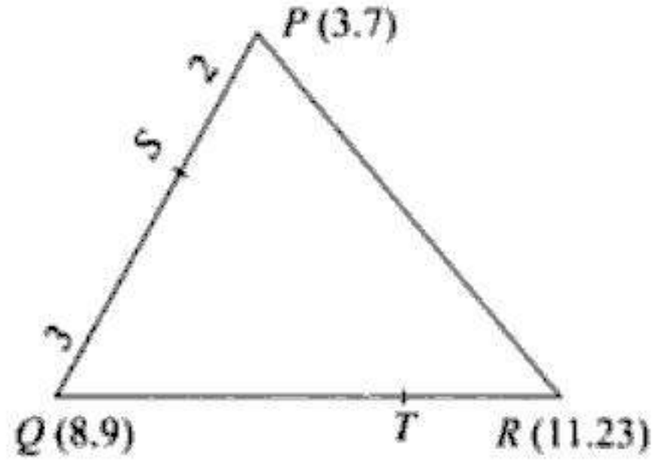

Fig. 14.5

So that $(PR)^2 + (QR)^2 + 51$

$= 320 + 205 + 51 = 576$

$100\ (PS)^2 = 4 \times 116 = 464$

and $(PQ)^2 + 2(QR)^2 + 25 = 29 + 410 + 25 = 464$

Next, $2(PR)^2 - 64 = 2 \times 320 - 64 = 576$ (r)

$50\ (QS)^2 - 2(PQ)^2 = 2 \times 261 - 2 \times 29 = 464$ (s)

and $5\ (QT)^2 + (QR)^2 + 2 = 369 + 205 + 2 = 576$ (t)

Combining these results we get the required answer.

SECTION VI

REASONING TYPE

Example 46 **Statement-1:** The points $A(3, 4)$, $B(2, 7)$, $C(4, 4)$ and $D(3, 5)$ are such that one of them lies inside the triangle formed by the other points.

Statement-2: Centroid of a triangle lies inside the triangle.

Ans. (a)

Solution $\dfrac{3+2+4}{3} = 3$ and $\dfrac{4+7+4}{3} = 5$

So D is the centroid of the triangle ABC, using statement–2 which is correct we find statement A is also correct.

Example 47 **Statement-1:** The points $A(-2, 2)$, $B(2, -2)$ and $C(1, 1)$ are the vertices of an obtuse angled isosceles triangle.

Statement-2: Every obtuse angled triangle is isosceles.

Ans. (c)

Solution Statement-2 is not correct as the triangle with angles 120°, 40°, 20° is obtuse angled but not isosceles.

In statement–1 $AB = 4\sqrt{2}$, $AC = \sqrt{10}$, $BC = \sqrt{10}$

$$\cos C = \frac{AC^2 + BC^2 - AB^2}{2(AC)(BC)} = \frac{10 \times 10 - 32}{2 \times \sqrt{10}\ \sqrt{10}} < 0$$

$\Rightarrow$ C is obtuse.

So the triangle is obtuse angled isosceles.

Example 48 **Statement-1:** The quadrilateral whose vertices (in order) are $A(1, 0)$, $B(0, 3)$, $C(-2, 0)$ and $D(0, 2)$ can not be convex.

Statement-2: A quadrilateral $ABCD$ (in order) is convex if and only if when any diagonal is taken then the remaining vertices must be on the opposite sides of it.

Ans. (a)

Solution Statement-2 is correct because if one angle say D of the quadrilateral $ABCD$ which is not covex is more than 180° then B and D lie on the same side of the diagonal AC of the quadrilateral. Statement-1 is true, using statement-2, the quadrilateral $ABCD$ is not convex.

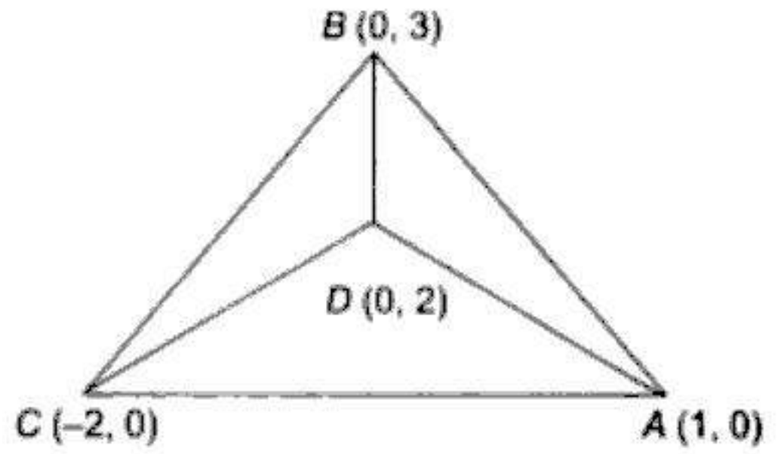

Fig. 14.6

Example 49 **Statement-1:** Three points $A(x_1, y_1)$ $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$.

Statement-2: The points $A(x, -x)$, $B(7, -5)$ $C(-5, 3)$ are collinear if $x = 1$.

Ans. (d)

Solution Statement-1 is False as the condition for collinearity is $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$, using in statement-2 we have $\begin{vmatrix} x & -x & 1 \\ 7 & -5 & 1 \\ -5 & 3 & 1 \end{vmatrix} = 0$

$\Rightarrow$ $4x - 4 = 0 \Rightarrow x = 1$ and statement-2 is True.

Example 50 **Statement-1:** $A(2, 2)$, $B(6, -1)$ and $C(7, 3)$ are the vertices of a triangle ABC. D, E, F are the mid-points of the sides BC, CA and AB respectively. Area of the ΔDEF is equal to 19/8 square units.

Statement-2: Lines joining the mid-points of the sides of a triangle divide the triangle into four triangles of equal areas.

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LEVEL 2

SECTION I

SINGLE CORRECT CHOICE TYPE

1. If the points $(a, 1)$, $(1, b)$ and $(a-1, b-1)$ are collinear, α, β are respectively the arithmetic and geometric means of a and b then $4\alpha - \beta^2$ is equal to
   (a) -1 (b) 0
   (c) 3 (d) 2

2. If $P(1, 0)$, $Q(-1, 0)$ and $R(2, 0)$ are three given points. The point S satisfies the relation $SQ^2 + SR^2 = 2SP^2$. The locus of S meets PQ at the point
   (a) (0, 0) (b) (2/3, 0)
   (c) (− 3/2, 0) (d) (0, − 2/3)

3. $A_1, A_2, \ldots A_n$ are points whose coordinates (x, y) satisfy $y = x$ and lie in the positive quadrant such that $OA_n = n\,OA_{n-1}$, O being the origin. If $OA_1 = 1$ and the coordinates of A_n are $(2520\sqrt{2}, 2520\sqrt{2})$ then $n =$
   (a) 5 (b) 6
   (c) 7 (d) 8

4. The coordinates of the point A_n are (na, a^n). If α denotes the arithmetic mean of x coordinates of the points $A_1, A_2 \ldots A_n$ and β denotes the geometric mean of the ordinates of these points then if a is a variable, locus of the point $P(\alpha, \beta)$ is
   (a) $x^n = n^n y^2$
   (b) $x^{n+1} = \left(\frac{n+1}{2}\right)^{n+1} y^2$
   (c) $x^n = n^n y$
   (d) none of these

5. Coordinates of the point A_n are $(n^2, 2n)$ and of B_n are $(n^2, -2n)$. Area of the quadrilateral with vertices $A_1 B_1 B_n A_n$ is
   (a) $(n+1)^2$ (b) $(n-1)^2$
   (c) $2(n^2+1)$ (d) $2(n^2-1)$

SECTION II

MULTIPLE CORRECT CHOICE TYPE

6. ABC is an equilateral triangle. If the coordinates of two of its vertices are (1, 3) and (− 2, 7), the coordinates of the third vertex can be
   (a) $\left(-\frac{1}{2}+2\sqrt{3}, 5+\frac{3\sqrt{3}}{2}\right)$
   (b) $\left(-\frac{1}{2}+2\sqrt{3}, 5-\frac{3\sqrt{3}}{2}\right)$
   (c) $\left(\frac{1}{2}-2\sqrt{3}, -5+\frac{3\sqrt{3}}{2}\right)$
   (d) $\left(\frac{1}{2}+2\sqrt{3}, -5-\frac{3\sqrt{3}}{2}\right)$

7. The coordinates of a point at a distance $\sqrt{5}$ from the point (3, 4) and having the same abcissa and ordinate are
   (a) (2, 2) (b) (− 2, − 2)
   (c) (5, 5) (d) (− 5, − 5)

8. If two vertices of a triangle of area a^2 are (0, 0) and $(a, 0)$ then the coordinates of the third vertex can be
   (a) $(a/2, 2a)$ (b) $(3a/2, -2a)$
   (c) $(a/2, a/2)$ (d) $(a/2, -2a)$

9. $P(-a, 0)$, $Q(a, 0)$ and $R(1, 1)$ are three points such that $|(PR)^2 - (QR)^2| = 12$ then
   (a) $(PR)^2 = 17$ (b) $(QR)^2 = 5$
   (c) $(PR)^2 = 5$ (d) $(QR)^2 = 17$

10. $A(0, b)$, $B(0, 0)$ and $C(a, 0)$ are the vertices of a triangle ABC. D, E, F are the mid-points of the sides BC, CA and AB respectively. If $a^2 + b^2 = 20$, then
    (a) $(AD)^2 = 9$
    (b) $(BE)^2 = 4$
    (c) $(AD)^2 + (CF)^2 = 25$
    (d) $(AD)^2 + (CF)^2 = (BE)^2$

SECTION III

LINKED COMPREHENSION TYPE

Paragraph for Questions Nos 11 to 13

Vertices of a parallelogram are
$A(0, 0)$, $B(-2/3, 7/3)$, $C(a, b)$ and $D(5/3, -4/3)$

11. Length of the larger diagonal of $ABCD$ is
    (a) $\sqrt{2}$ (b) $\sqrt{58}/3$
    (c) $\sqrt{41}/3$ (d) $\sqrt{170}/3$

12. If P and Q are two points on the diagonal DB such that $DP = PQ = QB$ then area of the ΔCPQ in square units is
    (a) 1/3 (b) 1/2
    (c) 1/4 (d) 1

13. Sum of the areas of the triangles ABP, APD, BQC and CQD is
    (a) 2 (b) 3/2
    (c) 1 (d) 1/2

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15. $ax + ary + ar^2 = 0 \Rightarrow x + ry + r^2 = 0$
ordinates of P and Q are the roots of $(r^2 - 1)\, y^2 + 2r^3\, y + r^4 - 4 = 0$

16. $\left(\dfrac{\alpha^2+\beta^2+\gamma^2}{3}, \dfrac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{3\alpha^2\beta^2\gamma^2}\right)$

$= \left(\dfrac{(3p)^2 - 2\times 3q}{3}, \dfrac{(3q)^2 - 2\times 3p}{3}\right)$

$= (3p^2 - 2q, 3q^2 - 2p)$

17. $a + c = 2b,\ a^2 + 1 + c^2 + 9 = 2(b^2 + 4)$

18. $\dfrac{\cos t + \sin t}{3} = \dfrac{2(\sin t - \cos t)}{3}$ or $\dfrac{\sin t - \cos t}{3}$

$= \dfrac{2(\cos t + \sin t)}{3}$

$3\cos t = \sin t$ or $\sin t = 3\cos t$

19. Area of the triangle $\dfrac{1}{2}\begin{vmatrix} m^2 & 2m & 1 \\ mm' & m+m' & 1 \\ (m')^2 & 2m' & 1 \end{vmatrix}$

$= \dfrac{1}{2}\begin{vmatrix} m^2 & 2m & 1 \\ (m'-m) & m'-m & 0 \\ m'^2 - m^2 & 2(m'-m) & 0 \end{vmatrix}$

$= \dfrac{1}{2}\,|(m'-m)^3|$

$= 32 \Rightarrow (m'-m)^3 = \pm 64$

20. $(OP)^2 + (OQ)^2 = (PQ)^2 \Rightarrow x_1x_2 + y_1y_2 = 0$

21. $P(0, 1),\ Q(1/2, 1),\ R(1/3, 7/3)$
$(PR)^2 = 17/9,\ (QR)^2 = 65/36$

22. Area of $\Delta\, OAB = 3/2$, Area of $\Delta\, PQR = 1/3$

23. $S\,(1/6, 10/6),\ QS = \sqrt{5}\,/3,\ OQ = \sqrt{5}\,/2$

$AB = \sqrt{5}\,,\ PQ = 1/2,\ PS = \sqrt{17}\,/6\,.$

24. $a^2 + 1 = b^2 + 1 = (a-1)^2 + (b-1)^2$
$\Rightarrow a = b$ [as $0 < a, b < 1$] and
$2(a^2 - 2a + 1) = a^2 + 1 \Rightarrow a^2 - 4a + 1 = 0$

$\Rightarrow a = 2 - \sqrt{3}$, Area of the triangle

$= (\sqrt{3}/4)\,(2-\sqrt{3})^2$

25. $AC = BC \Rightarrow a = b$
$5[2(a-1)^2] = 2(a^2 + 1) \Rightarrow a = 1/2 = b$

26. $q^2 - 4p \geq 0 \Rightarrow q^2 \geq 4p$, distance of the point is least if $p^2 + q^2$ is least
$\Rightarrow \quad p = 50,\ q = 15$

27. $\displaystyle\sum_{i=1}^{n}(OAi)^2 = \sum_{i=1}^{n}(i^2 + i^3) = \Sigma n^2 + \Sigma n^3$

$= \dfrac{n(n+1)(2n+1)}{6} + \left(\dfrac{n(n+1)}{2}\right)^2$

28. $a + b = 2 \times 28 = 56,\ b - 25a = 4 \Rightarrow a = 2,\ b = 54$
$x = 2r,\ y = 2r^2,\ b = 2r^3 \Rightarrow r = 3$
and $Q(x, y) = (6, 18) \Rightarrow OP^2 + OQ^2 = 2^2 + 6^2 + 18^2 + 54^2$

29. $(p-1)^2 + (p+1)^2 = 4[p^2 + p^2] \Rightarrow \dfrac{1}{p^2} = 3$

30. $(A_n A_{n+1})^2 = 2$ for all n. Required sum
$= 2(10 + 11 + \ldots + 20) = 530$

31. $(OA_n)^2 = n^2 + (2n+1)^2 = 5n^2 + 4n + 1$
$(AA_n)^2 = (n-1)^2 + (2n)^2 = 5n^2 - 2n + 1$
$(A_1 A_n)^2 = (n-1)^2 + (2n+1-3)^2 = 5n^2 - 10n + 5$
$(A_{n-1} A_n)^2 = 5$

32. $\dfrac{a + a + 2}{2} = 0 \Rightarrow a = -1$

$\dfrac{b + b + 6}{2} = 0 \Rightarrow b = -3$

coordinates of $P\ (0, -3/2)$, $Q\ (5/2, 0)$
$R(7/2, 3/2)$. Centroid of ΔABC and ΔPQR is same $(a + 3, b + 3) = (2, 0)$

33. **Statement-2** is True. Using it in statement-1 each side of the square $PQRS$ is equal to $\sqrt{(1/3)^2 + (2/3)^2}$ and hence its area is 5/9.

34. Centroid of ABC is $(a - 1/3, b - 1/3)$. if (X, Y) are the new coordinate of A the $X = a - (a - 1/3) = 1/3$.
$Y = b - (b - 1/3) \Rightarrow$ statement-1 is true.
Statement-2 is also true but is not a correct explanation for statement-1.

35. $\alpha + \beta = 7,\ r = 7/2.$
locus is $(x - \alpha)^2 + (y - \beta)^2 = [x - (\alpha + r)]^2 + [y - (\beta + r)]^2$
$\Rightarrow \quad x + y - (\alpha + \beta) = r \Rightarrow 2x + 2y - 21 = 0$
Statement-1 is true.
For statement-2, Let the fixed points be $(-1, 0)$, $(1, 0)$ whose mid point is $(0, 0)$.
Locus is $\sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2} = c$ (constant)
which does not pass through $(0, 0)$ for all values of c. So statement-2 is False.

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be the equations of a line and a *conic*, respectively. Writing the equation of the line as

$\frac{lx+my}{-n} = 1$ and making $S = 0$ homogeneous with its help, we get $S = ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$

which being a homogeneous equation of second degree, represents a pair of straight lines through the origin and passing through the points common to $S = 0$ and $L = 0$.

7. *Some important results regarding pair of lines*

(i) Equation of the pair of lines through the origin perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

(ii) The product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$$

(iii) The product of the perpendiculars drawn from the origin to the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\frac{c}{\sqrt{(a-b)^2 + 4h^2}}$$

(iv) Equations of the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are given by

$$\frac{(x-x_1)^2 - (y-y_1)^2}{a-b} = \frac{(x-x_1)(y-y_1)}{h}$$

where (x_1, y_1) is the point of intersection of the lines represented by the given equation.

(v) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel straight lines, then the distance between them is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$.

Every nth degree homogeneous equation $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y + \ldots + a_{n-1}xy^{n-1} + a_ny^n = 0$ in x and y represents *n straight lines through the origin*. If $m_1, m_2, \ldots m_n$ are the slopes of these n lines and $a_n \neq 0$, then

$$m_1 + m_2 + \ldots + m_n = -\frac{a_{n-1}}{a_n}$$

and $$m_1 m_2 \ldots m_n = (-1)^n \frac{a_0}{a_n}$$

SOLVED EXAMPLES

SECTION I

SINGLE CORRECT CHOICE TYPE

Example 1 If a, b, c form a G.P. with common ratio r, the sum of the ordinates of the points of intersection of the line $ax + by + c = 0$ and the curve $x + 2y^2 = 0$ is

(a) $-r^2/2$ (b) $-r/2$
(c) $r/2$ (d) $r^2/2$

Ans. (c)

Solution The equation of the given line is $ax + by + c = 0$
$\Rightarrow ax + ary + ar^2 = 0 \Rightarrow x + ry + r^2 = 0$ (i)

(i) intersects the curves $x + 2y^2 = 0$ at the points whose ordinates are given by

$-2y^2 + ry + r^2 = 0$ or $2y^2 - ry - r^2 = 0$

Therefore required sum of the ordinates = $r/2$.

Example 2 Orthocentre of the triangle with vertices (0, 0), (3, 4) and (4, 0) is

(a) (3, 5/4) (b) (3, 12)
(c) (3, 3/4) (d) (3, 9)

Ans. (c)

Solution Let $O(0, 0)$, $A(3, 4)$ and $B(4, 0)$ then equation of OB is $y = 0$. Equations of the altitudes from A and O are respectively $x = 3$ and $y = (1/4)x$ so the orthocentre is (3, 3/4), the point of intersection of these altitudes.

Example 3 The number of integral points (integral point means both the coordinates should be integer) that lie exactly in the interior of the triangle with vertices (0, 0), (0, 21) and (21, 0) is

(a) 133 (b) 190
(c) 233 (d) 105

Ans. (b)

Solution If (x, y) is an interior point of the triangle OAB then $x + y < 21$. Possible values of x are 1, 2, ... 19 and the corresponding values of y are 19 for $x = 1$, 18 for $x = 2$ and so on 1 for $x = 19$ as $x + y \leq 20$, the total number of points are $19 + 18 + \ldots + 1 = \frac{19 \times 20}{2} = 190$

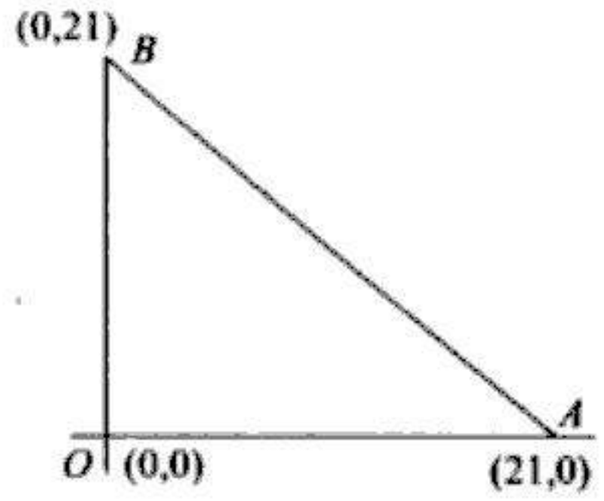

Fig. 15.1

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Example 18 If a line joining two points A (2, 0) and B (3, 1) is rotated about A in anticlockwise direction through an angle 15°, then equation of the line in the new position is

(a) $\sqrt{3}\,x + y = 2\sqrt{3}$ (b) $\sqrt{3}\,x - y = 2\sqrt{3}$

(c) $x + \sqrt{3}y = 2\sqrt{3}$ (d) $x - \sqrt{3}y = 2\sqrt{3}$

Ans. (b)

Solution Slope of $AB = \dfrac{1-0}{3-2} = 1$

$\Rightarrow$ $\angle BAX = 45°$ (Fig. 15.7)

If AC is the new position of the line AB then $\angle CAX = 45° + 15° = 60°$.

and thus its equation is

$$y = \tan 60°\,(x - 2)$$

$$\Rightarrow \quad y = \sqrt{3}\,(x - 2) \quad \Rightarrow \quad \sqrt{3}x - y = 2\sqrt{3}$$

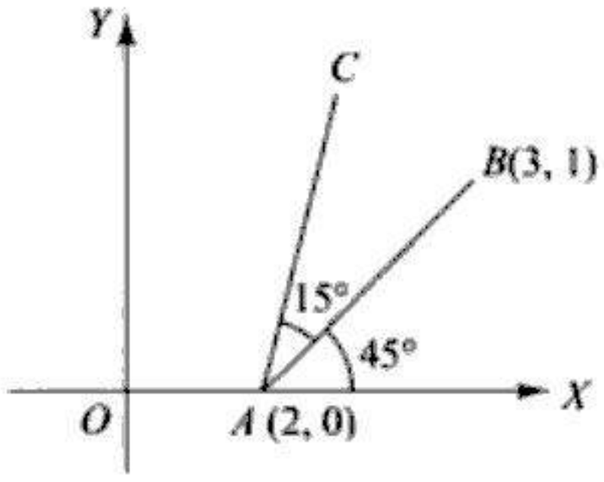

Fig. 15.7

Example 19 An equation of a line through the point (1, 2) whose distance from the point (3, 1) has the greatest value is

(a) $y = 2x$ (b) $y = x + 1$

(c) $x + 2y = 5$ (d) $y = 3x - 1$

Ans. (a)

Solution Let the equation of the line through (1, 2) be $y - 2 = m(x - 1)$

If p denotes the length of the perpendicular from (3, 1) on this line, then

$$p = \left|\frac{2m+1}{\sqrt{m^2+1}}\right|$$

$$\Rightarrow \quad p^2 = \frac{4m^2+4m+1}{m^2+1} = 4 + \frac{4m-3}{m^2+1} = s \text{ (say)},$$

then p^2 is greatest if and only if s is greatest

Now $$\frac{ds}{dm} = \frac{(m^2+1)(4) - 2m(4m-3)}{(m^2+1)^2}$$

$$= \frac{-2(2m+1)(m-2)}{(m^2+1)^2}$$

$\dfrac{ds}{dm} = 0 \Rightarrow m = -1/2, 2$. Also $\dfrac{ds}{dm} < 0$ if $m < -1/2$

> 0 if $-1/2 < m < 2$

< 0 if $m > 2$.

So s is greatest for $m = 2$

and thus the equation of the required line is $y = 2x$.

Note
p will be least when (3, 1) lies on the line through (1, 2) and the equation of the line in that case is $x + 2y = 5$.

Example 20 Let $0 < \alpha < \pi/2$ be a fixed angle. If $P = (\cos\theta, \sin\theta)$ and $Q = (\cos(\alpha - \theta), \sin(\alpha - \theta))$ then Q is obtained from P by

(a) clockwise rotation around the origin through an angle α

(b) anticlockwise rotation around the origin through an angle α

(c) reflection in the line through origin with slope $\tan\alpha$

(d) reflection in the line through the origin with slope $\tan(\alpha/2)$

Ans. (a)

Solution OP makes an angle θ with the positive direction of x-axis and OQ makes an angle $(\alpha - \theta)$ with the positive direction of x-axis.(Fig.15.8)

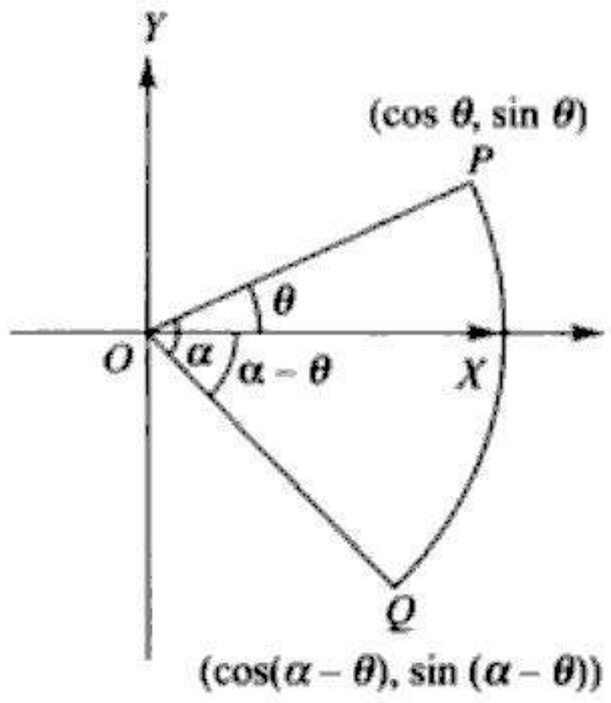

Fig. 15.8

So that $\angle POQ = \alpha$ and thus Q is obtained from P by clockwise rotation through an angle α around the origin.

Note
In the anticlockwise rotation, OQ makes $(2\pi - \alpha + \theta)$ with the + ve direction of x-axis.

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(a) $(-8/3)\sin^2\alpha$ (b) $(-8/3)\operatorname{cosec} 2\alpha$

(c) $-8\sqrt{3}\operatorname{cosec} 2\alpha$ (d) $-4\operatorname{cosec} 2\alpha$

Ans. (b)

Solution The given equation can be written as

$(x^2+y^2)(\cos^2\theta\sin^2\alpha+\sin^2\theta)=x^2\tan^2\alpha-2xy\tan\alpha\sin\theta+y^2\sin^2\theta$ or $(\cos^2\theta\sin^2\alpha+\sin^2\theta-\tan^2\alpha)x^2+2(\tan\alpha\sin\theta)xy+\cos^2\theta\sin^2\alpha y^2=0$

Since the slope of these lines are given as $\tan\theta_1$ and $\tan\theta_2$.

$$\text{and sum of the slopes} = \frac{-2\tan\alpha\sin\theta}{\cos^2\theta\sin^2\alpha}$$

$$\Rightarrow \quad \tan\theta_1+\tan\theta_2=\frac{-2\tan\alpha}{2\times(3/4)\times\sin^2\alpha} \quad (\because\ \theta=\pi/6)$$

$$=-(8/3)\operatorname{cosec} 2\alpha.$$

Example 35 If two of the lines represented by

$$x^4+x^3y+cx^2y^2-xy^3+y^4=0$$

bisect the angle between the other two, then the value of c is

(a) 0 (b) -1 (c) 1 (d) -6

Ans. (d)

Solution Since the product of the slopes of the four lines represented by the given equation is 1 and a pair of lines represent the bisectors of the angles between the other two, the product of the slopes of each pair is -1. So let the equation of one pair be $ax^2+2hxy-ay^2=0$.

The equation of its bisectors is $\dfrac{x^2-y^2}{2a}=\dfrac{xy}{h}$.

By hypothesis

$$x^4+x^3y+cx^2y^2-xy^3+y^4$$
$$=(ax^2+2hxy-ay^2)(hx^2-2axy-hy^2)$$
$$=ah(x^4+y^4)+2(h^2-a^2)(x^3y-xy^3)-6ahx^2y^2$$

Compairing the respective co-efficients we get

$$ah=1 \text{ and } c=-6ah=-6$$

SECTION II

MULTIPLE CORRECT CHOICE TYPE

Example 36 The point of intersection of the lines

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{and} \quad \frac{x}{b}+\frac{y}{a}=1$$

lies on the line

(a) $x-y=0$

(b) $(x+y)(a+b)=2ab$

(c) $(lx+my)(a+b)=(l+m)ab$

(d) $(lx-my)(a+b)=(l-m)ab$.

Ans. (a), (b), (c) and (d)

Solution Solving the given equations, we find that the point of intersection of the given lines is

$$\left(\frac{ab}{a+b},\frac{ab}{a+b}\right)$$

which lies on all the lines given by (a), (b), (c) and (d).

Example 37 Equations

$$(b-c)x+(c-a)y+(a-b)=0$$

and $(b^3-c^3)x+(c^3-a^3)y+a^3-b^3=0$

will represent the same line if

(a) $b=c$ (b) $c=a$

(c) $a=b$ (d) $a+b+c=0$.

Ans. (a), (b), (c) and (d)

Solution The two lines will be identical if there exists some real number k, such that

$$b^3-c^3=k(b-c),\ c^3-a^3=k(c-a)$$

and $a^3-b^3=k(a-b)$.

$\Rightarrow$ $b-c=0$ or $b^2+c^2+bc=k$,

$c-a=0$ or $c^2+a^2+ca=k$

and $a-b=0$ or $a^2+b^2+ab=k$

That is, $b=c$ or $c=a$ or $a=b$.

Next $b^2+c^2+bc=c^2+a^2+ca \Rightarrow b^2-a^2=c(a-b)$

Hence, $a=b$ or $a+b+c=0$.

Example 38 The straight lines $x+y=0$, $3x+y-4=0$ and $x+3y-4=0$ from a triangle which is

(a) isosceles (b) right-angled

(c) obtuse-angled (d) equilateral.

Ans. (a), (c)

Solution Let the given lines be represented by AB, BC and CA, respectively. Then $A(-2, 2)$, $B(2, -2)$ and $C(1, 1)$ are the vertices of a triangle ABC. Also, note that $AB=4\sqrt{2}$, $AC=\sqrt{10}$ and $BC=\sqrt{10}$. This shows that ΔABC is isosceles; it is clearly not right-angled or equilateral. Since

$$\cos C=\frac{BC^2+AC^2-AB^2}{2(BC)(AC)}=\frac{10+10-32}{2\sqrt{10}\sqrt{10}}<0$$

$\Rightarrow \Delta ABC$ is obtuse-angled.

Example 39 Equation of a straight line passing through the point (4, 5) and equally inclined to the lines $3x=4y+7$ and $5y=12x+6$ is

(a) $9x-7y=1$ (b) $9x+7y=71$

(c) $7x+9y=73$ (d) $7x-9y+17=0$.

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Equation of any line parallel to (ii) is

$$3x + 4y = k$$

Since its distance from (iii) is 7

$$\left|\frac{k}{\sqrt{3^2+4^2}}\right| = 7 \quad \Rightarrow \quad k = \pm(7 \times 5) = \pm 35$$

Example 53 The lines joining the origin to the point of intersection of $3x^2+\lambda xy-4x+1=0$ and $2x+y-1=0$ are at right angles for

(a) $\lambda = -4$ (b) $\lambda = 4$

(c) $\lambda = 7$ (d) no value of λ

Ans. (a), (b) and (c)

Solution Equation of the lines joining the origin to the points of intersection of the given lines is

$$3x^2 + \lambda xy - 4x(2x + y) + 1\cdot(2x + y)^2 = 0$$

(Making the equation of the pair of lines homogeneous with the help of the equation of the line)

$\Rightarrow \quad x^2 - \lambda xy - y^2 = 0$

which are perpendicular for all values of λ.

Example 54 If $x^2 + 2hxy + y^2 = 0$ represents the equations of the straight lines through the origin which make an angle α with the straight line $y+x=0$, then

(a) $\sec 2\alpha = h$ (b) $\cos\alpha = \sqrt{\frac{1+h}{2h}}$

(c) $2\sin\alpha = \sqrt{\frac{1+h}{h}}$ (d) $\cot\alpha = \sqrt{\frac{h+1}{h-1}}$

Ans. (a), (b) and (d)

Solution Let equation of the lines given by $x^2 + 2hxy + y^2 = 0$ be $y = m_1x$ and $y = m_2x$. Since these make an angle α with $y + x = 0$ whose slope is -1,

$$\frac{m_1+1}{1-m_1} = \tan\alpha = \frac{-1-m_2}{1-m_2}$$

$$\Rightarrow \quad m_1 = \frac{\tan\alpha - 1}{\tan\alpha + 1} \text{ and } m_2 = \frac{\tan\alpha + 1}{\tan\alpha - 1}$$

$$\Rightarrow \quad m_1 + m_2 = \frac{(\tan\alpha-1)^2 + (\tan\alpha+1)^2}{\tan^2\alpha - 1}$$

$$= -\frac{2\sec^2\alpha \times \cos^2\alpha}{\cos 2\alpha} = -2\sec 2\alpha = -2h.$$

$$\Rightarrow \quad \sec 2\alpha = h$$

$$\Rightarrow \quad \cos 2\alpha = \frac{1}{h} \quad \Rightarrow \quad 2\cos^2\alpha - 1 = \frac{1}{h}$$

$$\Rightarrow \quad \cos\alpha = \sqrt{\frac{1+h}{2h}} \text{ and } \cot\alpha = \sqrt{\frac{h+1}{h-1}}$$

Example 55 If $y = mx$ bisects the angle between the lines $x^2(\tan^2\theta + \cos^2\theta) + 2xy\tan\theta - y^2\sin^2\theta = 0$ when $\theta = \pi/3$ the value of m is

(a) $\frac{-2-\sqrt{7}}{\sqrt{3}}$ (b) $\frac{\sqrt{7}-2}{\sqrt{3}}$

(c) $2\sqrt{7}$ (d) $2\sqrt{3}$

Ans. (a) and (b)

Solution Equation of the bisectors of the angles between the given lines is

$$\frac{x^2-y^2}{\tan^2\theta + \cos^2\theta + \sin^2\theta} = \frac{xy}{\tan\theta}$$

$$\Rightarrow \quad \frac{x^2-y^2}{1+\tan^2\theta} = \frac{xy}{\tan\theta}$$

$$\Rightarrow \quad \frac{x^2-y^2}{1+3} = \frac{xy}{\sqrt{3}} \text{ when } \theta = \pi/3$$

which is satisfied by $y = mx$ if

$$\frac{1-m^2}{4} = \frac{m}{\sqrt{3}}$$

$$\Rightarrow \quad \sqrt{3}\,m^2 + 4m - \sqrt{3} = 0.$$

$$\Rightarrow \quad m = \frac{-2 \pm \sqrt{7}}{\sqrt{3}}$$

SECTION III

LINKED COMPREHENSION TYPE

Paragraph for Question Nos. 56 to 58

$A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of a triangle ABC. $lx + my + n = 0$ is an equation of the line L.

Example 56 If L intersects the sides BC, CA and AB of the triangle ABC at P, Q, R respectively then $\frac{BP}{PC}\times\frac{CQ}{QA}\times\frac{AR}{RB}$ is equal to

(a) -1 (b) $-1/2$

(c) $1/2$ (d) 1

Example 57 If the centroid of the triangle ABC is at the origin and algebraic sum of the lengths of the perpendiculars from the vertices of the triangle ABC on the line L is equal to 1 then sum of the squares of the intercepts made by L on the coordinate axes is equal to

(a) 0 (b) 4

(c) 9 (d) 16

Example 58 If P divides BC in the ratio 2 : 1 and Q divides CA in the ratio 1 : 3 then R divides AB in the ratio (P,Q,R are the points as in Ex. 46)

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(b) $\dfrac{2\sqrt{(5/2)^2-(2)(3)}}{2+3}=\dfrac{1}{5}$

(c) $\dfrac{2\sqrt{4-1}}{1+1}=\sqrt{3}$, lines in (d) are perpendicular to those in (a)

Example 88

(a)	If the slope of one of the lines represented by $ax^2 - 6xy + y^2 = 0$ is square of the other then	(p) $a = -4$
(b)	If $9x^2 + 2(a - 2)xy + 4y^2 + 6x + 4y - 3$ represent two parallel lines, then	(q) $a = 8$
(c)	If a denotes the slope of a line given by $108x^2 + 31xy + y^2 = 0$, then	(r) $a = -27$
(d)	If $a^2x^2 + 15xy - 16y^2$ represent a pair of two perpendicular lines then	(s) $a = 4$

Ans.

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

Solution

(a) $m + m^2 = 6$, $m^3 = a \Rightarrow$ $m = 2$ or -3 and $a = 8$ or -27.

(b) The lines are parallel if
$9 \times 4 = (a - 2)^2 \Rightarrow a - 2 = \pm 6$
$\Rightarrow$ $a = 8$ or $a = -4$
But the equation represents a pair of lines if

$$\begin{vmatrix} 9 & a-2 & 3 \\ a-2 & 4 & 2 \\ 3 & 2 & -3 \end{vmatrix} = 0 \text{ or if } a = \pm 8$$

So we get $a = 8$

(c) We have $(27x + y)(4x + y) = 0$
Slopes are -27 or -4 *i.e.* $a = -27$ or $a = -4$

(d) Lines are perpendicular if $a^2 - 16 = 0$
$\Rightarrow$ $a = \pm 4$

Example 89

The lines given by

(a)	$6x^2 + 5xy - 6y^2 - x + 5y - 1 = 0$	(p) are parallel
(b)	$30x^2 + 36xy + 6y^2 - 35x - 11y + 5 = 0$	(q) are perpendicular
(c)	$30x^2 + 41xy + 6y^2 - 35x - 11y + 5 = 0$	(r) make an intercept 5/6 on x-axis
(d)	$6x^2 + 12xy + 6y^2 - 7x - 7y + 1 = 0$	(s) make an intercept 1/6 on y-axis.

Ans.

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

Solution

(a) Co-efficient of a^2 + co-efficient of $y^2 = 0$
$\Rightarrow$ line are perpendicular
Meet x-axis at $6x^2 - x - 1 = 0 \Rightarrow (2x - 1)(3x + 1) = 0$
$\Rightarrow$ $x = 1/2$ or $x = -1/3 \Rightarrow$ intercept on x-axis $= \dfrac{1}{2}+\dfrac{1}{3}=\dfrac{5}{6}$
Meet y-axis at $-6y^2 + 5y - 1 = 0$
$\Rightarrow$ $(2y - 1)(3y - 1) = 0$
$\Rightarrow$ $y = 1/2$ or $y = 1/3 \Rightarrow$ intercept on y-axis $= 1/2 - 1/3 = 1/6$.

(b) The lines are neither parallel nor perpendicular meet x-axis at $30x^2 - 35x + 5 = 0 \Rightarrow$ $x = 1$ or $x = 1/6$ so the intercept on x-axis $= 1 - 1/6 = 5/6$.
Similarly intercept on y-axis $= 1 - 5/6 = 1/6$.

(c) As in (b).

(d) The lines are parallel as $6 \times 6 = (6)^2$ ($h^2 = ab$) meet x-axis at $6x^2 - 7x + 1 = 0$
$\Rightarrow$ $x = 1$ or $x = 1/6$
so intercept on x-axis $= 1 - 1/6 = 5/6$.
intercept on y-axis is also $5/6$.

SECTION VI

REASONING TYPE

Example 90 **Statement-1:** If the circumcentre of a triangle lies at the origin and centroid is the middle point of the line joining the points (2, 3) and (4, 7), then its orthocentre lies on the line $5x - 3y = 0$.

Statement-2: The circumcentre, centroid and the orthocentre of a triangle lie on the same line.

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Statement-2: If a and b are constants ($a, b \neq 0$) and c is a variable, then from the lines $ax + by + c = 0$ and $bx - ay - c = 0$ the line farthest from the origin is the one for which $|c|$ is least.

52. **Statement-1:** If the area of the triangle formed by the lines $y = x$, $x + y = 4$ and the lines through $P(h, k)$ parallel to y-axis is k^2, then P lies on $x^2 + y^2 - 4x + 4 = 0$
Statement-2: Area of the triangle formed by the lines $y = x$, $x + y = 4$ and the x-axis is equal to half the area of the triangle formed by the line $x + y = 4$ and the coordinate axes.

53. **Statement-1:** Reflexion of the point (6, 1) in the line $x + y = 0$ is $(-1, -6)$.
Statement-2: Reflexion of a point $P(\alpha, \beta)$ in the line $ax + by + c = 0$ is $Q(\alpha', \beta')$ if $\left(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2}\right)$ lies on the line.

54. **Statement-1:** The product of the perpendiculars from the point (1, 3) on the lines $15x^2 + 14xy - 8y^2 = 0$ is $3\sqrt{2}/4$.
Statement-2: The product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is $\left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$.

55. **Statement-1:** Angle between the lines represented by $3x^2 + 10xy + 3y^2 - 2x + 2y - 1 = 0$ and $3x^2 + 10xy + 3y^2 - 11x + 7y - 20 = 0$ is same.
Statement-2: Equations of the bisectors of the angles between the lines $3x^2 + 10xy + 3y^2 = 0$ is $x^2 - y^2 = 0$

LEVEL 2

SECTION I

SINGLE CORRECT CHOICE TYPE

1. Triangle ABC has vertices (0, 0), (11, 60) and (91, 0). If the line $y = kx$ cuts the triangle into two triangles of equal area, then k is equal to
   (a) 30/51 (b) 4/7
   (c) 7/4 (d) 30/91
2. The distance of the point (1, 2) from the line $x + y + 5 = 0$ measured along the line parallel to $3x - y = 7$ is equal to
   (a) $4/\sqrt{10}$ (b) 40
   (c) $\sqrt{40}$ (d) $10\sqrt{2}$
3. A ray of light travels along the line $2x - 3y + 5 = 0$ and strikes a plane mirror lying along the line $x + y = 2$. The equation of the straight line containing the reflected ray is
   (a) $2x - 3y + 3 = 0$ (b) $3x - 2y + 3 = 0$
   (c) $21x - 7y + 1 = 0$ (d) $21x + 7y - 1 = 0$
4. Given two points $A = (-2, 0)$ and $B = (0, 4)$, the coordinates of a point M lying on the line $x = y$ so that the perimeters of the ΔAMB is least; is
   (a) (1, 1) (b) (0, 0)
   (c) (2, 2) (d) (3, 3)
5. If the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$, then
   (a) $-1/2 < \alpha < 1/2$ (b) $-3/2 < \alpha < 1$
   (c) $-3/2 < \alpha < 3/2$ (d) $-3/2 < \alpha < -1/2$
6. If $u = a_1x + b_1y + c_1 = 0$ and $v = a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then $u + kv = 0$ represents
   (a) $u = 0$
   (b) a family of concurrent lines
   (c) a family of parallel lines
   (d) none of these
7. The point (4, 1) undergoes the following successive transformations:
   (i) reflection about the line $y = x$
   (ii) translation through a distance 2 units along the positive x-axis.
   then, the final coordinates of the point are
   (a) (4, 3) (b) (3, 4)
   (c) (1, 4) (d) (4, 4)
8. A line has intercepts a and b on the coordinate axes. When the axes are rotated through an angle α, keeping the origin fixed, the line makes equal intercepts on the coordinate axes, then $\tan \alpha =$
   (a) $\frac{a+b}{a-b}$ (b) $\frac{a-b}{a+b}$
   (c) $a^2 - b^2$ (d) none of these
9. If h denotes the arithmetic mean and k denotes the geometric mean of the intercepts made on the coordinate axes by the lines passing through the point (1, 1), then the point (h, k) lies on
   (a) a circle (b) a parabola
   (c) a straight line (d) a hyperbola
10. From the vertex of the isosceles triangle with vertices (0, 0), $(p, 0)$ and $(p/2, q)$ an arc is described with a radius equal to half the base, meeting the base at a point which divides it into two parts whose lengths are the roots of the equation.

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36. Point of intersection of P is (0, 1). Required equation is $(x + y - 3)(x - y + 1) = 0$

37. Given equations represent a rhombus with sides $3x \pm 4y = 12$ and $3x \pm 4y = -12$, Length of the diagonals are 6 and 8 units. $A = \frac{1}{2} \times 6 \times 8 = 24$

$4A^2 + A + 1 = 2329$

38. $h = \frac{p}{2\cos\alpha}, k = \frac{p}{2\sin\alpha}$

$\Rightarrow \frac{1}{h^2} + \frac{1}{k^2} = \frac{4}{p^2} \Rightarrow$ locus of (h, k) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$

it passes through $(p + 1, p - 1)$

$$\frac{1}{(p+1)^2} + \frac{1}{(p-1)^2} = \frac{4}{p^2} \Rightarrow p^4 - 5p^2 + 2 = 0$$

So $71(4p^4 - 20p^2 + 99) = 71(-8 + 99) = 71 \times 91 = 6461$

39. Locus is $x^2 + y^2 = a^2 + b^2$, since it passes through (3, 4), $a^2 + b^2 = 25$. Also $3a - 4b = 0 \Rightarrow a^2 = 16$, $b^2 = 9$ and $9a^4 + 16b^4 + 34 = 9 \times 256 + 16 \times 81 + 34 = 3634$

40. Let the slopes be $\frac{m}{r}$, m and mr,

product of slopes $= m^3 = \frac{24}{3} = 8 \Rightarrow m = 2$

$\therefore \quad 24 - am + 26m^2 - 3m^3 = 0$

$\Rightarrow \quad 24 - 2a + 26 \times 4 - 3 \times 8 = 0 \Rightarrow a = 52$

$24x^3 - 52x^2y + 26xy^2 - 3y^3 = 0$

$\Rightarrow \quad (2x - y)(12x^2 - 20xy + 3y^2) = 0$

$\Rightarrow \quad (2x - y)(6x - y)(2x - 3y) = 0$

The lines are $2x - y = 0$, $6x - y = 0$ and $2x - 3y = 0$

Perpendicular distances from (1, 1) are

$\frac{1}{\sqrt{5}}, \frac{5}{\sqrt{37}}, \frac{1}{\sqrt{13}}$, so $p = \frac{5}{\sqrt{37}}$

$a^2 + 37p^2 = (52)^2 + (5)^2 = 2729$

41. Vertices of the parallelogram are $A(2, 1)$, $B(2, 5)$, $C(3, 5)$, $D(3, 1)$

Equation of AC, $4x - y = 7$;

Equation of $BD = 4x + y = 13$

$a = \frac{13}{4} - \frac{7}{4} = \frac{3}{2}$, $b = 13 + 7 = 20$

$\Rightarrow \quad (2a + 3b)^2 = (63)^2 = 3969$

42. If $y = mx + c$ then (a) $m = \sqrt{2}$, $c = 5$, (b) $m = -\frac{1}{\sqrt{2}}$, $c = -3$.

similarly for (c) and (d)

43. Write the equation in the form $\frac{x}{a} + \frac{y}{b} = 1$ and the required sum is $a^2 + b^2$

44. (a) The lines are concurrent if

$$\begin{vmatrix} 1 & 3 & -5 \\ 3 & -K & -1 \\ 5 & 2 & -12 \end{vmatrix} = 0 \Rightarrow K = 5$$

(b) Slope of $L_2 = \frac{3}{K} = -\frac{5}{2}$ if $K = -6/5$

and $\frac{3}{K} = -\frac{1}{3}$ if $K = -9$

(c), (d) The lines do not form a triangle if they are concurrent or any two of them are parallel.

45. (a) $3.1 + 4.1 + 7 = 14$

$3(-3) + 4(-3) + 7 = -14$

Similarly for other parts.

46. Match with the point of intersection of the lines

47. Area of the triangle formed by the lines in

(a) $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \frac{5}{2}$

Similarly for other parts.

48. (a) $(x + y)(2x - 3y)$

(b) $(2x - 3y)(3x + 4y)$

(c) $(x - y)(2x - 3y)$

(d) $(3x + 4y)(x - y)$

49. Vertices are (0, 0), $(k/2, 0)$, $(0, k/3)$. It is a right angled triangle, right angle at the origin. So orthocentre is at the origin for all values of k

Circum-centre is $\left(\frac{k}{4}, \frac{k}{6}\right) = (3, 2)$ if $k = 12$

Centroid is $\left(\frac{k}{6}, \frac{k}{9}\right) = (3, 2)$ if $k = 18$

Area of the triangle $= \frac{1}{2} \times \frac{k}{2} \times \frac{k}{3} = 192$ if $k^2 = (48)^2$

50. Statement-2 is true (See Text)

Equation in statement-1 is

$x + \sqrt{3}y + 5 + \cot\theta(\sqrt{3}x - y - 7) = 0$ where $\cot\theta$ is non-zero finite real number.

As $\theta \neq n\pi/2$

Using statement-2, Statement-1 is True.

51. Statement-2 is false as the distance of any such line from the origin is

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$S + \lambda L = 0$ represents a family of circles passing through these points.

5. If $L = 0$ is a tangent to the circle $S = 0$ at P, then $S + \lambda L = 0$ represents a family of circles touching $S = 0$ at P, and having $L = 0$ as the common tangent at P.

6. *Coaxal Circles* A system of circle is said to be coaxal if every pair of circles of the system have the same radical axis.

The simplest form of the equation of a coaxal system of circles is

$x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is constant, the common radical axis of the system being y-axis and the line of centres being x-axis.

The Limiting points of the coaxal system of circles are the members of the system which are of zero radius. Thus the limiting points of the coaxal system of circles

$$x^2 + y^2 + 2gx + c = 0 \text{ are } \left(\pm\sqrt{c}, 0\right) \text{ if } c \geq 0.$$

The equation $S + \lambda S' = 0$ $(\lambda \neq -1)$ represents a family of coaxal circles, two of whose members are given to be $S = 0$ and $S' = 0$.

Conjugate systems (or orthogonal systems) of circles Two system of circles such that every circle of one system cuts every circle of the other system orthogonally are said to be conjugate system of circles. For instance,

$$x^2 + y^2 + 2gx + c = 0 \text{ and } x^2 + y^2 + 2fy - c = 0,$$

where g and f are variables and c is constant, represent two systems of coaxal circles which are conjugate.

16.6 COMMON TANGENTS TO TWO CIRCLES

If $(x - g_1)^2 + (y - f_1)^2 = a_1^2$ and $(x - g_2)^2 + (y - f_2)^2 = a_2^2$ are two circles with centres C_1 (g_1, f_1) and C_2 (g_2, f_2) and radii a_1 and a_2 respectively, then we have the following results regarding their common tangents.

1. When $C_1\ C_2 > a_1 + a_2$ i.e. distance between the centres is greater than the sum of their radii (Fig. 16.2), the two circles do not intersect with each other and *four* common tangents can be drawn to two circles. Two of them are *direct common tangents* and the other two are *transverse common tangents*. The points T_1, T_2 of intersection of direct common tangents and transverse common tangents respectively, always lie on the line joining the centres of the two circles and divide it externally and internally respectively in the ratio of their radii.

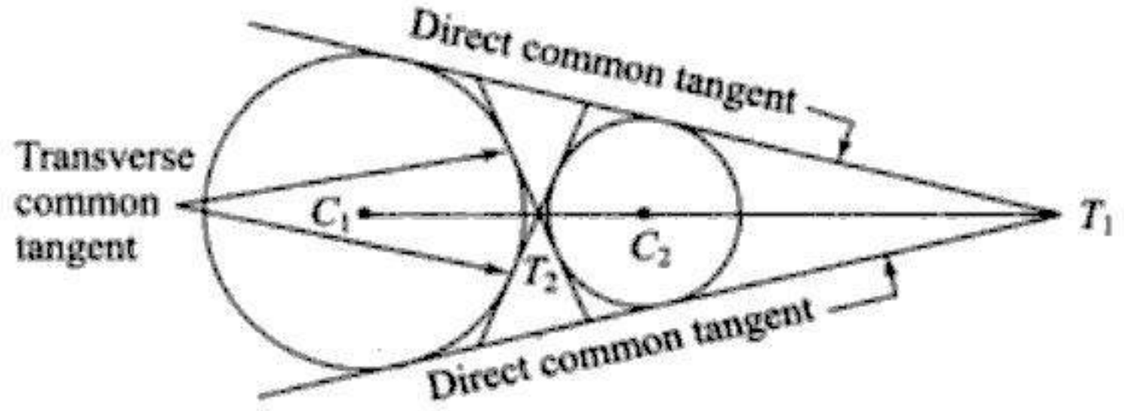

Fig. 16.2

$$\frac{C_1T_1}{T_1C_2} = \frac{a_1}{a_2} \text{ (externally) } \frac{C_1T_2}{T_2C_2} = \frac{a_1}{a_2} \text{ internally}$$

2. When $C_1\ C_2 = a_1 + a_2$ i.e. the distance between the centres is equal to the sum of the radii (Fig. 16.3), the two circles touch each other externally, two direct tangents are real and distinct and the tranverse tangents coincide.

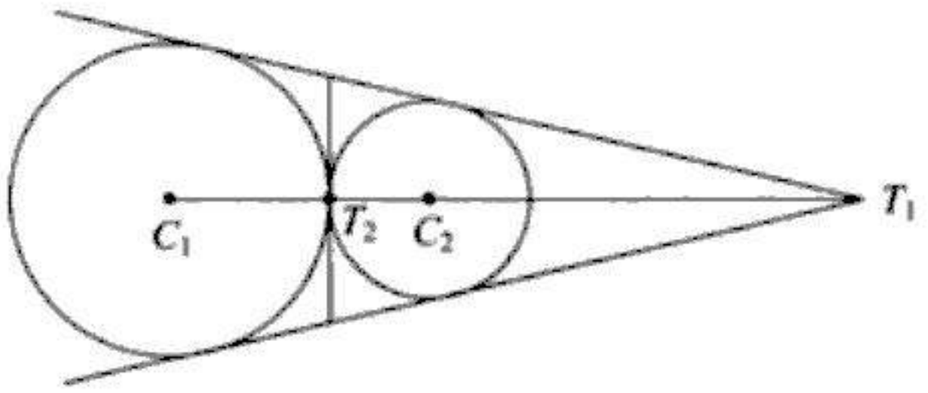

Fig. 16.3

3. When $C_1\ C_2 < a_1 + a_2$ i.e. the distance between the centres is less than the sum of the radii (Fig . 16.4), the circles intersect at two real and distinct points, the two direct common tangents are real and distinct while the transverse common tangents are imaginary.

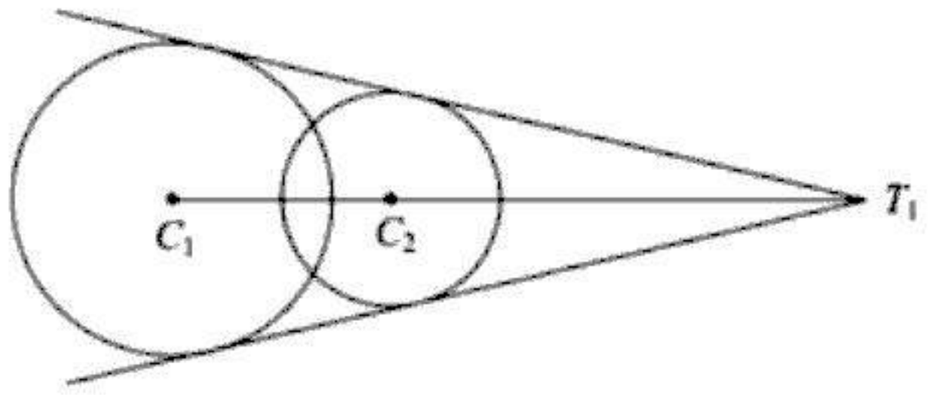

Fig. 16.4

4. When $C_1\ C_2 = |a_1 - a_2|$ $(a_1 \neq a_2)$ i.e. the distance between the centres is equal to the difference of their radii (Fig. 16.5), the circles touch each other internally, two direct common tangents are real and coincident while the transverse common tangents are imaginary.

5. When $C_1\ C_2 < |a_1 - a_2|$ $a_1 \neq a_2$ i.e. the distance between the centres is less than the difference of the radii (Fig. 16.6), one circle with smaller radius lies inside the other and the four common tangents are all imaginary.

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Example 11 Equation of a circle through the origin and belonging to the co-axial system, of which the limiting points are (1, 2), (4, 3) is

(a) $x^2 + y^2 - 2x + 4y = 0$
(b) $x^2 + y^2 - 8x - 6y = 0$
(c) $2x^2 + 2y^2 - x - 7y = 0$
(d) $x^2 + y^2 - 6x - 10y = 0$

Ans. (c)

Solution Since the limiting points of a system of co-axial circles are the point circles (radius being zero), two members of the system are

$(x - 1)^2 + (y - 2)^2 = 0 \Rightarrow x^2 + y^2 - 2x - 4y + 5 = 0$
and $(x - 4)^2 + (y - 3)^2 = 0 \Rightarrow x^2 + y^2 - 8x - 6y + 25 = 0.$

The co-axial system of circles with these as members is

$x^2 + y^2 - 2x - 4y + 5 + \lambda (x^2 + y^2 - 8x - 6y + 25) = 0$

It passes through the origin if $5 + 25\lambda = 0$

or $\lambda = -(1/5),$

which gives the equation of the required circle as

$5(x^2 + y^2 - 2x - 4y + 5) - (x^2 + y^2 - 8x - 6y + 25) = 0$
$\Rightarrow 4x^2 + 4y^2 - 2x - 14y = 0$
$\Rightarrow 2x^2 + 2y^2 - x - 7y = 0.$

Example 12 If a line segment $AM = a$ moves in the plane XOY remaining parallel to OX so that the left end point A slides along the circle $x^2 + y^2 = a^2$, the locus of M is

(a) $x^2 + y^2 = 4a^2$ (b) $x^2 + y^2 = 2ax$
(c) $x^2 + y^2 = 2ay$ (d) $x^2 + y^2 - 2ax - 2ay = 0$

Ans. (b).

Solution Let the coordinates of A be (x, y) and M be (α, β) (Fig. 16.13).

Since AM is parallel to OX,

$\alpha = x + a$ and $\beta = y \quad \Rightarrow \quad x = \alpha - a$ and $y = \beta$

As $A(x, y)$ lies on the circle $x^2 + y^2 = a^2$ we have

$$(\alpha - a)^2 + \beta^2 = a^2$$
$$\Rightarrow \quad \alpha^2 - 2a\alpha + \beta^2 = 0$$
$$\Rightarrow \quad \text{locus of } M(\alpha, \beta) \text{ is } x^2 + y^2 = 2ax.$$

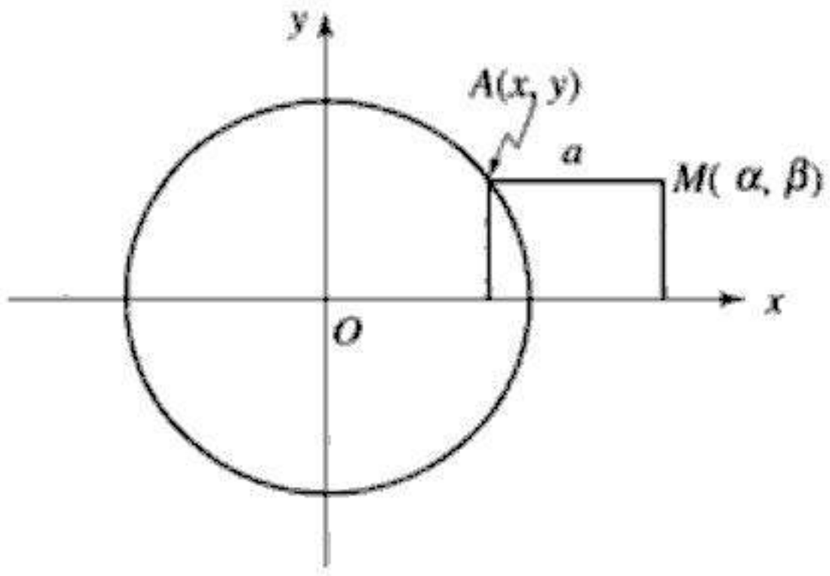

Fig. 16.13

Example 13 If common chord of the circle C with centre at (2, 1) and radius r and the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a diameter of the second circle, then value of r is

(a) 3 (b) 2
(c) 3/2 (d) 1

Ans. (a)

Solution Equation of the circle C is

$$(x - 2)^2 + (y - 1)^2 = r^2$$
$$\Rightarrow \quad x^2 + y^2 - 4x - 2y + 5 - r^2 = 0$$

Equation of the common chord is

$(x^2 + y^2 - 4x - 2y + 5 - r^2) - (x^2 + y^2 - 2x - 6y + 6) = 0$
$\Rightarrow 2x - 4y + r^2 + 1 = 0$

If it is a diameter of the second circle, it passes through the centre (1, 3) of the circle

So $2 - 4 \times 3 + r^2 + 1 = 0 \Rightarrow r^2 = 9 \Rightarrow r = 3$

Example 14 Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle in PAB is

(a) $x^2 + y^2 + 4x - 6y + 19 = 0$
(b) $x^2 + y^2 - 4x - 10y + 19 = 0$
(c) $x^2 + y^2 - 2x + 6y - 29 = 0$
(d) $x^2 + y^2 - 6x - 4y + 19 = 0$

Ans. (b)

Solution Equation of AB the Chord of contact of p is

$1 \cdot x + 8y - 3(x + 1) - 2(y + 8) - 11 = 0$
$\Rightarrow -2x + 6y - 30 = 0$
$\Rightarrow x - 3y + 15 = 0$

Equation of any circle through. AB is

$x^2 + y^2 - 6x - 4y - 11 + \lambda(x - 3y + 15) = 0$

It will pass through $P(1, 8)$ if

$1 + 64 - 6 - 32 - 11 + \lambda (1 - 24 + 15) = 0$
$\Rightarrow 16 - 8\lambda = 0 \quad \Rightarrow \quad \lambda = 2$

Thus, equation of the required circle is

$x^2 + y^2 - 6x - 4y - 11 + 2(x - 3y + 15) = 0$
$\Rightarrow x^2 + y^2 - 4x - 10y + 19 = 0$

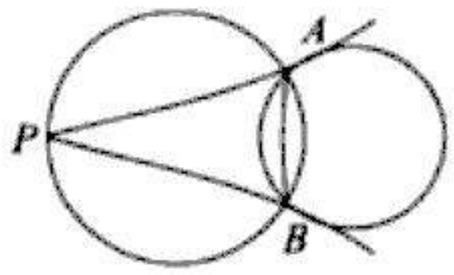

Fig. 16.14

Example 15 Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

(a) 3 (b) 2
(c) 3/2 (d) 1

Ans. (b)

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$\Rightarrow \quad \angle OBP = \theta$

Equation of the tangent OT at O to the given circle is

$$2x + 3y = 0$$

$\Rightarrow$ slope of the tangent $= -2/3$

So that, if $\angle XOT = \alpha$, $\tan \alpha = 2/3$

From geometry, $\angle POT = \angle OBP = \theta \Rightarrow \angle POX = \theta - \alpha$

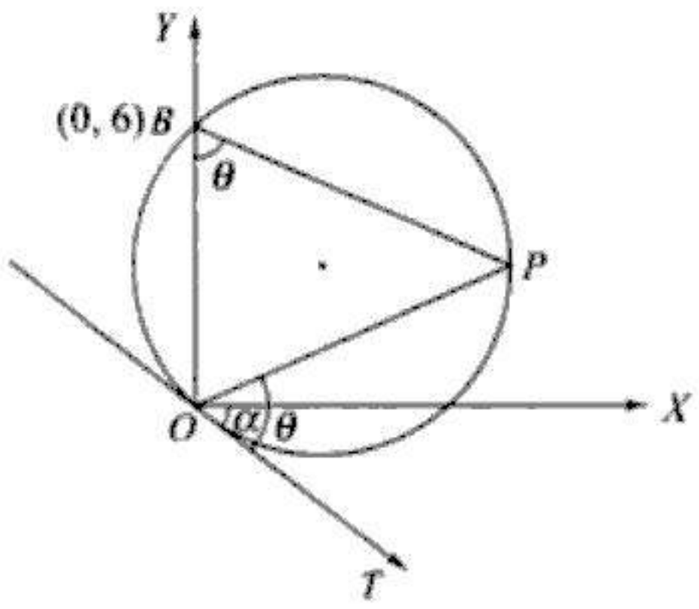

Fig. 16.26

$$\text{and } \tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = \frac{\frac{7}{4} - \frac{2}{3}}{1 + \frac{7}{4} \times \frac{2}{3}} = \frac{13}{26} = \frac{1}{2}$$

Hence the equation of OP is $y = x \tan(\theta - \alpha) \Rightarrow x - 2y = 0$

Example 38 On the line joining the points A (0, 4) and B (3, 0), a square $ABCD$ is constructed on the side of the line away from the origin. Equation of the circle having centre at C and touching the axis of x is

(a) $x^2 + y^2 - 14x - 6y + 49 = 0$
(b) $x^2 + y^2 - 14x - 6y + 9 = 0$
(c) $x^2 + y^2 - 6x - 14y + 49 = 0$
(d) $x^2 + y^2 - 6x - 14y + 9 = 0$

Ans. (a)

Solution Let $\angle ABO = \theta$ then $\angle CBL = 90° - \theta$, CL being perpendicular to x-axis (Fig. 16.27). The coordinates of C are (OL, LC)

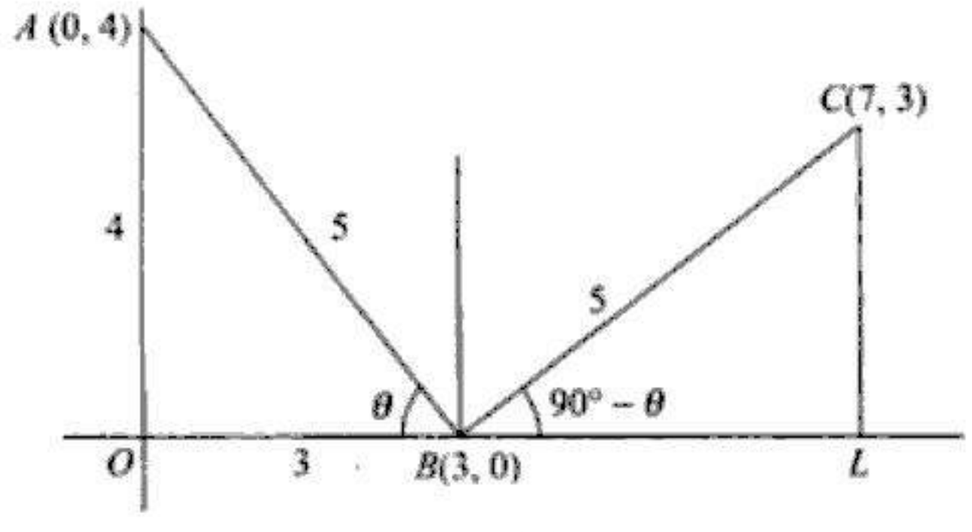

Fig. 16.27

$$OL = OB + BL = 3 + 5 \sin\theta$$
$$= 3 + 5 \times (4/5) = 7$$
$$CL = 5 \cos\theta = 5 \times (3/5) = 3$$

So the coordinate of C are (7, 3) and the equation of the circle having C as centre and touching x-axis is

$$(x - 7)^2 + (y - 3)^2 = (CL)^2 = 9$$
$\Rightarrow \quad x^2 + y^2 - 14x - 6y + 49 = 0.$

Example 39 A circle with centre at the origin and radius equal to a meets the axis of x at A and B. $P(\alpha)$ and $Q(\beta)$ are two points on this circle so that $\alpha - \beta = 2\gamma$, where γ is a constant. The locus of the point of intersection of AP and BQ is

(a) $x^2 - y^2 - 2ay \tan\gamma = a^2$
(b) $x^2 + y^2 - 2ay \tan\gamma = a^2$
(c) $x^2 + y^2 + 2ay \tan\gamma = a^2$
(d) $x^2 - y^2 + 2ay \tan\gamma = a^2$

Ans. (b)

Solution Coordinates of A are $(-a, 0)$ and of P are $(a\cos\alpha, a\sin\alpha)$

$\therefore$ Equation of AP is

$$y = \frac{a\sin\alpha}{a(\cos\alpha + 1)}(x + a)$$

or $\quad y = \tan(\alpha/2)(x + a) \quad$ (i)

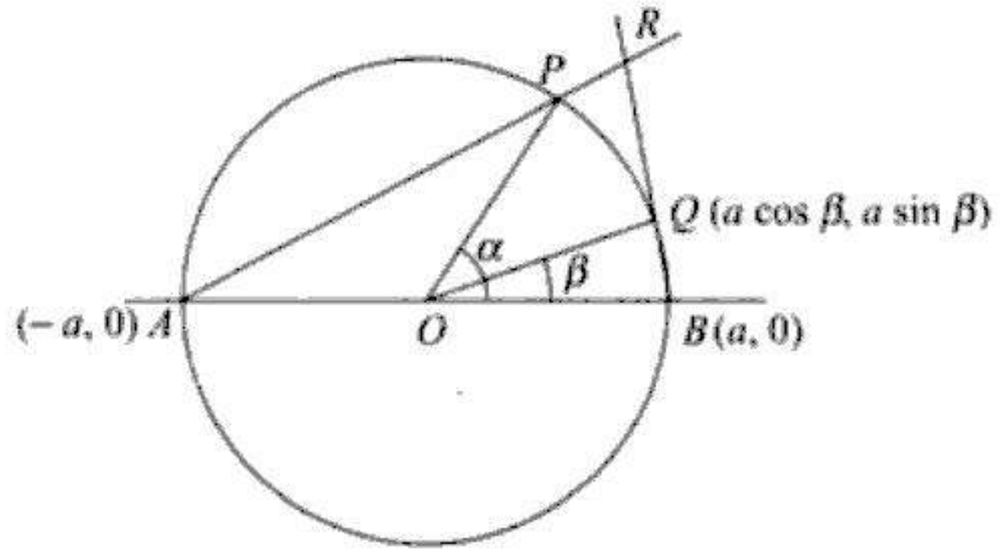

Fig. 16.28

Similarly equation of BQ is

$$y = \frac{a\sin\beta}{a(\cos\beta - 1)}(x - a)$$

or $\quad y = -\cot(\beta/2)(x - a) \quad$ (ii)

We now eliminate α, β from (i) and (ii)

From (i) and (ii) $\tan(\alpha/2) = \frac{y}{a + x}$, $\tan(\beta/2) = \frac{a - x}{y}$

Now $\alpha - \beta = 2\gamma$

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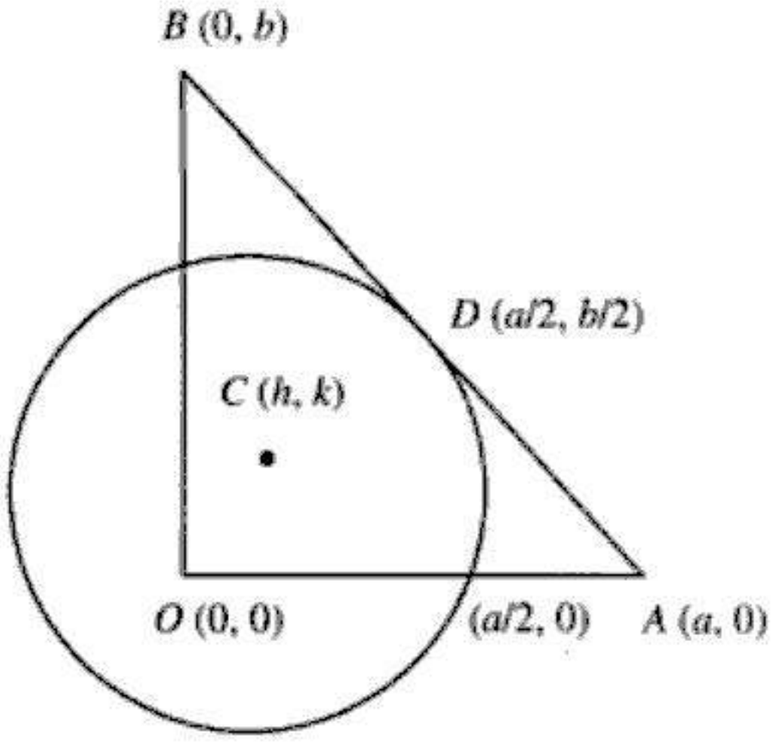

Fig. 16.34

$$r^2 = \left(\frac{2a^2 - b^2}{4a} - \frac{a}{2}\right)^2 + \left(\frac{b}{4}\right)^2 = \frac{b^4}{16a^2} + \frac{b^2}{16}$$

$$= \frac{b^4 + a^2b^2}{16a^2} = \frac{b^2(a^2 + b^2)}{(4a)^2}$$

$$\Rightarrow \quad r = \frac{b\sqrt{a^2 + b^2}}{4a}$$

Example 64 A and B are two points on the x-axis and y-axis respectively. Two circles are drawn passing through the origin and having centre at A and B.

(a) Equation of the common chord is $ax - by = 0$

(b) mid-point of the common chord is

$$\left(\frac{ab^2}{a^2 + b^2}, \frac{a^2 b}{a^2 + b^2}\right)$$

(c) AB bisects the common chord.

(d) AB is perpendicular to the common chord.

Ans. (a), (b), (c), (d)

Solution Let $OA = a$ and $OB = b$ (Fig. 16.35), so that the equation of the line AB is $x/a + y/b = 1$. Equations of the circles passing through the origin and having centres at $A(a, 0)$ and $B(0, b)$ are, respectively,

$$x^2 - 2ax + y^2 = 0 \qquad (1)$$

and $$x^2 - 2by + y^2 = 0 \qquad (2)$$

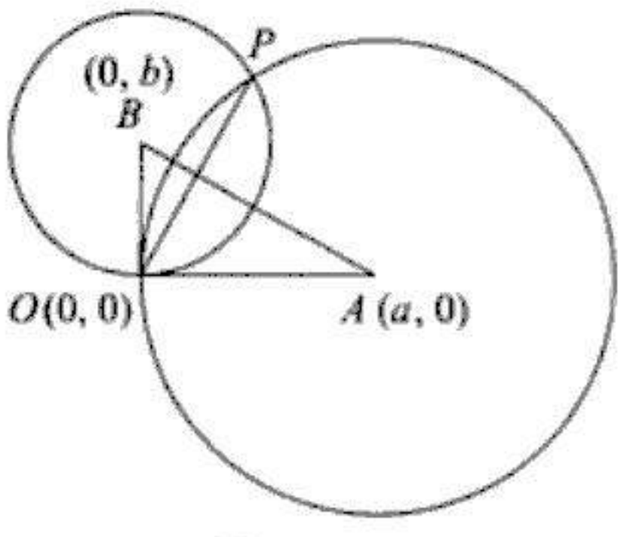

Fig. 16.35

so that the equation of the common chord OP is

$$ax - by = 0 \qquad (3)$$

which is perpendicular to AB

This has one end at the origin O and the other end P is given by solving (2) and (3).

$$\left(\frac{b}{a}y\right)^2 - 2by + y^2 = 0$$

$$\Rightarrow \qquad y = \frac{2a^2 b}{a^2 + b^2} \quad \Rightarrow \quad x = \frac{2ab^2}{a^2 + b^2}$$

Therefore, the mid-point of the common chord OP is

$$\left(\frac{ab^2}{a^2 + b^2}, \frac{a^2 b}{a^2 + b^2}\right)$$

which lies on AB

Example 65 Equation of the straight line which meets the circle $x^2 + y^2 = a^2$ at points which are at a distance d from a point $A(\alpha, \beta)$ on the circle is

(a) $2\alpha x + 2\beta y = 2a^2 - d^2$ (b) $2\alpha x - 2\beta y = 2a^2 + d^2$

(c) $2\alpha x + 2\beta y = 2a^2 + d^2$ (d) $2\alpha x + 2\beta y + 2a^2 = d^2$

Ans. (a), (d)

Solution Let M and N be the points on the circle $x^2 + y^2 = a^2$ such that $AM = AN = d$ (Fig. 16.36). The line joining $A(\alpha, \beta)$ and $O(0, 0)$, the centre of the circle, meets the chord MN at B, the mid-point of MN. Now, the slope of OA is β/α, so that the slope of MN is $-\alpha/\beta$. An equation of MN is $\alpha x + \beta y = k$. Then, since (α, β) lies on the given circle, we have

$$OB = \left|\frac{-k}{\sqrt{\alpha^2 + \beta^2}}\right| = \left|\frac{-k}{a}\right|,$$

$$AB = a \pm \frac{k}{a}, \; BM^2 = OM^2 - OB^2 = a^2 - \frac{k^2}{a^2}$$

and $$AM^2 = AB^2 + BM^2$$

$$\Rightarrow \quad d^2 = \left(a \pm \frac{k}{a}\right)^2 + a^2 - \frac{k^2}{a^2} \quad \Rightarrow \quad d^2 = 2a^2 \pm 2k$$

$$\Rightarrow \quad k = \pm \frac{2a^2 - d^2}{2}$$

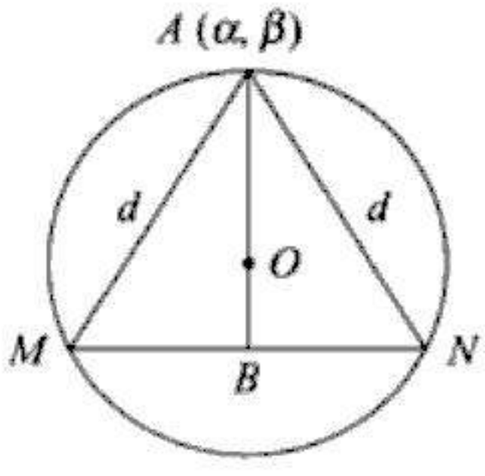

Fig. 16.36

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Thus $(AE)^2 = (a + a)^2 + \left(2\sqrt{3}a\right)^2 = 16a^2$

$\Rightarrow$ $AE = 4a = 2AB$.

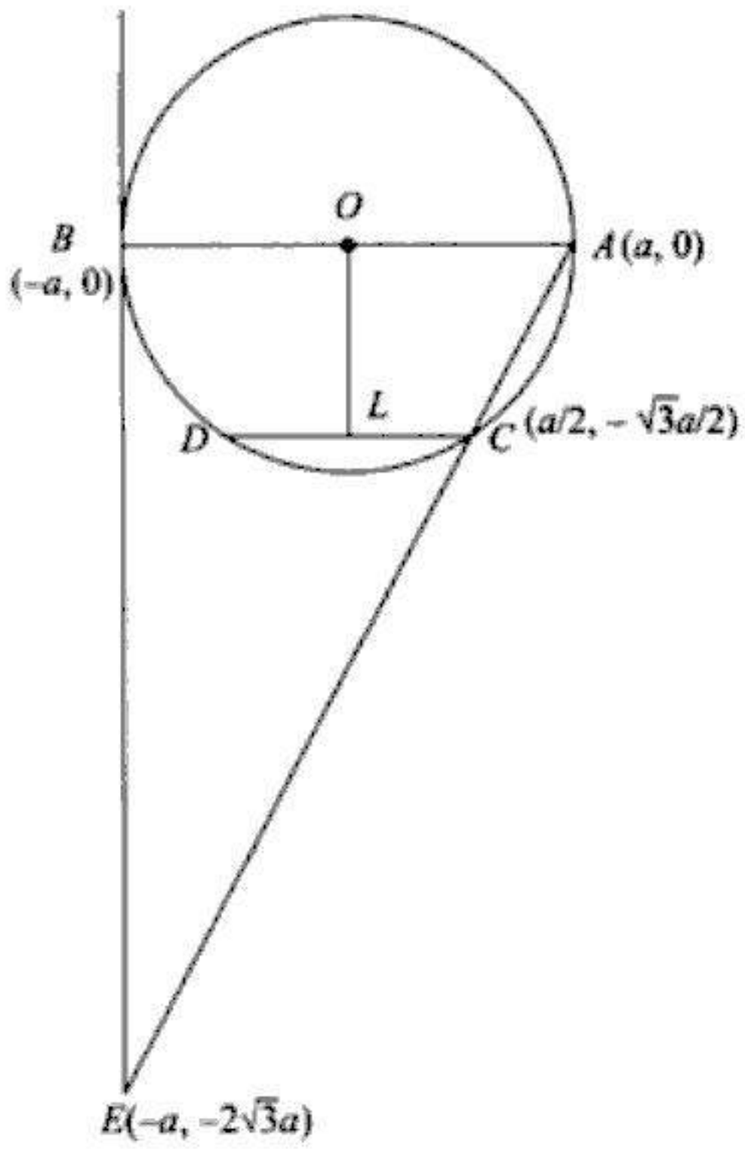

Fig. 16.42

Area of the trapezium $ACDB = \frac{1}{2}(AB + CD) \times OL$

$$= \frac{3\sqrt{3}}{4}a^2$$

$= \frac{3\sqrt{3}}{4}6^2 = 27\sqrt{3}$, as 6 is an integral root of $4x^2 - 3x - 126 = 0$

Paragraph for Question Nos. 79 to 81

$x^2 + y^2 - 2x - 2ay - 8 = 0$, a is a variable.

Example 79 The equation represents a family of circles passing through two fixed points whose coordinates are

(a) $(-2, 0)$ (b) $(2, 0)$
(c) $(4, 0)$ (d) $(-4, 0)$

Example 80 Equation of a circle C of this family, tangents to which at these fixed points intersects on the line $x + 2y + 5 = 0$ is

(a) $x^2 + y^2 - 2x - 8y - 8 = 0$
(b) $x^2 + y^2 - 2x + 6y - 8 = 0$
(c) $x^2 + y^2 - 2x + 8y - 8 = 0$
(d) $x^2 + y^2 - 2x - 6y - 8 = 0$

Example 81 If the chord joining the fixed points subtends an angle θ at the centre of the circle C, then $\theta =$

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

Ans. 79. (a), (c) 80. (d) 81. (d)

Solution Equation of the given circle can be written as

$$(x^2 + y^2 - 2x - 8) - 2a(y) = 0$$

which represents a family of circles passing through the intersection of the circle $x^2 + y^2 - 2x - 8 = 0$ and the line $y = 0$.

The circle and the line intersect at the points. $P(-2, 0)$ and $Q(4, 0)$.

Let the tangents at P and Q to a member of this family intersect at (h, k), then PQ is the chord of contact of (h, k) and its equation is

$$hx + ky - (x + h) - a(y + k) - 8 = 0$$

or $x(h - 1) + y(k - a) - (h + ak + 8) = 0$

Comparing this with equation $y = 0$ of PQ, we get

$$h = 1, \quad h + ak + 8 = 0,$$

Since (h, k) lies on the given line $x + 2y + 5 = 0$

$1 + 2k + 5 = 0 \Rightarrow k = -3$

and $1 - 3a + 8 = 0 \Rightarrow a = 3$

Hence the equation of the required member C of the family is $x^2 + y^2 - 2x - 6y - 8 = 0$.

Centre of the circle C is $R(1, 3)$

$\therefore$ Slope of $PR = 1$ and of $QR = -1$

$\Rightarrow \theta = \pi/2$

Paragraph for Question Nos. 82 to 84

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}\,x + y - 6 = 0$ and the point D is $(3\sqrt{3}/2, 3/2)$. Further it is given that the origin and the centre of C are on the same side of PQ.

Example 82 The equation of circle C is

(a) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$
(b) $(x - 2\sqrt{3})^2 + (y + 1/2)^2 = 1$
(c) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$
(d) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Example 83 Points E and F are given by

(a) $(\sqrt{3}/2, 3/2), (\sqrt{3}, 0)$
(b) $(\sqrt{3}/2, 1/2), (\sqrt{3}, 0)$
(c) $(\sqrt{3}/2, 3/2), (\sqrt{3}/2, 1/2)$
(d) $(3/2, \sqrt{3}/2), (\sqrt{3}/2, 1/2)$

Example 84 Equations of the sides QR, RP are

(a) $y = (2/\sqrt{3})x + 1$, $y = (-2/\sqrt{3})x - 1$
(b) $y = (1/\sqrt{3})x$, $y = 0$

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be a pair of perpendicular tangents to the circle $x^2 + y^2 = a^2$. Eliminating m we get

$(y - mx)^2 + (my + x)^2 = 2a^2 (1 + m^2) \Rightarrow x^2 + y^2 = 2a^2$

(c) $x \cos \theta = y \cot \theta = a$

$\Rightarrow x = a \sec \theta, y = a \tan \theta$

Eliminating θ we get $x^2 - y^2 = a^2$

(d) Equation of the chord of the circle $x^2 + y^2 - 2ax = 0$ in terms of its middle point (α, β) is

$$\alpha x + \beta y - a(x + \alpha) = \alpha^2 + \beta^2 - 2a\alpha.$$

Since it passes through the origin.

$\alpha^2 + \beta^2 - a\alpha = 0$. Locus of (α, β) is $x^2 + y^2 = ax$

Example 94 For the circle $x^2 + y^2 + 4x + 6y - 19 = 0$

(a) Length of the tangent from (6, 4) to the circle.	(p) $\frac{72\sqrt{226}}{113}$
(b) Length of the chord of contact from (6, 4) to the circle	(q) $\sqrt{113}$
(c) Distance of (6, 4) from the centre of the circle	(r) $\sqrt{113} - \sqrt{32}$
(d) Shortest distance of (6, 4) from the circle	(s) 9

Ans.

	p	q	r	s
a				●
b	●			
c		●		
d			●	

(a) Required length of the tangent

$$PT = \sqrt{36 + 16 + 24 + 24 - 19} = 9.$$

(b) Equation of the chord of contact TT' is

$$6x + 4y + 2(x + 6) + 3(y + 4) - 19 = 0$$

$$\Rightarrow \quad 8x + 7y + 5 = 0$$

perpendicular distance of TT' from the centre C $(-2, -3)$ of the circle is

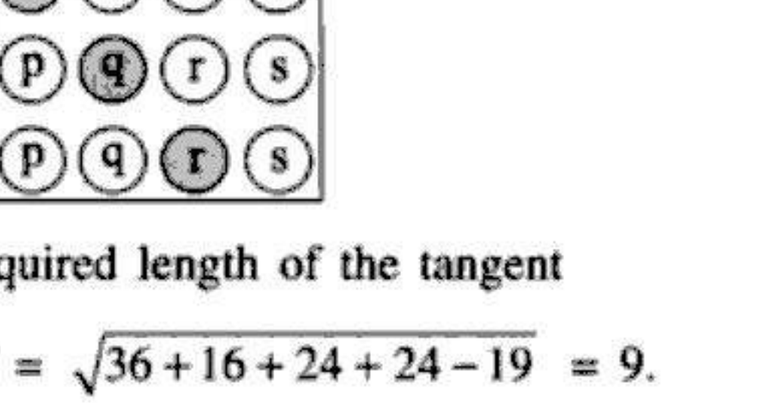

Fig. 16.46

$$CS = \left|\frac{-16 - 21 + 5}{\sqrt{64 + 49}}\right| = \frac{32}{\sqrt{113}}$$

and the radius of the circle is

$$CT = \sqrt{4 + 9 + 19} = \sqrt{32}$$

Now $(ST)^2 = (CT)^2 - (CS)^2$

$$= (32) - \frac{(32)^2}{113} = \frac{32 \times 81}{113}$$

Length of $TT' = 2ST = 2 \times 36 \sqrt{\frac{2}{113}} = \frac{72\sqrt{226}}{113}$

(c) required distance is

$$CP = \sqrt{(6 + 2)^2 + (4 + 3)^2} = \sqrt{113}$$

(d) required distance is PR where R is the point of intersection of CP with the circle.

Now $PR = CP - CR = \sqrt{113} - \sqrt{32}$

Example 95 If $S \equiv x^2 + y^2 + x - y - 2 = 0$, then

(a) (− 2, 1) lies	(p) on the circle S.
(b) (2, − 1) lies	(q) outside the circle S.
(c) (0, 1) lies	(r) on the tangent at (1, 0) to S
(d) (2, 3) lies	(s) inside the circle S.

Ans.

	p	q	r	s
a	●			
b		●		
c				●
d		●	●	

Solution Let $S(x, y) = x^2 + y^2 + x - y - 2$

then $S_{(-2, 1)} = 4 + 1 - 2 - 1 - 2 = 0 \Rightarrow (-2, 1)$ lies on S.

$S_{(2, -1)} = 4 + 1 + 2 + 1 - 2 > 0 \Rightarrow (2, -1)$ lies outside S.

$S_{(0, 1)} = 0 + 1 + 0 - 1 - 2 < 0 \Rightarrow (0, 1)$ lies inside S.

$S_{(2, 3)} = 4 + 9 + 2 - 3 - 2 > 0$

Tangent at (1, 0) is $3x - y - 3 = 0$ which passes through (2, 3) but not through (2, − 1)

Example 96

(a) The circle $x^2 + y^2 + 2x + c = 0$ and $x^2 + y^2 + 2y + c = 0$ touch each other	(p) if $c = 1$

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27. *A* circle C passes through $(2a, 0)$ and has the line $2x - a = 0$ as the radical axis with the circle $x^2 + y^2 = a^2$, then
    (a) centre of the circle C is $(a, 0)$
    (b) centre of the circle C is $(-a, 0)$
    (c) C passes through $(0, 0)$
    (d) C passes through (a, a)
28. $C_1: x^2 + y^2 + 2x - 3 = 0$, $C_2: x^2 + y^2 - 2x - 3 = 0$, $C_3: x^2 + y^2 - 2y - 3 = 0$ are three circles
    (a) C_1 passes through the centre of C_2
    (b) C_2 passes through the centre of C_1
    (c) centres of the three circles form an equilateral triangle.
    (d) sum of the radii of C_1 and C_2 is twice the radius of C_3.
29. If P is a point on the circle $x^2 + y^2 = 9$, Q is a point on the line $7x + y + 3 = 0$, and the line $x - y + 1 = 0$ is perpendicular bisector of PQ, then the coordinates of P are
    (a) $(3, 0)$ (b) $(-3, 0)$
    (c) $(-72/25, 21/25)$ (d) $(72/25, 21/25)$
30. If the circle $C_1: x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope 3/4, the coordinates of the centre of C_2 are
    (a) $(9/5, -12/5)$ (b) $(-9/5, 12/5)$
    (c) $(9/5, 12/5)$ (d) $(-9/5, -12/5)$
31. If a chords of the circle $x^2 + y^2 = 8$ makes equal intercepts of length a on the coordinate axis then a can be
    (a) 2 (b) 3
    (c) 4 (d) 5
32. Two equal chords of the circle $x^2 + y^2 - 2x + 4y = 0$, passing through the origin are perpendicular to each other; their equations are
    (a) $x - 2y = 0$ (b) $2x + y = 0$
    (c) $x + 3y = 0$ (d) $3x - y = 0$
33. Equation of a circle of radius 3, touching the coordinate axes, no portion of which lies below x-axis is
    (a) $x^2 + y^2 + 6x + 6y + 9 = 0$
    (b) $x^2 + y^2 - 6x + 6y + 9 = 0$
    (c) $x^2 + y^2 + 6x - 6y + 9 = 0$
    (d) $x^2 + y^2 - 6x - 6y + 9 = 0$
34. Equation of a tangent to the circle $x^2 + y^2 = a^2$ is
    (a) $y = ax + a\sqrt{1 + a^2}$
    (b) $y = -ax + a\sqrt{1 + a^2}$
    (c) $y = ax - a\sqrt{1 + a^2}$
    (d) $y = -ax - a\sqrt{1 + a^2}$
35. The locus of a point, which is such that the tangents from its two concentric circles of radii a and b are inversely proportional to their radii, is a circle C of area 16π.
    (a) centre of C is at the centre of the given circles.
    (b) centre of C is at a distance 4 from the centre of the given circles
    (c) $a^2 + b^2 = 16$
    (d) $a^2 + b^2 = 4$

SECTION III

LINKED COMPREHENSION TYPE

Paragraph for Question Nos. 36 to 38

L_1 is a line intersecting x and y axis at $P(a, 0)$ and $Q(0, b)$. L_2 is a line perpendicular to L_1 intersecting x and y axis at R and S respectively.

36. Locus of the point of intersection of the line PS and QR is a circle with
    (a) centre at $(a/2, b/2)$
    (b) $ax + by = 0$ as a tangent
    (c) a and b as the intercepts on x-axis and y-axis respectively
    (d) radius equal to $(1/2)\sqrt{a^2 + b^2}$
37. Common chord of the circles on QS and PR as diameters passes through the point (a, b) if
    (a) $a = 2b$ (b) $2a = b$
    (c) $a = b$ (d) $a = -b$
38. If the area of the triangle ORS is 4 times the area of the triangle OPQ then equation of PS is
    (a) $x + 2y = 2b$ (b) $2x + y = 2a$
    (c) $x - 2y + 2b = 0$ (d) $2x - y - 2a = 0$

Paragraph for Question Nos. 39 to 41

$A(3, 7)$ and $B(6, 5)$ are two points.
$C: x^2 + y^2 - 4x - 6y - 3 = 0$ is a circle.

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(a) 3 (b) 2
(c) 3/2 (d) 1 [2004]

60. Find an equation of the circle which touches the line $2x + 3y + 1 = 0$ at point $(1, -1)$ and is orthogonal to the circle which has $(0, -1)$ and $(-2, 3)$ as the end points of a diameter. [2004]

61. A circle C touches the x-axis and the circle $x^2 + (y-1)^2 = 1$ externally, then locus of the centre of the circle C is given by
(a) $\{(x, y): x^2 = 4y\} \cup \{(0, y): y \le 0\}$
(b) $\{(x, y): y = x^2\} \cup \{(0, y): y \le 0\}$
(c) $\{(x, y): x^2 + (y-1)^2 = 4\} \cup \{(0, y): y \le 0\}$
(d) $\{(x, y): x^2 + 4y = 0\} \cup \{(0, y): y \le 0\}$
[2005]

62. Three circles with radii 3 cm, 4 cm and 5 cm touch each other externally. If A is the point of intersection of tangents to these circles at their points of contact, find the distance of A from the points of contact. [2005]

63. Let $ABCD$ be a quadrilateral with area K, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is
(a) 3 (b) 2
(c) 3/2 (d) 1 [2007]

REASONING TYPE

64. Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$.
Statement-1: The tangents are mutually perpendicular.
Because
Statement-2: The locus of the point from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$. [2007]

Paragraph for Question Nos. 65 to 67

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $(3\sqrt{3}/2, 3/2)$. Further it is given that the origin and the centre C are on the same side PQ.

65. The equation of circle C is
(a) $(x - 2\sqrt{3})^2 + (y-1)^2 = 1$
(b) $(x - 2\sqrt{3})^2 + (y + 1/2)^2 = 1$
(c) $(x - \sqrt{3})^2 + (y+1)^2 = 1$
(d) $(x - \sqrt{3})^2 + (y-1)^2 = 1$ [2008]

66. Points E and F are given by
(a) $(\sqrt{3}/2, 3/2), (\sqrt{3}, 0)$
(b) $(\sqrt{3}/2, 1/2), (\sqrt{3}, 0)$
(c) $(\sqrt{3}/2, 3/2), (\sqrt{3}/2, 1/2)$
(d) $(3/2, \sqrt{3}/2), (\sqrt{3}/2, 1/2)$ [2008]

67. Equations of the sides QR, RP are
(a) $y = (2/\sqrt{3})x + 1, \; y = (-2/\sqrt{3})x - 1$
(b) $y = (1/\sqrt{3})x, \; y = 0$
(c) $y = (\sqrt{3}/2)x + 1, \; y = (-\sqrt{3}/2)x - 1$
(d) $y = \sqrt{3}x, \; y = 0$ [2008]

REASONING TYPE

68. Consider L_1: $2x + 3y + p - 3 = 0$
L_2: $2x + 3y + p + 3 = 0$
where p is a real number, and C: $x^2 + y^2 + 6x - 10y + 30 = 0$
Statement-1: If L_1 is a chord of the circle C, then line L_2 is not always a diameter of circle C.
and
Statement-2: If L_1 is a diameter of circle C, the line L_2 is not a chord of circle C. [2008]

MULTIPLE CORRECT CHOICE TYPE

69. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then
(a) $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$
(b) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$
(c) $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$
(d) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$ [2008]

70. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is
(a) $x^2 + y^2 + 4x - 6y + 19 = 0$
(b) $x^2 + y^2 - 4x - 10y + 19 = 0$
(c) $x^2 + y^2 - 2x + 6y - 29 = 0$
(d) $x^2 + y^2 - 6x - 4y + 19 = 0$ [2009]

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Ist circle lies in the first quadrant as it touches both the axes and centre is also in this quadrant.

16. The sides of the square be $x = 0, x = 1, y = 0, y = 1$ (h, k) be any point on the locus, then
$(h)^2 + (1-h)^2 + k^2 + (1-k)^2 = 9$
$\Rightarrow h^2 + k^2 - h - k - 7/2 = 0$
Locus $x^2 + y^2 - x - y - 7/2 = 0$ is a circle with centre (1/2, 1/2), the centre of the square and radius equal to 4, greater than the diagonal of the square and hence contains the square.

17. The radical centre is the point of intersection of $2x + 2y = 4$ and $4x + 6y = 10$ i.e. (1, 1) which lies on $y = mx$ if $m = 1$.

18. The point lies on the line joining the origin to the centre (1, 2) i.e. $y = 2x$ which meets the given circle at the points. $(1 \pm 4/\sqrt{5}, 2 \pm 8/\sqrt{5})$, the one which is farthest from the origin is $(1 + 4/\sqrt{5}, 2 + 8/\sqrt{5})$.

19. Equation of AB is $x/a + y/b = 1$. If $P(h, k)$ is the reflexion of O in AB then $(h/2, k/2)$ lies on AB and AB and OP are at right angles.

$$\therefore \frac{h}{a} + \frac{k}{b} = 2 \text{ and } -\frac{b}{a} \times \frac{k}{h} = -1 \Rightarrow h = \frac{2ab^2}{a^2 + b^2}$$

$$k = \frac{2a^2 b}{a^2 + b^2}$$

20. Length of the longest ray is equal to the length of the tangent from the point (4, 3) which is
$\sqrt{16 + 9 + 64 + 54 + 1} = 12$
radius of the circle $= \sqrt{16 + 81 - 1} = 12$

21. Equation of the tangent is $\alpha x + \beta y = r^2$, which meets the coordinate axes at $A(r^2/\alpha, 0)$ and $B(0, r^2/\beta)$ and the area of $\Delta OAB = \dfrac{r^4}{2\alpha\beta}$ where $r^2 = 7$ or 8.

22. $2(x^2 + y^2 - b^2) = x^2 + y^2 - a^2 + x^2 + y^2 - c^2$
$\Rightarrow 2b^2 = a^2 + c^2 \Rightarrow a^2, b^2, c^2$ are in A.P. $\dfrac{1}{a^2}, \dfrac{1}{b^2}, \dfrac{1}{c^2}$ are in H.P.

23. Required length is $\sqrt{x^2 + y^2 + 15x - 17y + 21}$ where $x^2 + y^2 + 15x - 17y + c^2 = 0$
$= \sqrt{21 - c^2} = \sqrt{5} \Rightarrow c^2 = 16 \Rightarrow c = \pm 4$

24. Area of the quadrilateral
$= \sqrt{4^2 + 5^2 - 4 \times 4 - 2 \times 5 - c} \times \sqrt{4 + 1 + c}$
$\Rightarrow 8^2 = (15 - c)(5 + c) \Rightarrow c^2 - 10c - 11 = 0$
$\Rightarrow c = 11, -1$

25. $P(\cos\alpha, \sin\alpha)$, $Q(\cos\beta, \sin\beta)$, $R(\cos\gamma, \sin\gamma)$
$\therefore AP = \sqrt{(\cos\alpha + 1)^2 + \sin^2\alpha} = \sqrt{2 + 2\cos\alpha}$
$= 2\cos(\alpha/2)$
$AQ = 2\cos(\beta/2)$, $AR = 2\cos(\gamma/2)$
Since AP, AQ, AR are in G.P.
$\cos(\alpha/2), \cos(\beta/2), \cos(\gamma/2)$ are in G.P.
$\Rightarrow \cos(\alpha/2)\cos(\gamma/2) = \cos^2(\beta/2)$
Since α, β, γ, are in A.P $\dfrac{\alpha + \gamma}{2} = \beta$
$\Rightarrow \cos(\alpha/2)\cos(\gamma/2) - \sin(\alpha/2)\sin(\gamma/2) = \cos\beta$
$\Rightarrow \cos^2(\beta/2) - \cos\beta = \sin(\alpha/2)\sin(\gamma/2)$
$\Rightarrow 1 - \cos^2(\beta/2) = \sin(\alpha/2)\sin(\gamma/2)$
$\Rightarrow \sin^2(\beta/2) = \sin(\alpha/2)\sin(\gamma/2)$
$\Rightarrow \sin(\alpha/2), \sin(\beta/2), \sin(\gamma/2)$ are in G.P.

26. Radical axis of the given circles is $x + 5y - 6 = 0$ which passes through (1, 1).
Let the given circles intersect the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ orthogonally
then $2g(-3) + 2f(-6) = c + 1$
$2g(-2) + 2f(-1) = c - 11$
$\Rightarrow 2g(-1) + 2f(-5) = 12 \Rightarrow -g - 5f - 6 = 0$
$\Rightarrow$ the radical axis passes through the centre $(-g, -f)$ of the third circle.
Verify it does not pass through the mid-point of the line joining the centres.

27. Equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$, it passes through $(2a, 0) \Rightarrow 4a^2 + 4ag + c = 0$
Radical axis is $2gx + 2fy + c + a^2 = 0$ which is $2x - a = 0$
$$\frac{g}{1} = \frac{f}{0} = \frac{c + a^2}{-a} \Rightarrow f = 0, -ag = c + a^2$$
$\Rightarrow a^2 + ag = c \Rightarrow c = 0, g = -a$ and the equation of the circle is $x^2 + y^2 - 2ax = 0$
Centre is $(a, 0)$, passes through (0, 0) and (a, a)

28. Centres $C_1(-1, 0)$ $C_2(1, 0)$, $C_3(0, 1)$
Vertices of an isosceles triangle.
C_1 passes through (1, 0) and C_2 passes through $(-1, 0)$ radii of each = 2.

29. $P(h, k)$, $Q(x_1, y_1)$ then $h^2 + k^2 = 9$, $7x_1 + y_1 + 3 = 0$
$$\frac{y_1 - k}{x_1 - h} = -1 \Rightarrow x_1 + y_1 = h + k$$

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$$\left|\frac{m(O)-O+\sqrt{25m^2+4}}{\sqrt{1+m^2}}\right| = 4$$

$\Rightarrow$ $25m^2 + 4 = 16\,(1 + m^2) \Rightarrow m^2 = 4/3$

$\Rightarrow$ $m = \pm 2/\sqrt{3}$ and equation of the common tangent is

$$y = \pm\frac{2}{\sqrt{3}}x + \sqrt{25\times\frac{4}{3}+4}$$

$\Rightarrow$ $\sqrt{3}\,y \pm 2x = 4\sqrt{7}$

$\Rightarrow$ $$l^2 = \left(\frac{4\sqrt{7}}{2}\right)^2 + \left(\frac{4\sqrt{7}}{\sqrt{3}}\right)^2$$

$\Rightarrow$ $3l^2 = 196 \Rightarrow 81l^2 + 3 = 5295$

Example 53 If d is the length of the tangent from the point (100, 81) to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, then d^2 is equal to

Ans. 3228

Solution $d^2 = \frac{100\times100}{4}+\frac{81\times81}{9}-1 = 3228$

Example 54 Tangents to the parabola at the extremities of a common chord AB of the circle $x^2 + y^2 = 5$ and the parabola $y^2 = 4x$ intersect at the point T. A square $ABCD$ is constructed on this chord lying inside the parabola then $[(TC)^2 + (TD)^2]^2$ is equal to

Ans. 6400

Solution The points of intersection the circle and the parabola are $A\,(1, 2)$, $B\,(1, -2)$

Equation of common chord is $x = 1$, which is the latus rectum of the parabola.

$\therefore$ Tangents at the extremities of AB intersect on the directrix $x = -1$.

Coordinates of T are $(-1, 0)$

Since length of $AB = 4$, sides of the square $ABCD$ are of length 4, coordinates of C are $(-5, 2)$ and of D are $(5, 2)$.
$(TC)^2 = (TD)^2 = (5 + 1)^2 + 4 = 40$.

$\Rightarrow$ $[(TC)^2 + (TD)^2]^2 = 80^2 = 6400$

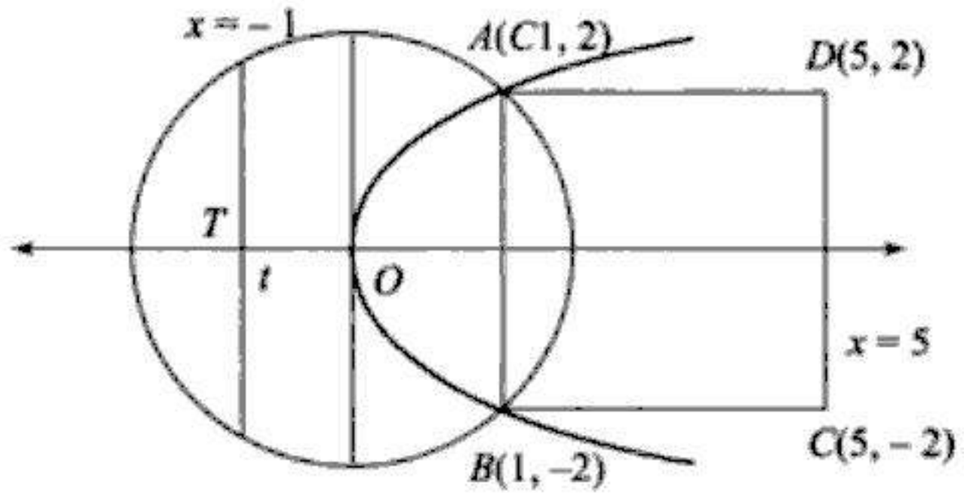

Fig. 17.17

Example 55 The point of intersection of the perpendicular tangents to the ellipse $\frac{x^2}{(41)^2}+\frac{y^2}{(35)^2}=1$ lies on a circle square of whose radius is

Ans. 2906

Solution Any tangent to the ellipse is

$y = mx + \sqrt{(41)^2 m^2 + (35)^2}$ and perpendicular to it is

$$y = -\frac{1}{m}x + \sqrt{\frac{(41)^2}{m^2}+(35)^2}$$

Eliminating m we get the locus of the point of intersection of these tangents.

$(y - mx)^2 + (my + x)^2 = (1 + m^2)\,[(41)^2 + (35)^2]$

$\Rightarrow x^2 + y^2 = (41)^2 + (35)^2$ which is a circle square of whose radius $= (41)^2 + (35)^2 = 1681 + 1225 = 2906$

Example 56 If $x + y = k$ is a normal to the parabola $y^2 = 12x$, p is the length of the perpendicular from the focus of the parabola on this normal; then $3k^3 + 2p^2$ is equal to

Ans. 2223

Solution For the parabola $y^2 = 12x$, equation of a normal with slope -1, is

$y = -x - 2\,(3)\,(-1) - (3)\,(-1)^3$ $\quad [m = -1, a = 3]$

$\Rightarrow$ $x + y = 9$ so $k = 9$

Focus of the parabola is (3, 0)

$\therefore$ $p = \left|\frac{3-9}{\sqrt{2}}\right| \Rightarrow 2p^2 = 36$

$\therefore$ $3k^3 + 2p^2 = 2223$

Example 57 If CF is perpendicular from the centre C of the ellipse $\frac{x^2}{49}+\frac{y^2}{25}=1$ on the tangent at any point P, and G is the point where the normal at P meets the minor axis, then $(C.F.\ PG)^2$ is equal to

Ans. 2401

Solution Equation of the tangent at $P\,(7\cos\theta, 5\sin\theta)$ on the ellipse is $\frac{x}{7}\cos\theta+\frac{y}{5}\sin\theta=1$

then $(CF)^2 = \frac{7^2\times5^2}{5^2\cos^2\theta+7^2\sin^2\theta}$

$= \frac{25\times49}{25\cos^2\theta+49\sin^2\theta}$

Equation of the normal at P is

$$\frac{7x}{\cos\theta}-\frac{5y}{\sin\theta} = 7^2 - 5^2$$

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Example 74 $y^2 = 4x$ is the equation of a parabola.
Statement-1: Through $(\lambda, \lambda + 1)$, 3 normals can be drawn to the parabola, if $\lambda < 2$
Statement-2: The point $(\lambda, \lambda + 1)$ lies outside the parabola for all $\lambda \neq -1$.

Ans. (d)

Solution Equation of a normal to the parabola is $y + tx = 2t + t^3$, if it passes through $(\lambda, \lambda + 1)$
then $$\lambda + 1 + t\lambda = 2t + t^3$$
or $$t^3 + (2 - \lambda)t - (\lambda + 1) = 0 = f(t) \text{ say.}$$
If $f(t) = 0$ has three real roots then $f'(t) = 0$ has two real roots $\Rightarrow 3t^2 + (2 - \lambda) = 0$ has the real roots which is not true as $\lambda < 2$
$\therefore$ $f(t) = 0$ has only one real root and thus there is only one normal through $(\lambda, \lambda + 1)$ to the parabola and the statement-1 is false.
Statement-2 is correct as $S \equiv y^2 - 4x = (\lambda + 1)^2 - 4\lambda > 0$ if $\lambda \neq 1$ showing that $(\lambda, \lambda + 1)$ lies outside the parabola $y^2 = 4x$, if $\lambda \neq -1$.

Example 75 **Statement-1:** If the tangent at the point $\left(a/\sqrt{2}, b/\sqrt{2}\right)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$ meets the auxillary circle in two points, the chord joining them subtends a right angle at the centre then the eccentricity e of the ellipse is given by $3e^2 - 2 = 0$
Statement-2. If e and e' are the eccentricities of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the hyperbola $x^2/a^2 - y^2/b^2 = 1$ respectively then $e^2 + e'^2 = 2$

Ans. (b)

Solution Equation of the tangent at $\left(a/\sqrt{2}, b/\sqrt{2}\right)$ to the ellipse is $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$. The joint equation of the lines joining the points of intersection of this tangent and the auxillary circle $x^2 + y^2 = a^2$ is

$$x^2 + y^2 = \frac{a^2}{2}\left[\frac{x}{a} + \frac{y}{b}\right]^2$$

which are at right angles,
if co-efficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \quad 1 - \frac{1}{2} + 1 - \frac{a^2}{2b^2} = 0 \Rightarrow \quad a^2 = 3b^2 = 3a^2(1 - e^2)$$

$$\Rightarrow \quad e^2 = 2/3 \text{ or } 3e^2 - 2 = 0$$

So statement-1 is true.

In Statement-2, $e^2 = \frac{a^2 - b^2}{a^2}$, $e'^2 = \frac{a^2 + b^2}{a^2}$

$\Rightarrow$ $e^2 + e'^2 = 2$ so statement-2 is also true but does not lead to statement-1.

EXERCISE

LEVEL 1

SECTION I

SINGLE CORRECT CHOICE TYPE

1. The axis of a parabola is along the line $y = x$ and the distance of its vertex from origin is $\sqrt{2}$ and that from its focus is $2\sqrt{2}$. If vertex and focus both lie in the first quadrant, then the equation of the parabola is
   (a) $(x + y)^2 = (x - y - 2)$
   (b) $(x - y)^2 = (x + y - 2)$
   (c) $(x - y)^2 = 4\ (x + y - 2)$
   (d) $(x - y)^2 = 8\ (x + y - 2)$
2. The normal $y = mx - 2am - am^3$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex if
   (a) $m = 1$ (b) $m = \sqrt{2} + 1$
   (c) $m = \pm\sqrt{2}$ (d) $m = \sqrt{2} - 1$
3. An equation of a tangent common to the parabolas $y^2 = 4x$ and $x^2 = 4y$ is
   (a) $x - y + 1 = 0$ (b) $x + y - 1 = 0$
   (c) $x + y + 1 = 0$ (d) $y = 0$
4. The coordinates of the end point of the latus rectum of the parabola $(y - 1)^2 = 2(x + 2)$, which does not lie on the line $2x + y + 3 = 0$ are
   (a) $(-2, 1)$ (b) $(-3/2, 1)$
   (c) $(-3/2, 2)$ (d) $(-3/2, 0)$
5. The point of contact of the tangent to the parabola $y^2 = 9x$ which passes through the point (4, 10) and makes an angle θ with the axis of the parabola such that $\tan\theta > 2$ is
   (a) (4/9, 2) (b) (36, 18)
   (c) (4, 6) (d) (1/4, 3/2)
6. Equation of the normal at a point on the parabola $y^2 = 36x$, whose ordinate is three times its abscissa is
   (a) $2x + 3y + 44 = 0$ (b) $2x - 3y + 44 = 0$
   (c) $2x + 3y - 44 = 0$ (d) $2x - 3y = 0$
7. In an ellipse, if the lines joining a focus to the extremities of the minor axis make an equilateral triangle with the minor axis, the eccentricity of the ellipse is
   (a) 3/4 (b) $\sqrt{3}/2$ (c) 1/2 (d) 2/3
8. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $x^2/a^2 + y^2/b^2 = 1$ are at right angles then $x_1 x_2/y_1 y_2$ is equal to
   (a) a^2/b^2 (b) $-b^2/a^2$ (c) $-a^4/b^4$ (d) $-b^4/a^4$

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$$\left(\frac{39}{h}\right)^2 - \left(\frac{k}{2h}\right)^2 = 1 \Rightarrow 4 \times (39)^2 = 4h^2 + k^2$$

Locus of (h, k) is $4x^2 + y^2 = 6084$ so $k = 6084$

25. (a) $\frac{x^2}{9} + \frac{y^2}{16} = 1$, $e^2 = \frac{16-9}{16} = \frac{7}{16}$, Focus $(0 \pm \sqrt{7})$

Equation of the tangent at $(9/4, \sqrt{7})$, the positive end of L.R. is $\frac{x}{4} + \frac{y\sqrt{7}}{16} = 1 \Rightarrow 4x + \sqrt{7}y - 16 = 0$

(b) $\frac{x^2}{16} - \frac{y^2}{25} = 1$, $e^2 = \frac{16+25}{16} = \frac{41}{16}$

Positive end of the $LR = \left(\sqrt{41}, 25/4\right)$

Equation of the tangent $\sqrt{41}\,x - 4y - 16 = 0$

(c) $x^2 - y^2 = 4$, $e^2 = 2$, positive end of LR $(2\sqrt{2}, 2)$

Equation of the tangent $\sqrt{2}\,x - y - 2 = 0$

(d) $y^2 = 8(x-4)$, focus $(6, 0)$, positive end of LR $(6, 4)$

Equation of the tangent $x - y - 2 = 0$

26. Equation of the normal is $y = mx - 2m - m^3$, since it passes through $(3, 0)$, $3m - 2m - m^3 = 0 \Rightarrow m = 0, \pm 1$

centroid of $\Delta PQR = \left(\frac{2}{3}, 0\right)$

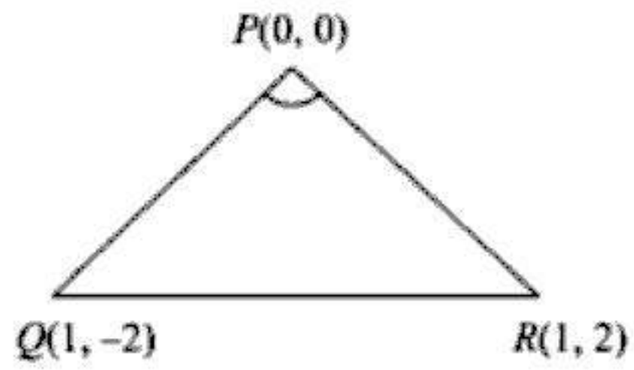


Fig. 17.22

ΔPQR is isosceles.

If (x, y) is the circumcentre, then

$x^2 + y^2 = (x-1)^2 + (y-2)^2 = (x-1)^2 + (y+1)^2$

$\Rightarrow x = 5/2,\ y = 0.$

radius $= 5/2$. Area $= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2.$

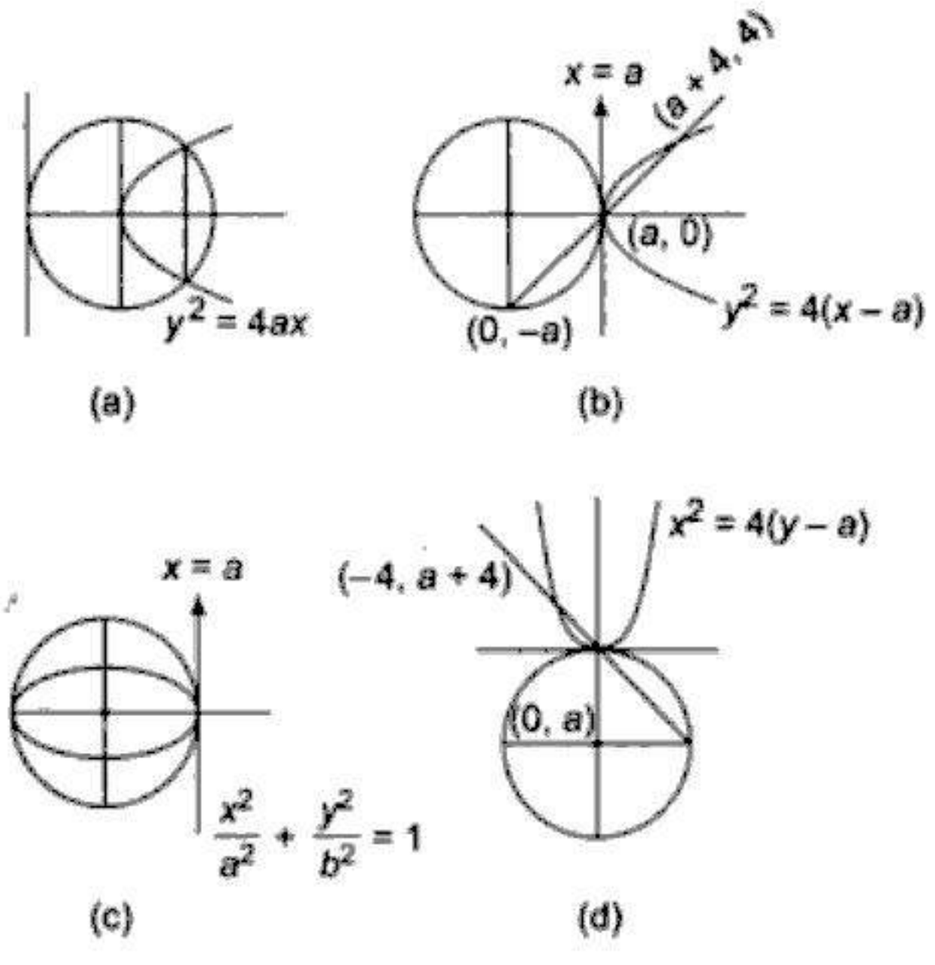
27. (a) $x^2 + 8x - 9 = 0 \Rightarrow x = -9$ or $x = 1$; For $x = -9$, y is not real.

(b) $x^2 + x - 2 = 0 \Rightarrow x = 1,\ y = \pm 2\sqrt{2}$

(c) $\frac{x^2}{8} + \frac{x^2}{16} = \frac{1}{2} + 1 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$, $y = \pm 1$

(d) $y^2 + 2y + 1 \Rightarrow y = -1$, $x = \pm 2\sqrt{2}$

28. Draw the figures in each case.



The line joining the centre of the circle and the focus of the conic in each case is a common normal.

(a) any tangent to the parabola $y^2 = 4ax$ is $ty = x + at^2$, which is not a tangent to the circle for any value of t. They have a common chord and a common point.

(b) $x = a$ is a common tangent, $x - y = a$ is a common chord. $(a, 0)$ is a common point.

(c) $x = \pm a$, are common tangents, each chord of the ellipse is a chord of the circle, $(\pm a, 0)$ are common points.

(d) interchange x and y in (b) to get (d).

29. $(2a, 0)$ is the focus in (a) and centre in (b), (p) and (q) pass through this point. Equation of any tangent to (c) is $y = mx - 2a/m$ which is satisfied by (p) for $m = 1$ and (r) for m $= 1/3$. (d) Equation of a normal is $y = mx - 2am - am^3$ and for $m = 1$, it represents (s).

30. (a) $x(x-a) + y(y-2a) = 0$, (b) $y = \pm 2x$, (c) $x = a \Rightarrow y = \pm 2a$ and $x = 0 \Rightarrow y = 0$, (d) $x = a \Rightarrow y = \pm 2a$, $x = 0 \Rightarrow y = \pm 2\sqrt{2}a$.

31. Statement-2 is true, perpendicular tangents to $y^2 = 4ax$ are $y = mx + a/m$, $y = m'x + a/m'$; $mm' = -1$, they intersect at a point where $(m - m')x = \frac{a(m-m')}{mm'} \Rightarrow x = -a$. Which is the directrix of the parabola. Slope of the tangents at the extremi-

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the normals to these planes, so that we have

$$l - cm - bn = 0$$
$$-cl + m - an = 0$$
$$-bl - am + n = 0$$

Eliminating l, m, n from these equations we get

$$\begin{vmatrix} 1 & -c & -b \\ -c & 1 & -a \\ -b & -a & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1 - a^2 + c(-c - ab) - b(ac + b) = 0$$

$$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$$

Example 12 The lines $x = ay + b$, $z = cy + d$, $x = a'y + b'$, $z = c'y + d'$ are perpendicular if $aa' + cc'$ equal to

(a) -1 (b) 0
(c) 1 (d) none of these

Ans. (a)

Solution Equations of the given lines can be written in the symmetric form as

$$\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$$

and $$\frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'}$$

which are perpendicular

if $aa' + 1 \times 1 + cc' = 0 \Rightarrow aa' + cc' = -1$

Example 13 The lines $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{k})$ and $\mathbf{r} = (2\mathbf{i} - \mathbf{j}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$ intersect for

(a) $\lambda = 1, \mu = 1$ (b) $\lambda = 2, \mu = 3$
(c) all values of λ and μ (d) no value of λ and μ.

Ans. (d)

Solution The given lines intersect, if the shortest distance between the lines is zero.

We know that the shortest distance between the lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ is

$$\frac{|(\mathbf{a}_1 - \mathbf{a}_2) \cdot \mathbf{b}_1 \times \mathbf{b}_2|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

So the shortest distance between the given lines is zero if

$$(\mathbf{i} - \mathbf{j} - (2\mathbf{i} - \mathbf{j})) \cdot (2\mathbf{i} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j} - \mathbf{k}) = 0$$

$$\text{L.H.S.} = \begin{vmatrix} -1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 1 \neq 0$$

Hence the given lines do not intersect.

Example 14 If p_1 and p_2 are the lengths of the perpendiculars from the points (2, 3, 4) and (1, 1, 4) respectively from the plane $3x - 6y + 2z + 11 = 0$, then p_1, p_2 are the roots of the equation

(a) $p^2 - 23p + 7 = 0$ (b) $7p^2 - 23p + 16 = 0$
(c) $p^2 - 17p + 16 = 0$ (d) $p^2 - 16p + 7 = 0$

Ans. (b)

Solution We have

$$p_1 = \frac{3\times2 - 6\times3 + 2\times4 + 11}{\sqrt{3^2 + (-6)^2 + (2)^2}} = \frac{7}{7} = 1$$

and $$p_2 = \frac{3\times1 - 6\times1 + 2\times4 + 11}{\sqrt{3^2 + (-6)^2 + (2)^2}} = \frac{16}{7}$$

so that p_1, p_2 are the roots of the equation

$$p^2 - \left(1 + \frac{16}{7}\right)p + \frac{16}{7} = 0$$

$$\Rightarrow \quad 7p^2 - 23p + 16 = 0$$

Example 15 A point moves so that the sum of the squares of its distances from the six faces of a cube given by $x = \pm 1, y = \pm 1, z = \pm 1$ is 10 units. The locus of the point is

(a) $x^2 + y^2 + z^2 = 1$ (b) $x^2 + y^2 + z^2 = 2$
(c) $x + y + z = 1$ (d) $x + y + z = 2$

Ans. (b)

Solution Let $P(x, y, z)$ be any point on the locus, then the distances from the six faces are

$$|x + 1|, |x - 1|, |y + 1|, |y - 1|, |z + 1|, |z - 1|$$

According to the given condition

$$|x + 1|^2 + |x - 1|^2 + |y + 1|^2 + |y - 1|^2 + |z + 1|^2 + |z - 1|^2 = 10$$

$$\Rightarrow \quad 2(x^2 + y^2 + z^2) = 10 - 6 = 4$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = 2.$$

Example 16 The reflection of the plane $2x + 3y + 4z - 3 = 0$ in the plane $x - y + z - 3 = 0$ is the plane

(a) $4x - 3y + 2z - 15 = 0$
(b) $x - 3y + 2z - 15 = 0$
(c) $4x + 3y - 2z + 15 = 0$
(d) none of these

Ans. (a)

Solution Using the fact that reflection of $a'x + b'y + c'z + d' = 0$ in the plane $ax + by + cz + d = 0$ is given by

$$2(aa' + bb' + cc')(ax + by + cz + d) = (a^2 + b^2 + c^2)(a'x + b'y + c'z + d')$$

We get the required equation as

$$2(2 + 3 + 4)(x - y + z - 3) = (1 + 1 + 1)(2x - 3y + 4z - 3)$$

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(vi) The domain of $f(x) = \text{cosec } x$ is $\mathbf{R} \sim \{m\pi, m \in \mathbf{I}\}$ and its range is the same as that of $\sec x$.

6. The domains and ranges of the inverse trigonometric functions are as follows:

(i) The domain of $f(x) = \sin^{-1} x$ (also written as $\arcsin x$) is $[-1, 1]$ and its range is $[-\pi/2, \pi/2]$.

(ii) The domain of $f(x) = \cos^{-1} x$ is $[-1, 1]$ and its range is $[0, \pi]$.

(iii) The domain of $f(x) = \tan^{-1} x$ is $\mathbf{R}$ and its range is

$$(-\pi/2, \pi/2) = \{x \in \mathbf{R}, -\pi/2 < x < \pi/2\}$$

(iv) The domain of $f(x) = \cot^{-1} x$ is $\mathbf{R}$ and its range is $(0, \pi)$.

(v) The domain of $f(x) = \sec^{-1} x$ is $\mathbf{R} \sim (-1, 1)$ and its range is

$$[0, \pi/2) \cup (\pi/2, \pi] = \{x \in \mathbf{R}, 0 \le x \le \pi, x \ne \pi/2\}$$

(vi) The domain of $f(x) = \text{cosec}^{-1} x$ is $\mathbf{R} \sim (-1, 1)$ and its range is

$$[-\pi/2, 0) \cup (0, \pi/2] = \{x \in R, -\pi/2 \le x \le \pi/2, x \ne 0\}$$

7. For any subset $A \subseteq \mathbf{R}$. The function $\varphi_A: \mathbf{R} \to \{0, 1\}$ defined by

$$\varphi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

is called the *characteristic function* of A. Its domain is $\mathbf{R}$ and the range is $\{0, 1\}$. The following are some simple properties:

(i) $\varphi_{A \cup B} = \varphi_A + \varphi_B - \varphi_{A \cap B}$

(ii) $\varphi_{A \cap B} = \varphi_A \varphi_B$

(iii) If A and B are disjoint sets then

$$\varphi_{A \cup B} = \varphi_A + \varphi_B$$

19.3 ALGEBRAIC OPERATIONS ON FUNCTIONS

1. If f and g are two functions, then *sum of the functions*, $f + g$ is defined for all $x \in \text{dom } f \cap \text{dom } g$ by

$$(f + g)(x) = f(x) + g(x)$$

2. If k is any real number and f is a function, then kf is defined for all $x \in \text{dom } f$ by $(kf)(x) = kf(x)$.

3. If f and g are two functions, then the *pointwise product* fg is defined for all $x \in \text{dom } f \cap \text{dom } g$ by

$$(fg)(x) = f(x)\, g(x)$$

4. If f and g are functions, then f/g is defined for all $x \in \text{dom } (f) \cap \text{dom}(g) \cap \{x: g(x) \ne 0\}$ by $(f/g)(x) = f(x)/g(x)$.

5. *Composition of functions* Let $f: A \to B$ and $g: B \to C$ be functions, then $gof: A \to C$ defined by $(gof)(x) = g(f(x))$.

Note that if gof is defined, then fog may not be defined. Let $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \cos x$ and $g(x) = x^3$. Then $gof(x) = g(f(x)) = g(\cos x) = \cos^3 x$ and $fog(x) = f(x^3) = \cos x^3$. Thus even if fog and gof are both defined it is not necessary that $fog = gof$.

We have the following formulae for domains of functions.

1. $\text{dom}(f + g) = \text{dom } f \cap \text{dom } g$
2. $\text{dom}(fg) = \text{dom } f \cap \text{dom } g$
3. $\text{dom}(f/g) = \text{dom } f \cap \text{dom } g \cap \{x : g(x) \ne 0\}$
4. $\text{dom } \sqrt{f} = \text{dom } f \cap \{x : f(x) \ge 0\}$

19.4 SOME CLASSES OF FUNCTIONS

1. *Rational function* This function is defined as the ratio of two polynomials

$$y = \frac{a_0 x^n + a_1 x^{n-1} + \cdots + a_n}{b_0 x^m + b_1 x^{m-1} + \cdots + b_m}$$

For example, $y = (x^2 + 3)/(x^3 + 4)$ is rational function.

2. *Irrational function* If in the $y = f(x)$, the operations of addition, subtraction, multiplication, division and raising to a power with rational non-integral exponents are performed on the right-hand side, the function $y = f(x)$ is said to be irrational. Examples are

$$y = (3x^2 + \sqrt{x})/\sqrt{2 + 4x} \text{ and } y = \sqrt{x}.$$

3. *Algebraic function* An algebraic function is any function $y = f(x)$ which satisfies an equation of the form

$$P_0(x) y^n + P_1(x) y^{n-1} + \cdots + P_n(x) = 0$$

where $P_0(x), P_1(x), \cdots, P_n(x)$ are certain polynomials in x. Clearly, every rational and irrational function is an algebraic function.

4. *Transcendental function* A function which is not an algebraic function is called a transcendental function. Examples are trigonometric, logarithmic and exponential functions.

5. *Explicit function* If the dependent variable, say y, is expressible explicitly in terms of the independent variable, the function $y = f(x)$ is called an *explicit function*. Otherwise it is said to be *an implicit function*. For example, $y = \cos^3 x + x^3$ is an explicit function, whereas $x^5 + y^5 - 3yx = 0$ is an implicit function.

6. *Even function* A function $y = f(x)$ is said to be an even function if $f(-x) = f(x)\ \forall\ x \in \text{dom}(f)$. Examples are $y = \cos x$ and $y = x^{2n}$. The graph of an even function $y = f(x)$ is symmetric about y-axis.

7. *Odd function* A function $y = f(x)$ is said to be an odd function if $f(-x) = -f(x)\ \forall\ x \in \text{dom}(f)$. Examples are $y = \sin x$ and $y = x^{2n+1}$. The graph of an odd function is symmetric about origin.

Clearly, $y = f(x) + f(-x)$ is always an even function,

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branches: either an upper or a lower branch.

7. *Exponential function*. The function $y = a^x$, where a is a positive constant number, is called an *exponential function*. The argument x adopts *any real* values; as the function values *only positive numbers* are considered, because otherwise we will have a multi-valued function. So, the function $y = 81^x$ has at $x = 1/4$ four different values: $y = 3$, $y = -3$, $y = 3i$ and $y = -3i$. But we consider as the function value only $y = 3$. Graphs of an exponential function for $a = 2$ and $a = 1/2$ are shown on Fig. 19.13. All they are going through the point (0, 1). At $a = 1$ we have as a graph a straight line, parallel to x-axis, i.e. the function becomes a constant value, equal to 1. At $a > 1$ an exponential function increases, and at $0 < a < 1$ – decreases.

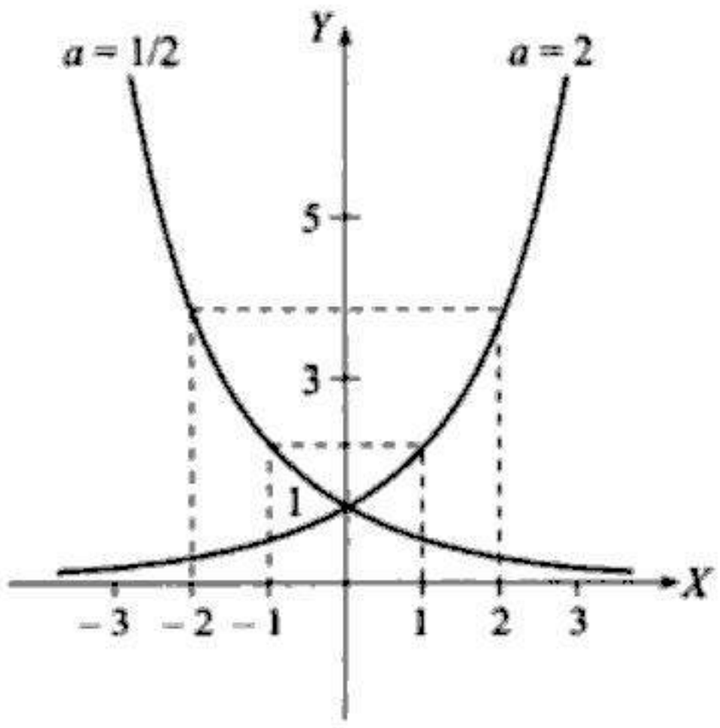

Fig.19.13

The main characteristics and properties of a exponential function:

- the function domain: $-\infty < x < +\infty$ (i.e. x is any real number) and its codomain: $y > 0$;
- this is a monotone function: it increases at $a > 1$ and decreases at $0 < a < 1$;
- the function is unbounded, continuous in everywhere, non-periodic;
- the function has no zeros.

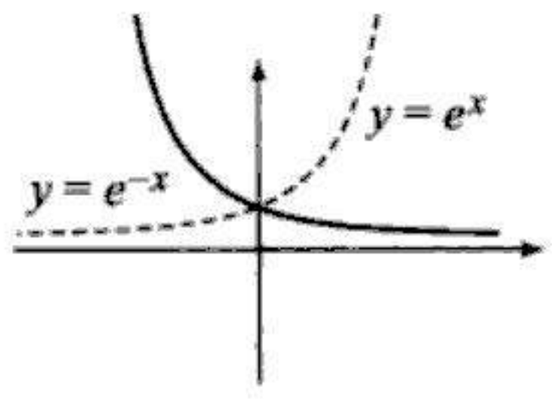

Fig.19.14

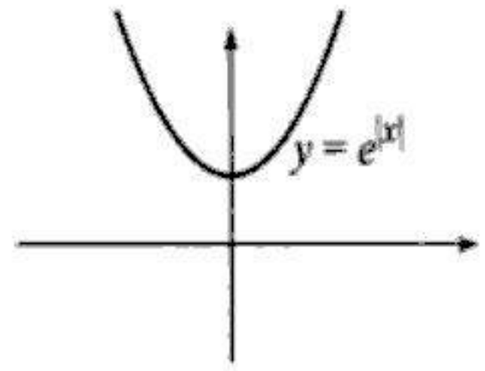

Fig.19.15

8. *Logarithmic function*. The function $y = \log_a x$, where a is a positive constant number, not equal to 1 is called a *logarithmic function*. This is an inverse function relatively to an exponential function; its graph (Fig.19.14) can be received by rotating a graph of an exponential function around of a bisector of the 1-st coordinate angle.

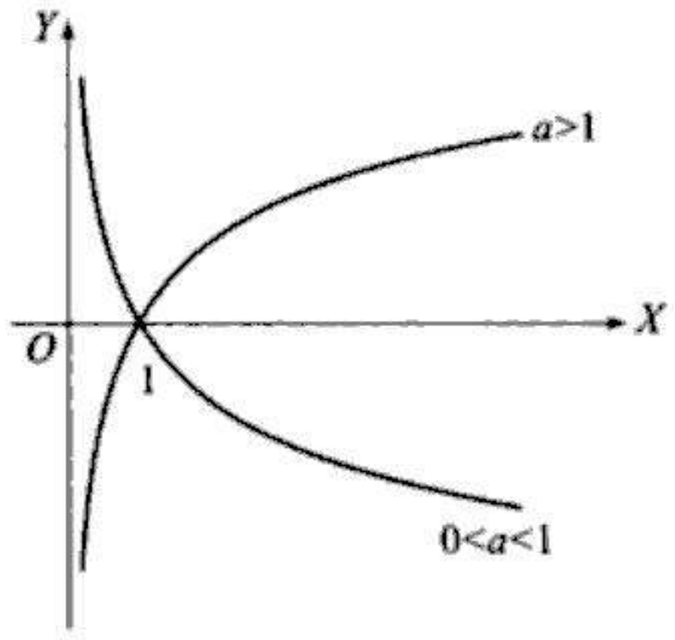

Fig.19.16

The main characteristics and properties of a logarithmic function:

- the function domain: $x > 0$ and its codomain: $-\infty < y < +\infty$ (i.e. y is any real number);
- this is a monotone function: it increases at $a > 1$ and decreases at $0 < a < 1$;
- the function is unbounded, continuous in everywhere, non-periodic;
- the function has one zero: $x = 1$.

9. *Absolute-value function* is given by

$$y = |x| = \max\{x, -x\} = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

It is depicted in Fig. 19.17.

10. *Greatest-integer function* $y = [x]$ denotes the greatest integer less than or equal to x. Examples are $[1.2] = 1$, $[1.3] = 1$, $[n] = n$ if n is an integer, and $[x] = n$ if $n \leq x < n + 1$. For positive value of x, this function might represent, for example, the legal age of a person is a function of his chronological age x. The graph of the greatest integer function lies on or below the line $y = x$.

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decreases so f takes values in $[0, \pi R^2]$. Hence the range of f is $[0, 4\pi R^2]$. Since f is strictly decreasing function on $[0, R] \cup [3R, 4R]$ so it one-one on this interval. Moreover, the function is constant on $[R, 3R]$, thus it cannot be one-one on $[R, 3R]$ or a set containing $[R, 3R]$.

Paragraph for Question Nos. 45 to 48

A cylinder is inscribed in a sphere of radius R. The volume V of the cylinder is written as $V = f(x)$, where x is the height of the cylinder.

Example 45 The domain of definition of f is

(a) $(0, R)$ (b) $(0, R^2)$

(c) $(0, 2R)$ (d) (R, ∞)

Example 46 The function V/x represents

(a) a straight line (b) an increasing function

(c) a circle (d) a decreasing function

Example 47 The function $g(x) = \frac{f(x)}{x} + \frac{\pi}{4}x^2$ is

(a) one-one (b) a constant function

(c) equal to $\frac{\pi}{4}R^2$ for all x (d) equal to $\pi R^2 x$

Example 48 If x_1 in the dom f is such that $f(x_1) = \frac{\pi x_1^2}{8}$.

Then number of such x_1 is

(a) 0 (b) 1 (c) 2 (d) 3

Ans. 45. (c), 46. (d), 47. (b), 48. (b)

Solutions 45 to 48

$V = f(x) = \pi x\left(R^2 - \frac{x^2}{4}\right)$ for $x \in (0, 2R)$. The function is not defined outside $(0, 2R)$. $V/x = \pi(R^2 - x^2/4)$ which is clearly a decreasing function. $g(x) = \frac{f(x)}{x} + \frac{\pi}{4}x^2 = \pi R^2$ which is constant. $f(x_1) = \pi x_1^3/8$ for $x_1 = 2\sqrt{2/3}\,R$, otherwise values satisfying $f(x_1) = \pi x_1^3/8$ don't lie in $(0, 2R)$.

Paragraph for Question Nos. 49 to 52

For $x \neq 0, 1$, define

$f_1(x) = x$, $f_2(x) = 1/x$, $f_3(x) = 1 - x$,

$f_4(x) = 1/(1 - x)$, $f_5(x) = (x - 1)/x$, $f_6(x) = x/(x - 1)$

This family of functions is closed under composition that is, the composition of any two of these functions is again one of these.

Example 49 Let F be a function such that $f_1 \, o \, F = f_4$. Then F is equal to

(a) f_1 (b) f_2 (c) f_3 (d) f_4

Example 50 Let G be a function such that $G \, o \, f_3 = f_6$. Then G is equal to

(a) f_5 (b) f_4 (c) f_3 (d) f_2

Example 51 Let H be a function such that $f_4 \, o \, M = f_5$. Then H is equal to

(a) f_2 (b) f_4 (c) f_5 (d) f_6

Example 52 Let J be a function such that $f_3 \, o \, J \, o \, f_2 = f_4$. Then J is equal to

(a) f_6 (b) f_5 (c) f_4 (d) f_3

Ans. 49. (d), 50. (a), 51. (b), 52. (c)

Solution 49 to 52

It is easy to see that $f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$, $f_4^{-1} = f_5$. For Q 49, $f_1 \, o \, F = f_4 \Rightarrow F = f_1^{-1} \, o \, f_4 = f_1 \, o \, f_4$. Thus $F(x) = f_1(1/(1 - x)) = 1/(1 - x) = f_4(x)$.

For Q 50, $G \, o \, f_3 = f_6 \Rightarrow G = f_6 \, o \, f_3^{-1} = f_6 \Rightarrow f_3$. So $G(x) = f_6(1 - x) = (1 - x)/(1 - x) - 1 = x - 1/x = f_5(x)$.

For Q 51, $f_4 \, o \, H = f_5 \Rightarrow H = f_4^{-1} \, o \, f_5 = f_5 \, o \, f_5$. Therefore,

$$H(x) = f_5(f_5(x)) = f_5((x - 1)/x) = \frac{(x-1)/x - 1}{(x-1)/x} = \frac{1}{1-x} = f_4(x).$$

For Q 52, $J = f_3^{-1} \, o \, f_4 \, o \, f_2^{-1} = f_3 \, o \, f_4 \, o \, f_2$. Thus $J(x) = f_3 \, o \, f_4 \, (1/x) = f_3 \, (1/(1 - 1/x)) = f_3 \, (x/x - 1) = 1 - x/x - 1 = 1/1 - x = f_4(x)$.

Paragraph for Question Nos. 53 to 56

The function $f(x) = mx$ satisfies $f(x + y) = f(x) + f(y)$ and $f(x) = a^x$ satisfies $f(x + y) = f(x) f(y)$.

From the given functional relations, we can determine several things about the functions. At times the function can be determined uniquely from the functional equation.

Example 53 If $f(x + y) = f(x) + f(y)$ for all x, y then $f(4)$ is equal to

(a) $f(1)$ (b) $4f(1)$ (c) $2f(1)$ (d) 0

Example 54 If $f(x + y) = f(x) + f(y)$ for all x, y and $f(1) = 1$ then $f(-9/8)$ is equal to

(a) 9/8 (b) 8/9 (c) $-9/8$ (d) 1

Example 55 If $f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

then

(a) $f(4x^2 + 3x) + 3f(x) = 0$

(b) $f(3x - 4x^3) + 3f(x) = 0$

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Ans.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

Solution For f to be defined we must have $-1 \le \frac{x-3}{2} \le 1$ and $6-x>0$ i.e. $1 \le x \le 5$ and $x<6$. So the domain of f is [1, 5]. Hence f is defined on [1, 3], [1, 2], [2, 5]. Similarly g is defined if $2k\pi < x < (2k+1)\pi$, $k \in \mathbf{I}$
For $k=0$, we must have $0<x<\pi$ so g is defined on [1, 3] [1, 2] f and g are continuous function wherever they are defined.

Example 73 Let $f(x) = \log_a\left(x+\sqrt{x^2+1}\right)$ $(a>0, a \neq 1)$. Then

(a) f is defined on	(p) $(0, \infty)$
(b) f increases on	(q) $(-\infty, \infty)$
(c) The inverse f is positive on	(r) $[0, a]$
(d) $(a^{f(x)} - x)^2$ increases on	(s) $(-\infty, a]$

Ans.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

Solution $f(x) = \log_a(x + \sqrt{x^2+1})$ is defined for all x, since $\sqrt{x^2+1} > |x|$ and is odd. It increases for all value of x and has an inverse. Solving

$$y = \log_a(x+\sqrt{x^2+1}),\ a^y = x+\sqrt{x^2+1}$$

so $a^{-y} = x - \sqrt{x^2+1}$. Adding we get $x = \frac{1}{2}(a^y - a^{-y})$.

Thus $f^{-1}(x) = \frac{1}{2}(a^x - a^{-x}) > 0$ for $x > 0$. $(a^{f(x)} - x)^2 = x^2 + 1$ which increases on $[0, \infty)$.

Example 74 Let $f(x) = a\cos x + b\sin x$ $(a^2+b^2>0)$ and $\tan\alpha = b/a$. Then

Column I	**Column II**
(a) f increases on	(p) $[7+\alpha, 8+\alpha]$
(b) f decreases on	(q) $[-\pi+\alpha, \alpha]$
(c) f is positive on	(r) $[\alpha, \pi/2+\alpha]$
(d) $f(x) \ge \sqrt{\frac{a^2+b^2}{2}}$ on	(s) $[\alpha, \alpha+\pi/4]$

Ans.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

Solution $f(x) = a\cos x + b\sin x = \sqrt{a^2+b^2}\cos(x-\alpha)$ where $\cos\alpha = a/\sqrt{a^2+b^2}$ and $\sin\alpha = b/\sqrt{a^2+b^2}$. It is known that $\cos x$ decreases on $2n\pi \le x \le (2n+1)\pi$, $n \in \mathbf{I}$ and increases on $(2n-1)\pi \le x \le 2n\pi$, $n \in \mathbf{I}$. Thus $\cos(x-\alpha)$ decreases on $2n\pi+\alpha \le x \le (2n+1)\pi+\alpha$ and increases on $(2n-1)\pi + \alpha \le x \le 2n\pi+\alpha$, $n \le \mathbf{I}$. For $n=1$, $[2n\pi+\alpha, (2n+1)\pi+\alpha] = [2\pi+\alpha, 3\pi+\alpha] \supset [7+\alpha, 8+\alpha]$. For $n=0$ f increases on $[-\pi+\alpha, \alpha]$. For $n=0$ f decreases $[\alpha, \pi+\alpha] \supset [\alpha, \pi/2+\alpha]$ and $[\alpha, \alpha+\pi/4]$. $f(x) > 0$ on $\left[\alpha, \frac{\pi}{2}+\alpha\right]$ and $f(x) \ge \sqrt{a^2+b^2}/2$ if $\cos(x-\alpha) \ge 1/\sqrt{2}$.

Example 75 Let $f(x) = \log\frac{1+x}{1-x}$ and $g(x) = \frac{3x+x^3}{1+3x^2}$, then

(a) $f\ o\ g\ \frac{1-e}{1+e}$	(p) 3
(b) $g\ o\ f\left(\frac{e-1}{e+1}\right)$	(q) -3
(c) $f\ o\ g(0)$	(r) 0
(d) $f\ o\ g\left(\frac{e-1}{e+1}\right)$	(s) 1

Ans.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

Solution $f\ o\ g(x) = f(g(x)) = \log\frac{1+g(x)}{1-g(x)}$

$$= \log\frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3} = \log\left(\frac{1+x}{1-x}\right)^3 = 3f(x)$$

and $g\ o\ f\left(\frac{e-1}{e+1}\right) = g(\log e) = g(1) = 1$.

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32. The function $h(x)$ is
    (a) one-one
    (b) one-one on $[-1, 1]$
    (c) a linear function on $[-2, 1]$
    (d) a linear function on $[1, 2]$
33. Which of the following is true
    (a) $h(x) = |x|$ on $[-2, 2]$
    (b) $h(x) = |x|\,\chi_{[-2, 0]}(x) + 2(x-1)\,\chi_{(1, 2]}(x)$
    (c) $h(x)$ is nonzero constant on $[0, 1]$
    (d) $h(x) = |x|\,\chi_{[-2, 0]}(x) + (x-1)\,\chi_{(0, 2]}(x)$

SECTION IV

INTEGER ANSWER TYPE

34. If $f(x) = \sin^2 x + \sin^2(x + \pi/3) + \cos x \cos(x + \pi/3)$ and $g(5/4) = 1298$ then $4\,g\,o\,f(x)$ is equal to.
35. Let $f(x) = 1 + x^2$ and g be a function such that $f(g(x)) = 1 + x^2 - 2x^3 + x^4$. The value of $g(18)$ is
36. If f: $\mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 + 1$, then the number of elements in $f^{-1}(17) \cup f^{-1}(-3)$ is
37. If f is an even function defined on the interval $(-5, 5)$, find the number of x satisfying
    $f(x) = f\left(\frac{x+1}{x+2}\right)$.
38. If $f(x) = 1 - x^3 - x^4 - 2x^5 = g(x) + h(x)$, where g is an even function and h is an odd function. Then $h(-5)$ is equal to

SECTION V

MATRIX MATCH TYPE

39. The period of the function

(a) $\sin 2\theta - \sqrt{3}\cos 2\theta$	(p) 4π
(b) $\operatorname{cosec}^2\frac{3}{4}x + \cot x$	(q) π
(c) $2\sin 3x + 3\sin 2x$	(r) 24
(d) $\sin\frac{\pi t}{3} + \sin\frac{\pi t}{4}$	(s) 2π

40. The domain of the function

(a) $\sqrt{3-x} + \sin^{-1}\frac{3-2x}{5}$	(p) $\bigcup_{k \in \mathbf{I}}(2k\pi, (2k+1)\pi)$
(b) $\log_{10}(1 - \log_{10}(x^2 - 5x + 16))$	(q) $[-4, -\pi] \cup [0, \pi]$
(c) $\cos^{-1}\frac{2}{2+\sin x}$	(r) (2, 3)
(d) $\sqrt{\sin x} + \sqrt{16 - x^2}$	(s) $[-1, 3]$

41.

(a) $y = x - \frac{x^3}{6} + \frac{x^5}{120}$	(p) even function .
(b) $y = \frac{x}{a^x - 1}$	(q) $f(0) = 1$ and odd function
(c) $y = x\frac{a^x - 1}{a^x + 1}$	(r) odd function
(d) $y = \frac{a^x + 1}{a^x - 1}$	(s) neither even nor odd

42. The functions defined have domain $\mathbf{R}$

(a) $7x + 1$	(p) onto $[-1, 1]$ but not one-one $[0, \pi]$
(b) $\cos x$	(q) one-one on $[0, \pi]$ but not onto $\mathbf{R}$
(c) $\sin x$	(r) one-one and onto $\mathbf{R}$
(d) $1 + \log x$	(s) one-one on $(0, \infty)$

43.

(a) Range of $f(x) = 3\sin x + 4\sin\left(x + \frac{\pi}{3}\right) + 7$ contains	(p) [0, 2]
(b) Range of $f(x) = \log_{\sqrt{5}}(\sqrt{2}(\sin x - \cos x) + 3)$ contains	(q) {0}
(c) domain of $f(x) = \sin^{-1}\left(\frac{1-\lvert x\rvert}{3}\right) + \cos^{-1}\left(\frac{\lvert x\rvert - 3}{5}\right)$ contains	(r) {1}
(d) domain of $f(x) = \sin^{-1}[1 - x^2]$ contains	(s) {0, 1}

44.

(a) $\{x : [\sin^{-1}x] > [\cos^{-1}x]\}$ contains	(p) {0, 1}
(b) Range of $[\lvert\sin x\rvert + \lvert\cos x\rvert]$ contains	(q) {1}
(c) Range of $[x + 1/2] + [x - 1/2] + 2[-x]$	(r) {sin1, 1}
(d) Range of $\left[\frac{1}{[x-3]}\right]$ contains	(s) {– 1}

SECTION VI

REASONING TYPE

45. **Statement-1:** The period of the function $f(x) = \sin\frac{\pi}{4}[x] + \cos\frac{\pi}{2}x + \cot\frac{\pi}{3}[x]$ is 24
    Statement-2: The period of $\sin x$, $\cos x$ is 2π and period $(f(x) + g(x)) = $ l.c.m (period of $f(x)$, period of $g(x)$)

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SECTION III

26. (a) 27. (b) 28. (d)
29. (d) 30. (c) 31. (d)
32. (d) 33. (b)

SECTION IV

34. 5192 35. 306 36. 2
37. 4 38. 6375

SECTION V

39.

	p	q	r	s
a	p	**q**	r	s
b	**p**	q	r	s
c	p	q	r	**s**
d	p	q	**r**	s

40.

	p	q	r	s
a	p	q	r	**s**
b	p	q	**r**	s
c	**p**	q	r	s
d	p	**q**	r	s

41.

	p	q	r	s
a	p	q	**r**	s
b	p	q	r	**s**
c	**p**	q	r	s
d	p	**q**	r	s

42.

	p	q	r	s
a	p	q	**r**	s
b	p	**q**	r	s
c	**p**	q	r	s
d	p	q	r	**s**

43.

	p	q	r	s
a	**p**	**q**	**r**	**s**
b	**p**	**q**	**r**	**s**
c	**p**	**q**	**r**	**s**
d	p	q	r	s

44.

	p	q	r	s
a	p	q	**r**	s
b	p	q	r	s
c	p	q	r	**s**
d	**p**	**q**	r	s

SECTION VI

45. (a) 46. (d) 47. (c)

LEVEL 2

SECTION I

1. (c) 2. (d) 3. (c)
4. (b) 5. (a) 6. (b)
7. (b) 8. (d) 9. (c)
10. (c) 11. (c) 12. (b)
13. (a) 14. (a) 15. (d)

SECTION II

16. (c) 17. (c), (d) 18. (b), (c), (d)
19. (c), (d) 20. (a), (d) 21. (b), (d)
22. (c), (d)

SECTION III

23. (d) 24. (c) 25. (d)
26. (a) 27. (a) 28. (b)
29. (a) 30. (d)

SECTION IV

31. 4 32. 1024 33. 1728
34. 871 35. 84 36. 5500

SECTION V

37.

	p	q	r	s
a	p	**q**	r	s
b	p	q	**r**	s
c	**p**	q	r	s
d	p	q	r	**s**

38.

	p	q	r	s
a	p	q	**r**	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	**s**

39.

	p	q	r	s
a	p	**q**	r	s
b	p	q	r	**s**
c	p	q	**r**	s
d	**p**	q	r	s

40.

	p	q	r	s
a	p	q	r	**s**
b	p	q	r	s
c	p	q	**r**	s
d	**p**	q	r	s

41.

	p	q	r	s
a	p	**q**	**r**	s
b	**p**	q	r	s
c	p	q	r	s
d	**p**	q	**r**	**s**

42.

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

SECTION VI

43. (a) 44. (d) 45. (c)

PAST YEAR'S IIT QUESTIONS

1. (c) 2. (a) 3. True
4. True 5. $[-2, -1] \cup [1, 2]$
6. $(-2, 1) : [-1, 1]$ 7. False
8. (b), (c) 9. $x + 1, -x + 1$
10. (i) (b) ; (ii) (c) 11. $a = 3$
12. (d) 13. (c) 14. $\frac{1}{2}(\pm 3 \pm \sqrt{5})$
15. (b) 16. (a) 17. (b)
18. (d) 19. (a) 20. (a)

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Ans. (c)

Solution $\lim_{x\to 1} \dfrac{x + x^2 + \ldots + x^n - n}{x-1}$ $\left(\dfrac{0}{0}\text{ form}\right)$

(L'Hopital Rule)

$$= \lim_{x\to 1} \frac{1 + 2x + \ldots + nx^{n-1}}{1} = 1 + 2 + \ldots + n$$

$$= \frac{n(n+1)}{2}.$$

Example 5 $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1} - \sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1} - \sqrt[5]{x^4+1}}$ equals

(a) 1 (b) 0
(c) –1 (d) none of these

Ans. (b)

Solution

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1} - \sqrt[3]{x^3+1}}{\sqrt[4]{x^4+1} - \sqrt[5]{x^4+1}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}} - \sqrt[3]{1+\frac{1}{x^3}}}{\sqrt[4]{1+\frac{1}{x^4}} - \sqrt[5]{\frac{1}{x}+\frac{1}{x^5}}} = \frac{1-1}{1-0}$$

because $1/x$, $1/x^2$, $1/x^3$, $1/x^4$ and $1/x^5$ all approach 0 as $x \to \infty$.

Example 6 $\lim_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$ is

(a) 0 (b) 1/2
(c) 2 (d) none of these

Ans. (b)

Solution $\lim_{x\to 0} \dfrac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \dfrac{\sin x - \sin x \cos x}{\cos x \cdot x^3}$

$$= \lim_{x\to 0} \frac{\sin x}{\cos x} \cdot \frac{1-\cos x}{x^3}$$

$$= \lim_{x\to 0} \frac{\tan x}{x} \cdot \frac{2\sin^2 (x/2)^2}{x^2}$$

$$= \lim_{x\to 0} \frac{\tan x}{x} \cdot \frac{2}{4}\left(\frac{\sin(x/2)}{x/2}\right)^2 = \frac{1}{2}$$

Example 7 $\lim_{n\to\infty}\left(\dfrac{1}{1-n^2} + \dfrac{2}{1-n^2} + \cdots + \dfrac{n}{1-n^2}\right)$ is equal to

(a) 0 (b) $-1/2$
(c) 1/2 (d) none of these

Ans. (b)

Solution

$$\left(\frac{1}{1-n^2} + \frac{2}{1-n^2} + \cdots + \frac{n}{1-n^2}\right) = \frac{1}{1-n^2} \times (1 + 2 + \cdots + n)$$

$$= \frac{1}{1-n^2} \cdot \frac{n(n+1)}{2} = \frac{n}{2(1-n)} = \frac{1}{2[(1/n)-1]}$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{1}{1-n^2} + \frac{2}{1-n^2} + \cdots + \frac{n}{1-n^2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2[(1/n)-1]} = -\frac{1}{2}$$

Example 8 If

$$f(x) = \begin{cases} \dfrac{[x]^2 + \sin[x]}{[x]} & \text{for } [x] \neq 0 \\ 0 & \text{for } [x] = 0 \end{cases}$$

where $[x]$ denotes the greatest integer less than or equal to x, then $\lim_{x\to 0} f(x)$ equals

(a) 1 (b) 0
(c) –1 (d) $\lim_{x\to 0+} f(x) = 0$

Ans. (d)

Solution As $x \to 0-$ (i.e., approaches 0 from the left), $[x] = -1$.

$$\therefore \quad \lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{1+\sin(-1)}{-1} = -1 + \sin 1$$

whereas, if $x \to 0+$, we get $[x] = 0$.

$$\therefore \quad f(x) = 0 \quad \Rightarrow \quad \lim_{x\to 0+} f(x) = 0$$

Thus $\lim_{x\to 0} f(x)$ does not exist.

Example 9 $\lim_{x\to \pi/3} \dfrac{\sin(\pi/3 - x)}{2\cos x - 1}$ is equal to

(a) 1/2 (b) $1/\sqrt{3}$
(c) $\sqrt{3}$ (d) $2/\sqrt{3}$

Ans. (b)

Solution

$$\lim_{x\to \pi/3} \frac{\sin(\pi/3 - x)}{2\cos x - 1} = \lim_{x\to \pi/3} \frac{-\cos(\pi/3 - x)}{-2\sin x} \text{ (\% form)}$$

$$= \frac{1}{2}\,\frac{1}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

Example 10 $\lim_{x\to 0} \dfrac{1 - \cos^3 x}{x \sin x \cos x}$ is equal to

(a) 2/5 (b) 3/5
(c) 3/2 (d) 3/4

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Ans. (c)

Solution Putting $x = 2$ and $y = 1$, we have

$$g(2)\, g(1) = g(2) + g(1) + g(2) - 2$$

$$\Rightarrow \qquad 5\, g(1) = 8 + g(1) \qquad \Rightarrow \qquad g(1) = 2.$$

Putting $y = 1/x$ in the given equation, we have

$$g(x)\, g(1/x) = g(x) + g(1/x) + g(1) - 2$$

$$= g(x) + g(1/x).$$

Hence as in Example 9 of Chapter 21, $g(x) = x^n + 1$ or $g(x) = -x^n + 1$. But $5 = g(2) = -2^n + 1$ so $-2^n = 4$ which is not possible. Thus $5 = g(2) = 2^n + 1$

$\Rightarrow$ $n = 2$. Hence, $\lim_{x\to 3} g(x) = \lim_{x\to 3} (x^2 + 1) = 10$.

Example 29 Let f: $\mathbf{R} \to \mathbf{R}$ be a function such that $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all x and y, and $f(0) = 3$ and $f'(0) = 3$. Then

(a) $f(x)/x$ is continuous on $\mathbf{R}$

(b) $f(x)$ is continuous on $\mathbf{R}$

(c) $f(x)$ is bounded on $\mathbf{R}$

(d) none of these

Ans. (b)

Solution $\lim_{h\to 0} f(x + h) = \lim_{h\to 0} f\left(\frac{2x+2h}{2}\right)$

$$= \lim_{h\to 0} \frac{1}{2}\,[f(2x) + f(2h)]$$

$$= \frac{1}{2}\, f(2x) + \frac{1}{2} \lim_{h\to 0} f(2h) = \frac{1}{2}\, f(2x) + \frac{1}{2}\, f(0)$$

(f is differentiable at 0 so continuous also)

Putting $y = 0$ in the given equation, we have

$$f(x) = f\left(\frac{2x}{2}\right) = \frac{f(2x)+f(0)}{2}.$$

Hence $$\lim_{h\to 0} f(x + h) = f(x)$$

Since $f(x)/x$ is not defined at $x = 0$, $f(x)/x$ is not continuous on $\mathbf{R}$. Clearly $f(x)$ need not be bounded on $\mathbf{R}$. e.g. $f(x) = x$. f satisfies the given equation but is not bounded on $\mathbf{R}$.

Example 30 Given the function $f(x) = 1/(1 - x)$, the number points of discontinuity of the composite function $y = f^{3n}(x)$, where $f^n(x) = fof \cdots of$ (n times) are ($n \in \mathbf{N}$)

(a) 0, 1 (b) $2n$

(c) $3n$ (d) $2n + 1$

Ans. (a)

Solution The point $x = 1$ is a discontinuity of the function $f(x) = 1/(1-x)$. If $x \neq 1$, then $u(x) = f(f(x))$ $= \frac{x-1}{x}$. Hence $x = 0$ is a point of discontinuity of the function u. If $x \neq 0$ and $x \neq 1$, then $fofof(x) = x$. Hence $y = f^{3n}(x) = (f^3(x)^n) = x$ is continuous everywhere.

Therefore, 0 and 1 are the only points of discontinuities of y.

Example 31 if

$$f(x) = \begin{cases} (1+|\sin x|)^{a/|\sin x|}, & -\pi/6 < x < 0 \\ b & x = 0 \\ e^{\tan 2x/\tan 3x} & 0 < x < \pi/6 \end{cases}$$

is a continuous function on $(-\pi/6, \pi/6)$, then

(a) $a = 2/3$, $b = e^2$ (b) $a = 1/3$, $b = e^{1/3}$

(c) $a = 2/3$, $b = e^{2/3}$ (d) none of these

Ans. (c)

Solution $b = f(0) = \lim_{x\to 0+} f(x) = \lim_{x\to 0} e^{\tan 2x/\tan 3x}$

$$= \lim_{x\to 0} e^{\frac{\tan 2x}{2x} \times \frac{3x}{\tan 3x} \times \frac{2}{3}} = e^{2/3},$$

Also, $b = f(0) = \lim_{x\to 0-} f(x) = \lim_{x\to 0} (1 + |\sin x|)^{\frac{1}{|\sin x|} \times a} = e^a$

Therefore $a = 2/3$.

Example 32 Let

$$f(x) = \lim_{n\to\infty} \frac{\log(2 + x) - x^{2n} \sin x}{1 + x^{2n}}.$$

Then

(a) f is continuous at $x = 1$

(b) $\lim_{x\to 1+} f(x) \neq \lim_{x\to 1-} f(x)$

(c) $\lim_{x\to 1+} f(x) = \sin 1$

(d) $\lim_{x\to 1-} f(x)$ doesn't exist

Ans. (b)

Solution For $|x| < 1$, $x^{2n} \to 0$ as $n \to \infty$ and for $|x| > 1$, $1/x^{2n} \to 0$ as $n \to \infty$. So

$$f(x) = \begin{cases} \log(2 + x) & \text{if } |x| < 1 \\ \lim_{n\to\infty} \dfrac{x^{-2n} \log(2 + x) - \sin x}{x^{-2n} + 1} = -\sin x & \text{if } |x| > 1 \\ \dfrac{1}{2}[\log(2 + x) - \sin x] & |x| = 1 \end{cases}$$

Thus $$\lim_{x\to 1+} f(x) = \lim_{x\to 1} (-\sin x) = -\sin 1$$

and $$\lim_{x\to 1-} f(x) = \lim_{x\to 1} \log(2 + x) = \log 3.$$

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(c) $a = \pi/6$ and $b - \pi/12$
(d) $a = \pi/6$ or $b - \pi/12$

Ans. (a), (b), (c), (d)

Solution $f(\pi/4) = \frac{\pi}{4} + a$. Also

$$f(\pi/4) = \lim_{x \to \pi/4+} = \lim_{x \to \pi/4+} (2x \cot x + b)$$

$$= \frac{\pi}{2} + b$$

so $a - b = \pi/4$

Also $f(\pi/2) = b$ and

$$f(\pi/2) = \lim_{x \to \frac{\pi}{2}+} f(x) = \lim_{x \to \frac{\pi}{2}+} (a \cos 2x - b \sin x)$$

$$= -a - b$$

so $2b = -a$ Hence $a = \pi/6$ and $b = -\pi/12$.

Example 49 Let $f(x) = g'(x) \frac{e^{a/x} - e^{-a/x}}{e^{a/x} + e^{-a/x}}$ where g' is the derivative of g and is a continuous function and $a > 0$ then $\lim_{x \to 0} f(x)$ exist if

(a) $g(x)$ is polynomial
(b) $g(x) = x$
(c) $g(x) = x^2$
(d) $g(x) = x^3 h(x)$ where $h(x)$ is a polynomial

Ans. (c) and (d)

Solution $\lim_{x \to 0+} \frac{e^{a/x} - e^{-a/x}}{e^{a/x} + e^{-a/x}} = \lim_{x \to 0+} \frac{1 - e^{-2a/x}}{1 + e^{-2a/x}} = 1$

and $\lim_{x \to 0-} \frac{e^{a/x} - e^{-a/x}}{e^{a/x} + e^{-a/x}} = \lim_{x \to 0-} \frac{e^{2a/x} - 1}{e^{2a/x} + 1} = -1$

Hence $\lim_{x \to 0} f(x)$ exists if $g'(0) = 0$

If $g(x) = Ax + B, A \neq 0$ then $\lim_{x \to 0+} f(x) = A$ and

$\lim_{x \to 0-} f(x) = -A$. Hence $\lim_{x \to 0} f(x)$ will exist if $g(x) = x^2$ or $g(x) = x^3 h(x)$ where $h(x)$ is a polynomial.

Example 50 Let $f(x) = [x] + [-x]$. Then for any integer n and $k \in \mathbf{R} \sim \mathbf{I}$

(a) $\lim_{x \to n} f(x)$ exists
(b) $\lim_{x \to k} f(x)$ exists
(c) f is continuous at $x = n$
(d) f is continuous at $x = k$

Ans. (a), (b) and (d)

Solution $f(n) = [n] + [-n] = n - n = 0$

$$\lim_{x \to n-} f(x) = \lim_{h \to 0+} [n-h] + [-n+h] = n - 1 + (-n) = -1$$

$$\lim_{x \to n+} f(x) = \lim_{h \to 0+} [n+h] + [-n-h] = n + (-n-1) = -1$$

If $k \notin \mathbf{I}$, the function $[x]$ and $[-x]$ are continuous at k. Hence $\lim_{x \to k} f(x)$ exists.

Example 51

Let $f(x) = \lim_{n \to \infty} \frac{x^{2n} - 1}{x^{2n} + 1}$, then

(a) $f(x) = 1$ for $|x| > 1$
(b) $f(x) = -1$ for $|x| < 1$
(c) $f(x)$ is not defined for any value of x
(d) $f(x) = 1$ for $|x| = 1$

Ans. (a), (b)

Solution

If $|x| > 1$, then $\lim_{n \to \infty} x^{2n} = \infty$. Thus for $|x| > 1$,

$$f(x) = \lim_{n \to \infty} \frac{1 - 1/x^{2n}}{1 + 1/x^{2n}} = 1.$$

If $|x| < 1$, then $\lim_{n \to \infty} x^{2n} = 0$, therefore for $|x| < 1, f(x) = -1$.

If $|x| = 1$ then

$x^{2n} = 1$ for any n and therefore $f(x) = 0$

Example 52 If

$$\lim_{x \to 0} \frac{a \sin x - bx + cx^2 + x^3}{2x^2 \log(1 + x) - 2x^3 + x^4}$$

exists and is finite, then

(a) a = b (b) c = 0 (c) a = b (d) c = 2

Ans. (a), (b), (c)

Solution

$$\lim_{x \to 0} \frac{a\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - bx + cx^2 + x^3}{2x^2\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) - 2x^3 + x^4}$$

$$= \lim_{x \to 0} \frac{(a-b)x + cx^2 + (1 - a/6)x^3 + ax^5/120 + \cdots}{2x^5/3 - x^6/2 + \cdots}$$

For this limit to exist we must have $a - b = 0$, $c = 0$, and $1 - a/6 = 0$, that is, $a = b = 6$ and $c = 0$.

Example 53 If $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$ then

(a) $a = c$ (b) $c = 1$
(c) $a = 2$ (d) $b = 2$

Ans. (a), (b), (d)

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$f\,o\,g(1/n) = 1/n - 1 \to -1$ as $n \to \infty$ and

$g\,o\,f(1/n) = 1 - 1/n \to 1$ as $n \to \infty$

For 64

$$\lim_{x\to\infty}\left\{\frac{a_1^{1/x}+\ldots\ldots+a_n^{1/x}}{n}\right\}^{nx}$$

$$= \lim_{y\to 0}\left\{\frac{a_1^{y}+a_2^{y}\ldots\ldots+a_n^{y}}{n}\right\}^{n/y}$$

$$= \lim_{y\to 0}\left\{1+\left((-1)+\frac{\sum_{i=1}^{n}a_i^{y}}{n}\right)\right\}^{\frac{n}{-n+a_1^{y}+\ldots\ldots+a_n^{y}}\times\frac{(a_1^{y}-1)+\ldots+(a_n^{y}-1)}{y}}$$

$$= e^{\log a_1+\log a_2+\ldots\ldots+\log a_n} = e^{\log a_1 a_2\ldots\ldots a_n} = a_1\,a_2\ldots a_n$$

Paragraph for Questions Nos. 65 to 68

Among various properties of continuous, we have if f is continuous function on $[a, b]$ and $f(a)\,f(b) < 0$, then there exists a point c in (a, b) such that $f(x) = 0$ equivalently if f is continuous on $[a, b]$ and $x \in \mathbf{R}$ is such that $f(a) < x < f(b)$ then there is $c \in (a, b)$ such that $x = f(c)$. It follows from the above result that the image of a closed interval under a continuous function is a closed interval.

Example 65 The number of continuous functions on **R** which satisfy $(f(x))^2 = x^2$ for all $x \in \mathbf{R}$ is

(a) 1 (b) 2 (c) 4 (d) 8

Example 66 Suppose that $f(1/2) = 1$ and f is continuous on [0, 1] assuming only rational value in the entire interval. The number of such function is

(a) infinite (b) 2 (c) 4 (d) 1

Example 67 Let f be a continuous function on $[-1, 1]$ satisfying $(f(x))^2 + x^2 = 1$ for all $x \in [-1, 1]$

The number of such functions is

(a) 2 (b) 1

(c) 4 (d) infinitely many

Example 68 Let $f(x) = x$, $x \neq 0$ and $f(0) = 1$. Then

(a) f is a continuous function

(b) Range of f is an interval

(c) Range of f is **R**

(d) hypothesis of the result in the comprehension is violated

Ans. 65. (c), 66. (d), 67. (a), 68. (d)

Solution 65–68

For 65 Clearly $f(0) = 0$, let $I = (0, \infty)$. Since f is continuous on I, and $f(x) \neq 0$ for any $x \in$ I, so $f(x) = x$ for all $x \in \mathbf{I}$ or $f(x) = -x$ for all $x \in \mathbf{I}$. Similarly, if $J = (-\infty, 0)$ $f(x) = x$ for all $x \in J$ and $f(x) = -x$. Combining these possibilities, we have $f(x) = x$ for all $x \in \mathbf{R}$; $f(x) = -x$ for all $x \in \mathbf{R}$; $f(x) = |x|$; $f(x) = -|x|$.

For 66 We claim that $f(x) = 1$ for all $x \in [0, 1]$. Let if possible there is c $\in [0, 1]$ such that $f(c) \neq 1$ so $f(c) < 1$ or $f(c) > 0$. Suppose that $c < 1/2$. Consider the interval $[c, 1/2]$, by above result (comprehension) f takes all values between $f(c)$ and 1 but f takes only rational values. This contradiction shows there is no $c \in [0, 1]$ such that $f(c) \neq 1$.

For 67 $f(x) = \sqrt{1-x^2}$, or $f(x) = -\sqrt{1-x^2}$ satisfy the given condition. Apply the result (comprehension) to show that these are the only possibilities.

For 68 f is not continuous at $x = 0$. Range of f is not an interval for $1, -1 \in$ Range f but $0 \in$ Range f.

Paragraph for Questions Nos. 69 to 72

The function $\sqrt[n]{1+x}-1$ and $\frac{x}{n}$ are equivalent infinitesimals as $x \to 0$. This follows from the Binomial theorem for rational indices. Taking advantage of this fact, we can approximately calculate the roots of the numbers.

Example 69 The value of $\sqrt[3]{1047}$ is

(a) 10.2 (b) 10.16

(c) 10.12 (d) 10.1

Example 70 The value of $\sqrt[3]{8144}$ is

(a) 20.2 (b) 20.16

(c) 20.12 (d) 20.4

Example 71 $\sqrt[5]{1.1}$ is equal to

(a) 1.02 (b) 1.04

(c) 1.06 (d) 1.03

Example 72 $\sqrt[5]{1080}$ is equal to

(a) 4.02 (b) 4.04

(c) 4.08 (d) 4.07

Ans. 69. (b), 70. (c), 71. (a), 72. (b)

Solution 69-72

For Q.69 $\sqrt[3]{1047} = \sqrt[3]{1000+47}$

$$= 10\sqrt[3]{1+\frac{47}{1000}} = 10\left(1+\frac{47}{3000}\right)$$

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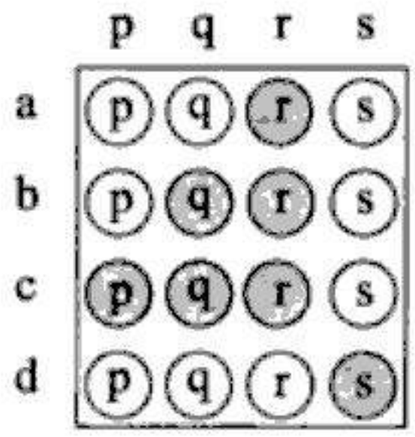
57. 1 58. 49 59. 24
60. 4

SECTION V

61.



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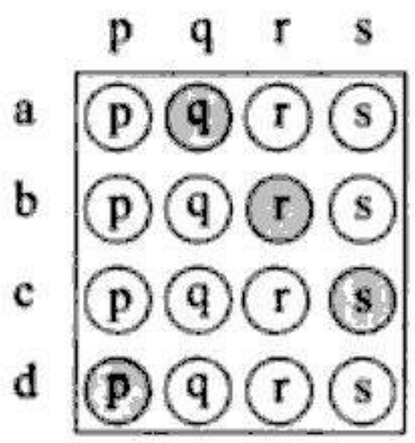

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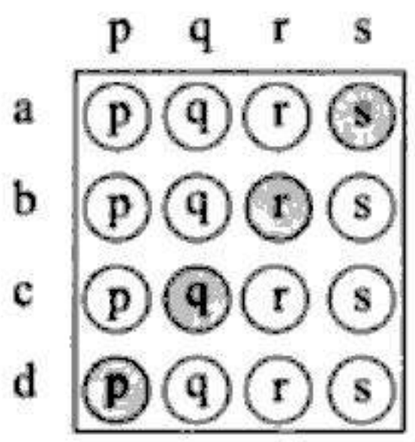

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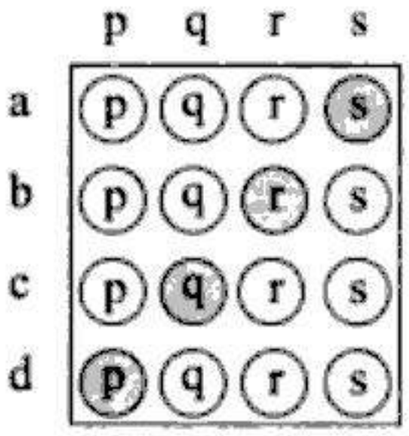
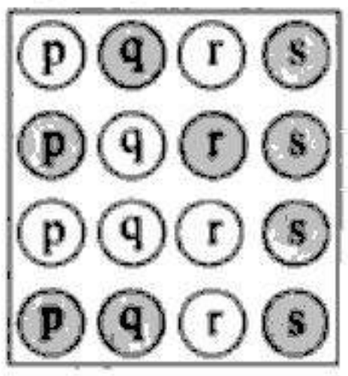


65.



SECTION VI

66. (b) 67. (d) 68. (a)

PAST YEARS' IIT QUESTIONS

1. False 2. 7

3. $f(x) = \begin{cases} \sqrt{4-x^2}, & -2 \le x \le 0 \\ -\sqrt{4-x^2}, & 0 < x \le 2 \end{cases}$

4. (b) 5. (b) 6. $2/\pi$
7. (d) 8. 1 9. -1
10. (c), (d) 11. $h\sqrt{2hr-h^2}$, $\frac{1}{128}r$
12. $a = \pi/6$; $b = -\pi/12$
13. e^5 14. $a = 8$ 15. (d)
16. (b), (c) 17. (d)
18. $\mathbf{R} \sim [-1, 0)$; $\mathbf{I} \sim \{0\}$
19. -1 20. 10 21. (d)
22. (c) 23. (c) 24. (b)
25. (c) 26. (d) 27. (c)
28. (a), (c)

HINTS AND SOLUTIONS

LEVEL 1

1. Reqd. limit $= \lim_{x\to 0} \frac{x^2}{x^2} \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1} = 4$

2. Write the given expression as

$$\frac{1}{(x+a)\left(\sqrt{x-b}+\sqrt{a-b}\right)}.$$

3. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log\left(e^{x^2}+2\sqrt{x}\right)}{\tan\sqrt{x}}$ (% form)

$$= \lim_{x\to 0} \frac{2xe^{x^2}+x^{-1/2}}{e^{x^2}+2\sqrt{x}} \times \frac{1}{\left(\sec^2\sqrt{x}\right).\frac{1}{2}x^{-1/2}}$$

$$= \lim_{x\to 0} \frac{2x^{3/2}e^{x^2}+1}{e^{x^2}+2\sqrt{x}} \times \frac{2}{\sec^2\sqrt{x}} = 2.$$

4. Divide the numerator and denominator by x^{10}.

5. Write $\frac{x^2+5x+3}{x^2+x+2} = 1+\frac{4x+1}{x^2+x+2}$ and the exponent $x = \frac{x^2+x+2}{4x+1} \times \frac{4+1/x}{1+1/x+2/x^2}$. Putting $u = (4x+1)/(x^2+x+2)$, we obtain reqd. limit as $\lim_{u\to 0}\left((1+u)^{1/u}\right)^{\lim \frac{4+1/x}{1+1/x+2/x^2}} = e^4$.

6. Write $f(x) = \frac{2\tan x}{(1+\tan x)^2}$ so $f(\pi/4) =$ $\lim_{x\to\pi/4} f(x) = 1/2$.

7. $\frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2} = \frac{2x\tan x\left(1-(1-\tan^2 x)\right)}{(1-\tan^2 x)(4\sin^2 x)}$

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SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos 44 to 46

Sometimes a function can be written as the product of two functions such that it is easy to determine the n-th derivative of two functions separately. In such cases, the nth derivative of the product can be written by applying Leibnitz's theorem which can be stated as below:

If u and v are functions of x having derivatives of nth order, and $y = uv$ then

$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_k u_{n-k} v_k + \ldots + {}^nC_n u v_n$

where suffixes denote derivative w.r.t.x

44. If $y = x^4 e^{2x}$ then $y_{10}(0)$ is equal to
    (a) 2^{10} (b) 315×2^{10}
    (c) 195×2^{10} (d) 315×2^8

45. If $y = x^2 \cos x$ then $y_8(0)$ is
    (a) 72 (b) 56
    (c) 0 (d) -56

46. If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$ then $y_n(0)$ is equal to
    (a) $(n+1)^2\, y_{n-2}(0)$ (b) $n^2\, y_{n-2}(0)$
    (c) $(n-1)^2\, y_{n-2}(0)$ (d) $(n-1)^2\, y_{n-1}(0)$

Paragraph for Question Nos 47 to 49

For a differentiable function f, define $D^*f(x) = \lim_{h\to 0} \dfrac{f^2(x+h) - f^2(x)}{h}$ where $f^2(x) = (f(x))^2$ e.g. $D^*f(x) = 2x$ if $f(x) = x$, $D^*f(x) = 2 \sin x \cos x$ if $f(x) = \sin x$ and $D^*f(x) = 2e^{2x}$ if $f(x) = e^x$.

47. If $f(x) = \tan x$ and $g(x) = \log x$ then $D^*(fg)(1)$ is equal to
    (a) $\tan 1$ (b) 0
    (c) $(\tan 1)^2$ (d) $2(\tan 1)^2$

48. If f and g are differentiable functions then $D^*(fg)$ is equal to
    (a) $f\, D^* g + g\, D^* f$ (b) $D^* f\, D^* g$
    (c) $f^2\, D^* g + g^2\, D^* f$ (d) $f(D^* g)^2 + g(D^* f)^2$

49. If $f(x) = \sin x$ and $g(x) = \cos x$ then $D^*(f \circ g)$ is equal to
    (a) $-\sin 2 (\cos x) \sin x$ (b) $-\sin(\cos x) \sin x$
    (c) $-\sin^2(\cos x) \sin x$ (d) $-\sin(\cos x) \sin^2 x$

Paragraph for Question Nos 50 to 52

Let $h \neq 0$. The difference $f(x + h) - f(x)$ is called the increment of f from x to $x + h$, and is denoted by Δf. The product $f'(x)\, h$ is called the differential of f at x with increment h and is denoted by df. The change in f from x to $x + h$ can be approximated by df.

50. Using the above approximation, the value $\sqrt{104}$ is
    (a) 10.18 (b) 10.49
    (c) 10.2 (d) 10.28

51. Using the above approximation, the value of $\cos 40°$ is
    (a) $\dfrac{1}{\sqrt{2}} - \dfrac{\pi}{\sqrt{2}\,36}$ (b) $\dfrac{1}{\sqrt{2}} + \dfrac{\pi}{36\sqrt{2}}$
    (c) 0.747 (d) 0.7267

52. A metal sphere with radius of 10 cm is to be covered with a 0.02 cm coating of silver approximately silver required is (in cu. cm)
    (a) 2π (b) 10π
    (c) 6π (d) 8π

SECTION IV

INTEGER ANSWER TYPE

53. For the function $f(x) = |x^2 - 5x + 6|$, the right hand derivative $f'(2+)$ is equal to.

54. If $f(x) = \sin^{-1}\left(x^2/\sqrt{x^4 + 16}\right)$ then $17f'(1)$ is equal to.

55. If $f(x) = (\log_{\cot x} \tan x)(\log_{\tan x} \cot x) + \tan^{-1} \dfrac{4x}{4 - x^2}$ then $2f'(2)$ is equal to.

56. If $y = \dfrac{x}{x^2 - 1}$ then $-y_5(0)$ is equal to.

57. If $y = \sin^4 x + \cos^4 x$ then $y_4(0)$ is equal to.

58. If $y = \sin(4 \sin^{-1} x)$ satisfies the relation $(1 - x^2)y_2 - xy_1 + Ay = 0$ then the value of A is.

59. If $y = e^t \cos t, x = e^t \sin t$ satisfies $y_2 (x + y)^2 = A(xy_1 - y)$ then A is equal to.

60. If $f: \mathbf{R} \to \mathbf{R}$ satisfies $|f(x) - f(y)| \leq |x - y|^3$ and $f(4) = 192$ then $f(7)$ is equal to.

61. If $a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) = 0$ where ai's are constants (not all zero) and f_1, f_2, f_3 are twice differentiable functions. Then

$$D = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ Df_1(x) & Df_2(x) & Df_3(x) \\ D^2 f_1(x) & D^2 f_2(x) & D^2 f_3(x) \end{vmatrix}$$ is equal to.

$Df_i(x) = \dfrac{d}{dx} f_i$

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$\lim_{x\to\mp\infty} y = \lim_{x\to\mp\infty} \frac{5x}{x-3} = 5$. So $x = 3$ and $y = 5$ are the only asymptotes.

For Example 55, the curve has no vertical asymptotes since it continuous at $x = 0$

$$k = \lim_{x\to\mp\infty} \frac{y}{x} = \lim_{x\to\mp\infty} \frac{|x|\sqrt{1+1/x^2}\ \sin\frac{1}{x}}{x} = \pm 1.0 = 0$$

$$b = \lim_{x\to\mp\infty} (y - kx) = \lim_{x\to\mp\infty} |x|\ \sqrt{1+1/x^2}\ \sin 1/x$$

$$= \begin{cases} 1 & as\ x \to \infty \\ -1 & as\ x \to -\infty \end{cases}$$

So curve has two horizontal asymptote.

SECTION IV

INTEGER ANSWER TYPE

Example 56 If the value of greater of $\sin x + \tan x$ and $2x$ $(0 < x < \pi/2)$ at $\pi/4$ is $g(\pi/4)$ then $g(\pi/4) - \frac{1}{\sqrt{2}}$ is equal to

Ans. 1

Solution Let $f(x) = \sin x + \tan x - 2x$. Then $g(x) = f'(x)$ $= \cos x + \sec^2 x - 2$.

$$\Rightarrow \qquad g'(x) = -\sin x + 2\sec^2 x \tan x$$

$$= \sin x \left(-1 + \frac{2}{\cos^3 x}\right)$$

$$= \sin x \left(\frac{1-\cos^3 x+1}{\cos^3 x}\right)$$

Since for $0 < x < \pi/2$, we have $0 < \cos^3 x < 1$, g is an increasing function. Hence $g(x) > g(0)$, i.e., $\cos x + \sec^2 x - 2 > 0$. Therefore, f is an increasing function, so $f(x) > f(0)$ for $0 < x < \pi/2$. Hence $\sin x + \tan x > 2x$.

Thus $g(\pi/4) = \frac{1}{\sqrt{2}} + 1$

Example 57 If the greatest value of $y = x/\log x$ on $[e, e^3]$ is u then e^3/u is equal to

Ans. 3

Solution

$$y' = \frac{\log x - 1}{(\log x)^2}$$

Since $(\log x)^2$ is always positive, we have $y' > 0$ if $\log x > 1$. That is, if $x > e$,. Therefore, y increases on $[e , e^3]$ Hence $u = e^3/3$.

Example 58 If (u, v) are the coordinates of the point on the curve $x^3 = y(x-4)^2$ where the ordinate is minimum then uv is equal to

Ans. 324

Solution The ordinate of any point on the curve is given by $y = \frac{x^3}{(x-4)^2}$

$$\frac{dy}{dx} = \frac{3x^2}{(x-4)^2} - \frac{2x^3}{(x-4)^3} = \frac{x^2(x-12)}{(x-4)^3}$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 12$$

$$(x-4)^3 (3x^2 - 24x)$$

and $$\frac{d^2y}{dx^2} = \frac{-3x^2(x-12)(x-4)^2}{(x-4)^6}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 0$$

and $$\left.\frac{d^2y}{dx^2}\right|_{x=12} = \frac{8\times 64(9\times 16) - 27\times 16\times 0}{(2\times 4)^6} > 0$$

Hence y is minimum at $x = 12$ and its value is

$$y = \frac{(3\times 4)^3}{(2\times 4)^2} = 27$$

Thus (u, v) = (12, 27)

Example 59 If $A > 0$, $B > 0$ and $A + B = \pi/3$ then the maximum value of $3 \tan A \tan B$ is

Ans. 1

Solution $y = 3 \tan A \tan B = 3 \tan A \tan(\pi/3 - A)$

$$= \frac{3x(\sqrt{3}-x)}{1+\sqrt{3}x} \quad \text{where } x = \tan A$$

$$\frac{dy}{dx} = 3\,\frac{(1+\sqrt{3}x)(\sqrt{3}-2x) - x(\sqrt{3}-x)(\sqrt{3})}{(1+\sqrt{3}x)^2}$$

$$= 3\,\frac{\sqrt{3}+3x-2x-2\sqrt{3}x^2-3x+\sqrt{3}x^2}{(1+\sqrt{3}x)^2}$$

$$= 3\,\frac{\sqrt{3}-2x-\sqrt{3}x^2}{(1+\sqrt{3}x)^2} = 3\frac{-(\sqrt{3}x-1)(x+\sqrt{3})}{(1+\sqrt{3}x)^2}$$

As $0 < A < \pi/3$, $x = \tan A > 0$.

Thus, $dy/dx = 0$ if $x = 1/\sqrt{3}$

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Solution Write $I = \int \frac{\sin x\,dx}{\sin^2 x(3+2\cos x)}$ and put

$3 + 2\cos x = t$, so that

$-\sin x\,dx = dt/2$ and

$$\sin^2 x = 1-\left(\frac{t-3}{2}\right)^2 = \frac{1}{4}\,[2^2 - (t-3)^2]$$

$$= \frac{1}{4}\,(t-1)\,(5-t)$$

$$\therefore \quad I = 2\int \frac{dt}{(t-1)(t-5)t}$$

$$= 2\int\left[\frac{1}{5}\frac{1}{t} - \frac{1}{4}\frac{1}{t-1} + \frac{1}{20}\frac{1}{t-5}\right]dt$$

$$= 2\left[\frac{1}{5}\log|t| - \frac{1}{4}\log|t-1| + \frac{1}{20}\log|t-5|\right] + C$$

$$= \frac{2}{5}\log(3+2\cos x) - \frac{1}{2}\log(1+\cos x) + \frac{1}{10}\log(1-\cos x) + C'$$

where $C' = C - \frac{1}{2}\log 2 + \frac{1}{10}\log 2$

Example 39 If $I = \int \frac{\sin 2x}{(3+4\cos x)^3}\,dx$, then I equals

(a) $\frac{3\cos x+8}{(3+4\cos x)^2} + C$ (b) $\frac{3+8\cos x}{16(3+4\cos x)^2} + C$

(c) $\frac{3+\cos x}{(3+4\cos x)^2} + C$ (d) $\frac{3-8\cos x}{16(3+4\cos x)^2} + C$

Ans. (b)

Solution Write $I = \int \frac{2\sin x\cos x}{(3+4\cos x)^3}\,dx$ and put

$3 + 4\cos x = t$, so that $-4\sin x\,dx = dt$ and

$$I = \frac{-1}{8}\int\frac{(t-3)}{t^3}\,dt = \frac{1}{8}\left(\frac{1}{t} - \frac{3}{2}\frac{1}{t^2}\right) + C$$

$$= \frac{2t-3}{16t^2} = \frac{8\cos x+3}{16(3+4\cos x)^2} + C$$

Example 40 If $I = \int e^{-x}\log(e^x+1)\,dx$, then I equals

(a) $x + (e^{-x}+1)\log(e^x+1) + C$
(b) $x + (e^x+1)\log(e^x+1) + C$
(c) $x - (e^{-x}+1)\log(e^x+1) + C$
(d) none of these

Ans. (c)

Solution $I = -e^{-x}\log(e^x+1) + \int\frac{e^{-x}e^x}{e^x+1}dx$

$$= -e^{-x}\log(e^x+1) + \int\frac{e^{-x}}{e^{-x}+1}dx$$

$$= -e^{-x}\log(e^x+1) - \log(e^{-x}+1) + C$$

$$= e^{-x}\log(e^x+1) - \log(1+e^x) + x + C$$

$$= -(e^{-x}+1)\log(e^x+1) + x + C$$

Example 41 If $I = \int \frac{dx}{x^3+1}$, then I equals

(a) $\frac{1}{3}\log\frac{|x+1|}{\sqrt{x^2-x+1}} + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

(b) $\frac{1}{3}\log\left(x^2-x+1\right) + \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$

(c) $\frac{1}{3}\log\left(x^2-x+1\right) + \frac{1}{3}\log|x+1| + \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

(d) $\frac{1}{3}\log\left(x^2-x+1\right) + \frac{1}{6}\log|x+1| + \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$

Ans. (a)

Solution $\frac{1}{x^3+1} = \frac{1}{(x+1)\left(x^2-x+1\right)}$

$$= \frac{1}{3}\left[\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right]$$

$$\Rightarrow \quad I = \frac{1}{3}\log|x+1| - \frac{1}{3}I_1$$

where $I_1 = \int\frac{x-2}{x^2-x+1}\,dx$

Put $t = x + \frac{b}{2a} = x - \frac{1}{2}$

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(b) $2\cos\alpha(1+\sin\alpha)\log|\sin(x-\alpha)| + (\cos 2\alpha - \cos\alpha)x + C$

(c) $2\sin\alpha(1+\cos\alpha)\log|\sin(x-\alpha)| + (\cos 2\alpha + \cos\alpha)x + C$

(d) none of these

36. If $I = \int e^x \frac{x^3+3x^2+4}{(x+1)^3}\,dx$, then I equals

(a) $e^x\left(\frac{x^2-x+1}{(x+1)^2}\right)+C$ (b) $e^x\left(\frac{x^2-x+1}{(x+1)^2}\right)+C$

(c) $e^x\left(\frac{x^2+x-1}{(x+1)^2}\right)+C$ (d) none of these

37. If $I = \int \tan x \tan 2x \tan 3x\,dx$, then I equals

(a) $\log|\cos x| + (1/2)\log|\cos 2x| + (1/3)\log|\cos 3x| + C$

(b) $\log|\cos x| - (1/2)\log|\cos 2x| - (1/3)\log|\cos 3x| + C$

(c) $\log|\cos x| + (1/2)\log|\cos 2x| - (1/3)\log|\cos 3x| + C$

(d) none of these

38. If $I = \int \frac{dx}{(2\sin x+\sec x)^4}$, then I equals

(a) $-\frac{1}{5\tan^5 x}+\frac{1}{3\tan^6 x}-\frac{2}{7\tan^7 x}+C$

(b) $\frac{1}{5\tan^5 x}+\frac{1}{3\tan^6 x}-\frac{1}{(2\sin x+\sec x)^3}+C$

(c) $\frac{-1}{3(2\sin x+\sec)^3}+\tan^{-1}\left(3\sqrt{\tan x}\right)+C$

(d) $\frac{-1}{3(2\sin x+\sec x)^3}-\tan^{-1}\left(3\sqrt{\tan x}\right)+C$

39. If $I = \int \tan^{-1}\left(1+\sqrt{x}\right)dx$, then I equals

(a) $\left(\sqrt{x}-1\right)^2\tan^{-1}\left(1+\sqrt{x}\right)-\sqrt{x} + \log\left(x+2\sqrt{x}+2\right)+C$

(b) $\left(2\sqrt{2}+1\right)\tan^{-1}\left(1+\sqrt{x}\right)+\sqrt{x} - \log\left(x+2\sqrt{x}+2\right)+C$

(c) $\left(\sqrt{x-1}\right)^2\tan^{-1}\left(1+\sqrt{x}\right)-\sqrt{x} + \log\left(\sqrt{x}+1\right)+C$

(d) none of these

40. If $I = \int \frac{2x+3}{(x^2+2x+3)\sqrt{x^2+2x+4}}\,dx$, then I equals

(a) $\log\left|\frac{\sqrt{x^2+2x+4}-1}{\sqrt{x^2+2x+4}+1}\right|+C$

(b) $\log\left|\frac{\sqrt{x^2+2x+4}-1}{\sqrt{x^2+2x+4}+1}\right|+\tan^{-1}\left(\frac{x+2}{3}\right)+C$

(c) $\log\tan^{-1}\left(\frac{\sqrt{x+3}}{2}\right)+C$

(d) none of these

41. If $I = \int e^x\left(\frac{x+2}{x+4}\right)^2 dx$, then I equals

(a) $\frac{x}{x+4}e^x+C$ (b) $\frac{x-2}{x+4}e^x+C$

(c) $\frac{x+1}{x+4}e^x+C$ (d) $\frac{x-1}{x+4}e^x+C$

42. If $I = \int \frac{dx}{(1+x^4)^{1/4}}$, then I equals

(a) $\frac{1}{2}\tan^{-1}\left(\frac{x}{(1+x^4)^{1/4}}\right)+C$

(b) $\frac{1}{4}\log\left|\frac{1-(1+x^4)^{1/4}}{1+(1+x^4)^{1/4}}\right|+C$

(c) $\frac{1}{2}\tan^{-1}\frac{(1+x^4)^{1/4}}{x} - \frac{1}{4}\log\left|\frac{1-(1+x^4)^{1/4}}{1+(1+x^4)^{1/4}}\right|+C$

(d) $\frac{1}{2}\tan^{-1}\frac{(1+x^4)^{1/4}}{x} - \frac{1}{4}\log\left|\frac{x-(1+x^4)^{1/4}}{x+(1+x^4)^{1/4}}\right|+C$

43. If $I = \int \cot^{-1}\left(\frac{a^2-ax+x^2}{a^2}\right)dx$, then I equals

(a) $x\tan^{-1}\left(\frac{x}{a}\right)-(x-a)\tan^{-1}\left(\frac{x-a}{a}\right)+C$

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2. If $I = \int \sin^3 x \sin \frac{x}{2}\, dx$, then I equals

(a) $\frac{16}{5} \sin^5 \frac{x}{2} - \frac{16}{7} \sin^7 \frac{x}{2} + C$

(b) $\frac{16}{5} \sin \frac{x}{2}\left(1 - \cos^4 \frac{x}{2}\right) + C$

(c) $\frac{16}{5} \cos^5 \frac{x}{2} - \frac{16}{7} \cos^7 \frac{x}{2} + C$

(d) none of these

3. If $I = \int \frac{dx}{x^{1/3}(x^{1/3} - 1)}$ then I equals

(a) $x^{2/3} + x^{1/3} \log |x^{1/3} - 1| + C$

(b) $3x^{1/3} + 3 \log |x^{1/3} - 1| + C$

(c) $3x^{2/3} + 3 \log |x^{1/3} - 1| + C$

(d) $3x^{1/3} + \log |x^{1/3} - 1| + C$

4. If $I = \int \frac{\sqrt{x-1}}{x+3}\, dx$, then I equals

(a) $2\sqrt{x-1} + 4 \tan^{-1}\left(\frac{\sqrt{x-1}}{2}\right) + C$

(b) $2\sqrt{x-1} - \tan^{-1}\left(\frac{\sqrt{x-1}}{2}\right) + C$

(c) $2\sqrt{x-1} + 4 \tan^{-1}\left(\frac{\sqrt{x-1}}{2}\right) + C$

(d) none of these

5. If $I = \int \frac{dx}{\sqrt{x} + \sqrt{1-x}}$, then I equals

(a) $\sqrt{x} - \sqrt{1-x} + \frac{1}{2\sqrt{2}} \log \left| \frac{\left(\sqrt{2x} - 1\right)\left(\sqrt{2(1-x)} + 1\right)}{\left(\sqrt{2x} + 1\right)\left(\sqrt{2(1-x)} - 1\right)} \right| + C$

(b) $\sqrt{x} + \sqrt{1-x} + \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2x} - 1}{\sqrt{2x} + 1} \right| + C$

(c) $\sqrt{x} - \sqrt{1-x} + \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2(1-x)} + 1}{\sqrt{2(1-x)} - 1} \right| + C$

(d) none of these

6. If $I = \int \frac{\tan x\, dx}{\sqrt{a + b \tan^2 x}}$ $(a > b > 0)$, then I equals

(a) $-\frac{1}{\sqrt{a-b}} \log \left| \sqrt{a-b} \cos x + \sqrt{a \sin^2 x + b \cos^2 x} \right| + C$

(b) $\frac{1}{\sqrt{a-b}} \log \left| \cos x + \sqrt{a \sin^2 x + b \cos^2 x} \right| + C$

(c) $\sqrt{a-b} \log \left| \cos x + \sqrt{a + b \tan^2 x} \right| + C$

(d) none of these

7. If $I = \int \frac{dx}{\sqrt{\cos^3 (x+\alpha) \sin (x+\beta)}}$, then I equals

(a) $2 \sec (\beta - \alpha) \sqrt{\tan (x+\alpha) \cos (\beta - \alpha) + \sin (\beta - \alpha)} + C$

(b) $2 \sec (\beta - \alpha) \sqrt{\tan (x+\alpha) \sin (\beta - \alpha)} + C$

(c) $2 \operatorname{cosec} (\alpha - \beta) \sqrt{\cot (x+\alpha) + \sin (\beta - \alpha)} + C$

(d) none of these

8. If $I = \int \sqrt{\frac{\cos (x-\alpha)}{\cos (x+\alpha)}}\, dx$, then I equals

(a) $2 \sin \alpha \log |\cos (x+\alpha) - \cos (x-\alpha)| + C$

(b) $-\sin \alpha \log \left| \cos x + \sqrt{\cos^2 x - \sin^2 \alpha} \right| + C$

(c) $\cos \alpha \sin^{-1}\left(\frac{\sin x}{\cos \alpha}\right) + C$

(d) none of these

9. If $I = \int \frac{e^x + e^{-x} + (e^x - e^{-x}) \sin x}{1 + \cos x}\, dx$

(a) $(e^x + e^{-x}) \tan (x/2) + C$

(b) $(e^x - e^{-x}) \cot (x/2) + C$

(c) $(e^x - e^{-x}) \sec (x/2) + C$

(d) $(e^x - e^{-x}) \operatorname{cosec} (x/2) + C$

10. If $I = \int x \sqrt{\frac{1-x}{1+x}}\, dx$, then I equals

(a) $x\sqrt{1-x^2} + (1/2) \sin^{-1} x + C$

(b) $(x-2) \sqrt{1-x^2} - (3/2) \sin^{-1} x + C$

(c) $(2-x) \sqrt{1-x^2} - (1/2) \sin^{-1} x + C$

(d) $(2+x) \sqrt{1-x^2} - (3/2) \sin^{-1} x + C$

11. If $I = \int (x+1) \sqrt{\frac{x+3}{x-3}}\, dx$, then I equals

(a) $(x+8) \sqrt{x^2-9} + 25 \log \left| x - \sqrt{x^2-9} \right| + C$

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(a) $\frac{1}{2n-1}$ (b) $\frac{1}{2n}$

(c) $\sqrt{2n}$ (d) $2n$

21. Value of $\int_0^{\pi} \theta \sin^2\theta \cos\theta \, d\theta$ is

(a) $-\frac{2}{3}$ (b) $-\frac{4}{3}$

(c) $-\frac{4}{9}$ (d) $-\frac{5}{9}$

22. Value of $I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$ is

(a) $\pi/4$ (b) $\pi/2$

(c) π (d) 2π

23. If $I = \int_0^{\pi/2} \left(\sqrt{\sin x}+\sqrt{\cos x}\right)^{-4} dx$, then I equals

(a) 1 (b) 1/2

(c) 1/3 (d) 1/4

24. If $\int_{1/4}^{x} \frac{dt}{\sqrt{t-t^2}} = \frac{\pi}{6}$, then x equals

(a) 1/2 (b) 1/3

(c) 1 (d) none of these

25. The value of $\int_0^{\pi/4} \sqrt{\tan\theta} \, d\theta$ is

(a) $\frac{\pi}{2\sqrt{2}}(1 + \log(3 - 2\sqrt{2}))$

(b) $\frac{\pi}{2\sqrt{2}}[1 + \log(\sqrt{2} - 1)]$

(c) $\sqrt{\pi} + \log 2$

(d) none of these

26. Value of $\int_{1/2}^{1} \frac{dx}{x\sqrt{3x^2+2x-1}}$ is

(a) $\pi/2$ (b) $\pi/3$

(c) $\pi/6$ (d) $\pi\sqrt{2}$

27. Value of $\int_{-\pi/2}^{\pi/2} \cos^3\theta(1+\sin\theta)^2 \, d\theta$ is

(a) 4/5 (b) 6/5

(c) 8/5 (d) 2

28. Value of $\int_0^{\pi} x \log \sin x \, dx$ is

(a) $-\frac{\pi}{2}\log 2$ (b) $\frac{\pi^2}{2}\log 2$

(c) $-\pi^2 \log 2$ (d) $\pi \log 2$

29. Value of $\int_0^{\pi/3} \cos^4 3\theta \sin^2 6\theta \, d\theta$ is

(a) $\pi/96$ (b) $\pi/32$

(c) $5\pi/96$ (d) $7\pi/96$

30. For a function f, $f(a+b-x) = f(x)$, and $\int_a^b xf(x)\,dx = k\int_a^b f(x)\,dx$, then value of k is

(a) $\frac{1}{2}(a+b)$ (b) $\frac{1}{2}(a-b)$

(c) $\frac{1}{2}(a^2+b^2)$ (d) $\frac{1}{2}(a^2-b^2)$

31. If a is a fixed real number such that $f(a-x)+f(a+x)=0$, then value of $\int_0^{2a} f(x)\,dx$ is

(a) $\frac{1}{2}a$ (b) 0

(c) $-\frac{1}{2}a$ (d) none of these

32. If $I = \int_0^{\pi} (\pi x - x^2)^{100} \sin 2x \, dx$, then value of I is

(a) π^{100} (b) $\frac{1}{2}(\pi^{100} - \pi^{97})$

(c) $\frac{1}{2}(\pi^{100} + \pi^{97})$ (d) 0

33. If $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$, value of $\int_{-\pi/2}^{\pi/2} \frac{\sin 2x}{\sqrt{1+\sin 2\alpha \sin x}} dx$ is

(a) $-\frac{4}{3}\tan\alpha\sec\alpha$ (b) $-\frac{4}{3}\cot\alpha\operatorname{cosec}\alpha$

(c) $-\frac{4}{3}\tan\alpha\operatorname{cosec}\alpha$ (d) $-\frac{4}{3}\cot\alpha\sec\alpha$

34. If $\int_0^{\pi} \theta^3 \log\sin\theta\,d\theta = k\int_0^{\pi} \theta^2 \log(\sqrt{2}\sin\theta)\,d\theta$, then k equals

(a) $\pi/2$ (b) π

(c) $3\pi/2$ (d) 2π

35. Value of $\int_0^{\pi/4} (\pi x - 4x^2)\log(1+\tan x)\,dx$ is

(a) $\frac{\pi^3}{192}\log 2$ (b) $\frac{\pi^3}{192}\log\sqrt{2}$

(c) $\frac{\pi^3}{36}\log 2$ (d) $\frac{\pi^3}{48}\log\sqrt{2}$

36. If $I_n = \int_0^{\pi/2} (\sin x + \cos x)^n \, dx$, then value of $nI_n - 2(n-1)I_{n-1}$ is

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Hence, prove that

$$\int_0^{\pi/2} \sin 2kx \cot x \, dx = \frac{\pi}{2}$$ [1990]

20. Evaluate

$$\int_0^{\pi} \frac{x \sin(2x) \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} \, dx$$ [1991]

21. Determine a positive integer $n \leq 5$, such that

$$\int_0^1 e^x (x-1)^n \, dx = 16 - 6e$$

22. The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is

(a) 0 (b) 1
(c) $\pi/2$ (d) $\pi/4$ [1993]

23. The value of $\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi} \, d\phi$ is _____. [1993]

24. Evaluate

$$\int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)} \, dx$$ [1993]

25. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx$ is _____. [1994]

26. Show that

$$\int_0^{n\pi+V} |\sin x| \, dx = 2n + 1 - \cos V$$

where n is a positive integer and $0 \leq V < \pi$. [1994]

27. The value of $\int_{\pi}^{2\pi} [2 \sin x] \, dx$ where [.] represents the greatest integer function, is

(a) $-\frac{5\pi}{3}$ (b) $-\pi$
(c) $\frac{5\pi}{3}$ (d) -2π [1995]

28. Evaluate the definite integral

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{2x}{1+x^2}\right) dx$$ [1995]

29. For $n > 0$

$$\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} \, dx = \text{_____}.$$ [1996]

30. Let f be a positive function. Let

$$I_1 = \int_{1-k}^{k} x f[x(1-x)] \, dx$$

$$I_2 = \int_{1-k}^{k} f[x(1-x)] \, dx$$

where $2k - 1 > 0$. Then $\frac{I_1}{I_2}$ is

(a) 2 (b) k
(c) 1/2 (d) 1 [1997]

31. $\lim_{x\to 0} \frac{\int_0^{x^2} \cos t^2 \, dt}{x \sin x} =$ _____. [1997]

32. If $g(x) = \int_0^x \cos^4 t \, dt$, $g(x + \pi)$ equals

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$
(c) $g(x)\, g(\pi)$ (d) $\frac{g(x)}{g(\pi)}$ [1997]

33. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} \, dx$ is _____ [1997]

34. Let $\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}$, $x > 0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x} \, dx = F(k) - F(1)$ then one of the possible values of k is _____. [1997]

35. Determine the value of

$$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} \, dx$$ [1997]

36. Evaluate $\int_0^{\pi/4} \log(1 + \tan x) \, dx$ [1997]

37. If $\int_0^x f(t) \, dt = x + \int_x^1 t f(t) \, dt$, then value of $f(1)$ is

(a) 1/2 (b) 0
(c) 1 (d) $-1/2$ [1998]

38. Let $f(x) = x - [x]$, for every real number x, where $[x]$ denotes the integral part of x. Then $\int_{-1}^{1} f(x) \, dx$ is

(a) 1 (b) 2
(c) 0 (d) 1/2 [1998]

39. Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = 2\int_0^1 \tan^{-1} x \, dx$. Hence, or otherwise evaluate the integral

$$\int_0^1 \tan^{-1}(1 - x + x^2) \, dx$$ [1998]

40. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to

(a) 2 (b) -2
(c) 1/2 (d) $-1/2$ [1999]

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74. Putting $\log x = t$, we have

$$\int_1^{e^{37}} \pi \frac{\sin(\pi \log x)}{x}\, dx = \pi \int_0^{37} \sin(\pi t)\, dt$$

$$= -\pi \frac{\cos \pi t}{\pi}\Big|_0^{37} = -[\cos 37\pi - 1] = 2.$$

75. $F(k) - F(1) = \int_1^4 \frac{2x e^{\sin x^2}}{x^2}\, dx = \int_1^{16} \frac{d}{dt} F(t) dt$

$(t = x^2)$

$= \int_1^{16} F'(t)\, dt = F(16) - F(1)$. Therefore $k = 16$.

76. $I = \int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi}\, d\phi =$

$$\int_{\pi/4}^{3\pi/4} \frac{\pi - \phi}{1 + \sin(\pi - \phi)} d\phi$$

$$= \pi \int_{\pi/4}^{3\pi/4} \frac{1}{1+\sin\phi} - I$$

Therefore $2I = \pi \int_{\pi/4}^{3\pi/4} \frac{d\phi}{1+\sin\phi}$

$$I = \frac{\pi}{2}\int_{\pi/4}^{3\pi/4} \frac{d\phi}{1+\cos(\pi/2-\phi)} = \frac{\pi}{4}\int_{\pi/4}^{3\pi/4} \sec^2(\pi/4 - \phi/2)\, d\phi$$

$$= \frac{\pi}{4} \times -2 \tan(\pi/4 - \phi/2)\Big|_{\pi/4}^{3\pi/4}$$

$$= -\frac{\pi}{2}[\tan(-\pi/8) - \tan(\pi/8)] = \pi\left(\sqrt{2}-1\right).$$

77. $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} (2-x) f((2-x)(2-(2-x)))\, dx$

$$= \int_{\sin^2 t}^{1+\cos^2 t} (2-x) f(x(2-x))\, dx$$

$$= 2\int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))\, dx - \int_{\sin^2 t}^{1+\cos^2 t} x f(x(2-x))\, dx$$

$= 2I_2 - I_1$.

Therefore, $2I_1 = 2I_2$ and so $I_1/I_2 = 1$.

78. $\frac{d}{dx}\left(\int_{x^2}^{x} \sqrt{\cos t}\, dt\right) = \sqrt{\cos x}\frac{dx}{dx} - \sqrt{\cos x^2}\frac{d}{dx}(x^2)$

$= \sqrt{\cos x} - 2x\sqrt{\cos x^2}$. Thus value of the differential at $x = 0$ is 1.

79. $\int_0^{1.5} x[x^2]\, dx = \int_0^1 x[x^2]\, dx + \int_0^{\sqrt{2}} x[x^2]\, dx$

$\int_{\sqrt{2}}^{1.5} x[x^2]\, dx$

$$= \int_0^1 x \cdot 0\, dx + \int_1^{\sqrt{2}} x \cdot 1 dx + \int_{\sqrt{2}}^{1.5} x \cdot 2dx = 0.75$$

80. Putting $\cos\theta = t$, the given integral reduces to

$3\int_{-1}^{1} (1 + 2t - t^2 - 2t^3)(1 + 2t + t^2)\, dt = 8.$

81. $\lim_{x\to 0} \frac{\int_{-x}^{x} f(t)\, dt}{\int_0^{2x} f(t+4)\, dt} = \lim_{x\to 0} \frac{f(x) + f(-x)}{2f(2x+4)}$

(L'Hospital Rule)

$$= \frac{1}{2}\frac{\lim_{x\to 0} f(x) + f(-x)}{\lim_{x\to 0} f(2x+4)} = \frac{1}{2}\frac{2f(0)}{f(4)} = \frac{f(0)}{f(4)} = 48.$$

82. Writing the integrand as $\frac{1}{\cos^2 x\left(\sqrt{\tan x}+1\right)^4}$ and putting $\tan x = t$, the given integral reduces to 636

$$\int_0^1 \frac{2t}{(t+1)^4}\, dt = 1272\int_0^1 \frac{1}{(t+1)^3} - \frac{1}{(t+1)^4}\, dt =$$

106.

83. Put $x = 2\sin^2\theta$ in I.

84. Use $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$ so that $I = \pi^2/4$.

85. Use the property as in Q82 and show that $I = \pi \int_0^{\pi/2} \cos^2 x \sin^4 x\, dx$ so that $I = \pi^2/32$.

86. $f(1/x) = \int_1^{\pi} \frac{\log t}{(t+1)t} dt$ so $f(x) + f(1/x) = \frac{1}{2}(\log x)^2$.

87. Use $\int_o^a f(x)\, dx = \frac{1}{2}\int_0^a (f(x) + f(a-x))\, dx$

88. $I_1 = \int_0^{\pi/2} \frac{dx}{2\cos x + \sin x + 1}$

$$= 2\int_0^1 \frac{dt}{2(1-t^2) + 2t + 1 + t^2} \qquad (t = \tan x/2)$$

$$= -2\int_0^1 \frac{dt}{t^2 - 2t - 3}$$

$$= 2\int_0^1 \frac{dt}{4 - (t-1)^2} = 2 \cdot \frac{1}{2.2} \log\left|\frac{2+t-1}{2-t+1}\right|_0^1$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{1}{3}\right] = \frac{1}{2}\log 3$$

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$$= \frac{X+1}{4(X-3)}$$

$$\Rightarrow \quad -\frac{X-3}{X+1}\,dX = dx \Rightarrow -\left(1-\frac{4}{X+1}\right)dX = dx$$

$$\Rightarrow \quad -X + 4\log(X+1) = x + \text{constant}$$

$$\Rightarrow \quad 4\log(3x-4y+1) = x + 3x - 4y + \text{constant}$$

$$\Rightarrow \quad \log(3x-4y+1) = x - y + C.$$

Example 8 The differential equation of all parabolas with axis parallel to the axis of y is

(a) $y_2 = 2y_1 + x$ (b) $y_3 = 2y_1$

(c) $y_2^3 = y_1$ (d) none of these

Ans. (d)

Solution Let (u, v) be the vertex of the parabola. The equation of the required parabolas are of the form $(x-u)^2 = A(y-v)$. There are three unknown constants.
Differentiating successively three times we have $2(x-u) = Ay_1$; $2 = Ay_2$ and $y_3 = 0$.

Example 9 The differential equation of all ellipses centred at the origin is

(a) $y^2 + xy_1^2 - yy_1 = 0$

(b) $xyy_2 + xy_1^2 - yy_1 = 0$

(c) $yy_2 + xy_1^2 - xy_1 = 0$

(d) none of these

Ans. (b)

Solution Ellipses centred at origin are given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(i)}$$

where a and b are unknown constants.

$$\frac{2x}{a^2} + \frac{2y}{b^2}y_1 = 0 \quad \Rightarrow \quad \frac{x}{a^2} + \frac{y}{b^2}y_1 = 0. \text{ (ii)}$$

Differentiating again, we have

$$\frac{1}{a^2} + \frac{1}{b^2}(y_1^2 + yy_2) = 0 \qquad \text{(iii)}$$

Multiplying (iii) with x and then subtracting from (ii), we have

$$(1/b^2)(yy_1 - xy_1^2 - xyy_2) = 0 \Rightarrow xyy_2 + xy_1^2 - yy_1 = 0$$

Example 10 The solution of the differential equation $\frac{d^2y}{dx^2} = \sin 3x + e^x + x^2$ when $y_1(0) = 1$ and $y(0) = 0$ is

(a) $\frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$

(b) $\frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x$

(c) $\frac{-\cos 3x}{3} + e^x + \frac{x^4}{12} + \frac{1}{3}x + 1$

(d) none of these

Ans.(a)

Solution Integrating the given differential equation, we have

$$\frac{dy}{dx} = \frac{-\cos 3x}{3} + e^x + \frac{x^3}{3} + C_1$$

but $y_1(0) = 1$ so $1 = (-1/3) + 1 + C_1 \Rightarrow C_1 = 1/3$.
Again integrating, we get

$$y = \frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x + C_2$$

but $y(0) = 0$ so $0 = 0 + 1 + C_2 \Rightarrow C_2 = -1$. Thus

$$y = \frac{-\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$$

Example 11 The equation of the curve satisfying the differential equation $y_2(x^2+1) = 2xy_1$ passing through the point $(0, 1)$ and having slope of tangent at $x = 0$ as 3 is

(a) $y = x^2 + 3x + 2$ (b) $y^2 = x^2 + 3x + 1$

(c) $y = x^3 + 3x + 1$ (d) none of these

Ans. (c)

Solution The given differential equation can be written as $y_2/y_1 = 2x/(x^2+1)$. Integrating both the sides we have $\log y_1 = \log(x^2+1) + C$, which implies $\log y_1(0) = \log 1 + C$, i.e., $C = \log 3$. Therefore, $\log y_1 = \log(x^2+1) + \log 3$ which implies $y_1 = 3(x^2+1)$ or $y = x^3 + 3x + A$, so $1 = y(0) = 0 + 0 + A$, i.e., $A = 1$. Hence the required equation of curve is $y = x^3 + 3x + 1$.

Example 12 A particular solution of $\log(dy/dx) = 3x + 4y$, $y(0) = 0$ is

(a) $e^{3x} + 3e^{-4y} = 4$ (b) $4e^{3x} - 3^{-4y} = 3$

(c) $3e^{3x} + 4e^{4y} = 7$ (d) $4e^{3x} + 3e^{-4y} = 7$

Ans. (d)

Solution $dy/dx = e^{3x+4y} = e^{3x}e^{4y} \Rightarrow e^{-4y}\,dy = e^{3x}\,dx$

$$\Rightarrow \quad \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

Putting $x = 0$, we have $-1/4 - 1/3 = C \Rightarrow C = -7/12$

Hence $\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12} \Rightarrow 7 = 3e^{-4y} + 4e^{3x}$.

Example 13 The solution of the differential equation

$y' = \frac{y}{x} + \frac{\phi(y/x)}{\phi'(y/x)}$ is

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Ans. 44 (c), 45 (b), 46(d)

Solution For $Q44$, we have $f(x, y, c) = y - cx^2 = 0$

$$\Rightarrow \quad \frac{y}{x^2} = c. \text{ Differentiating } \frac{x^2y' - 2xy}{x^4} = 0$$

$$\Rightarrow \qquad y' = \frac{2y}{x}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we have $y' = \frac{-x}{2y}$

$$\Rightarrow \quad 2y\,dy + x\,dx = \text{constant}$$

$$\Rightarrow \quad y^2 + \frac{1}{2}x^2 = \text{constant}$$

which represent a family of ellipses

For Q45, we consider $yx^{-3/2} = C$ and differentiate this equation $y'x^{-3/2} - \frac{3}{2}x^{-5/2}y = 0$. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we have

$$1 + \frac{3}{2}x^{-1}y\frac{dy}{dx} = 0$$

$$ydy = -\frac{2}{3}xdx$$

$$\Rightarrow \qquad \frac{y^2}{2} = -\frac{1}{3}x^2 + \text{const}$$

$$\Rightarrow \qquad 2x^2 + 3y^2 = \text{const}$$

For Q46, we consider $ye^x = C$ and differentiate $y'e^x + e^xy = 0$. Replace y' by $-\frac{dx}{dy}$,

we have $1 - y\frac{dy}{dx} = 0 \Rightarrow ydy = dx \Rightarrow \frac{y^2}{2} = x + C$

which represent a family of parabolas.

SECTION IV

INTEGER ANSWER TYPE

Example 47 If $y = e^{4x} + 2e^{-x}$ satisfies the equation $y_3 + Ay_1 + By = 0$ then the value of AB is

Ans. 156

Solution $y_1 = 4e^{4x} - 2e^{-x} \Rightarrow y_2 = 16e^{4x} + 2e^{-x} \Rightarrow y_3 = 64e^{4x} - 2e^{-x}$

Putting these values in $y_3 + Ay_1 + By = 0$, we have

$$64e^{4x} - 2e^{-x} + A(4e^{4x} - 2e^{-x}) + B(e^{4x} + 2e^{-x}) = 0$$

$$(64 + 4A + B)e^{4x} + (-2 - 2A + 2B)e^{-x} = 0$$

This is true for all x only if $64 + 4A + B = 0$ and $2 + 2A - 2B = 0$. Solving we get $A = -13$ and $B = -12$.
i.e. $AB = 156$.

Example 48 If the differential equation satisfied by $y = A\sin(98t + w_0) + B\cos(98t + w_0)$ (w_0 is a known constant) is $y_2 + Cy = 0$ then the value of C is

Ans. 9604

Solution $\frac{dy}{dt} = 98A\cos(98t + w_0) - 98B\sin(98 + w_0)$

$$\Rightarrow \frac{d^2y}{dt^2} = -9604\,A\sin(98t + w_0) - 9604\,B\cos(98t + w_0)$$

$$= -9604\,y \Rightarrow y_2 + 9604y = 0. \text{ Hence } A = 9604.$$

Example 49 If the differential equation of all straight lines which are at a fixed distance of 10 units from origin is

$(y - xy_1)^2 = A(1 + y_1^2)$ then A is equal to ... units.

Ans. 100

Solution The given family of lines can be represented by the equation

$$x\cos\alpha + y\sin\alpha = 10 \qquad (1)$$

where α is an arbitrary constant. Differentiating we have

$$\cos\alpha + \sin\alpha\,y_1 = 0 \qquad (2)$$

Multiplying (2) by x and subtracting it from (1)

$$y\sin\alpha - x\sin\alpha\frac{dy}{dx} = 10$$

$$\Rightarrow \qquad (y - xy_1)\sin\alpha = 10 \qquad (3)$$

Multiplying (1) by y_1 and (2) by y and subtract

$$xy_1\cos\alpha - y\cos\alpha = 10y_1$$

$$\Rightarrow \qquad (xy_1 - y)\cos\alpha = 10y_1 \qquad (4)$$

Square and add (3) and (4) we get, $(y - xy_1)^2 = 100(1 + y_1^2)$.

Example 50 Let $\frac{dy}{dx} = \frac{y\phi(x) - y^2}{\phi(x)}$, where $\phi(x)$ is a specified function satisfying $\phi(1) = 1$, $\phi(4) = 1296$. If $y(1) = 1$ then $y(4)$ is equal to

Ans. 324

Solution The given equation can be written as

$$\frac{dy}{dx} - y\frac{\phi'(x)}{\phi(x)} = \frac{-y^2}{\phi(x)}$$

$$\Rightarrow \qquad -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{\phi'(x)}{\phi(x)} = \frac{1}{\phi(x)} \qquad (i)$$

Put $\frac{1}{y} = u$ so that $-\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$. Now (i) becomes

$$\frac{du}{dx} + u\frac{\phi'(x)}{\phi(x)} = \frac{1}{\phi(x)}$$

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(c) obtained by putting $p = C/x$ in the given equation

(d) obtained by putting $p = C/x^2$ in the given equation

30. The solution of $\frac{dy}{dx} + x = xe^{(n-1)y}$

(a) $\frac{1}{n-1} \log \left(\frac{e^{(n-1)y} - 1}{e^{(n-1)y}} \right) = x^2/2 + C$

(b) $e^{(n-1)y} = Ce^{(n-1)y + (n-1)x2/2} + 1$

(c) $\log \left(\frac{e^{(n-1)y} - 1}{(n-1)e^{(n-1)y}} \right) = x^2 + C$

(d) $e^{(n-1)y} = ce^{(n-1)y2/2+x} + 1$

SECTION III

COMPREHENSION TYPE

Paragraph for Question Nos 31 to 33

A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant $= k > 0$). Suppose that $r(t)$ is the radius of liquid cone at time t.

31. The time after which the cone is empty is

(a) $H/2k$ (b) H/k

(c) $H/3k$ (d) $2H/k$

32. The radius of water cone at $t = 1$ is

(a) $R\ [1 - k/H]$ (b) $R\ [1 - H/k]$

(c) R [1 + H/k] (d) R[1 = k/H]

33. The value of $\sum_{i=1}^{10} r(i)$ is equal to

(a) $10\ R \left[2 - \frac{k}{H}\right]$ (b) $5\ R \left[2 + \frac{11k}{H}\right]$

(c) $5\ R \left[2 - \frac{11k}{H}\right]$ (d) $4\ R \left[2 - \frac{11k}{H}\right]$

Paragraph for Question Nos 34 to 36

Newton's law of cooling states that the rate of change of the temperature T of an object is proportional to the difference between T and the (constant) temperature T of the surrounding medium. We can write it as

$$\frac{dT}{dt} = -k(T - t),\ k > 0 \text{ constant}$$

A cup of coffee is served at 185° F in a room where the temperature is 65°F. Two minutes later the temperature of the coffee has dropped to 155°F. (log 3/4 = 0.144, log 3 = 1.09872).

34. The temperature of any object at $t = 2$ is

(a) $\tau e^{-k} + [T(0)]e^{-2k}$

(b) $\tau e^{k} + (T(0) + \tau)e^{-2k}$

(c) $\tau + [T(0) - \tau]\ e^{-2k}$

(d) $\tau + 2[T(0) - \tau]\ e^{-k}$

35. Time required for coffee to have 105°F temperature is

(a) 6 min (b) 6.43 min

(c) 7.23 min (d) 7.63 min

36. Temperature of the coffee at time t is given by

(a) $65 + 120e^{-kt}$ (b) $75 + 110e^{-kt}$

(c) $65 + 140e^{-2kt}$ (d) $75 + 140e^{-kt}$

SECTION IV

INTEGER ANSWER TYPE

37. The differential equation of parabola with axis parallel to y-axis and length of latus rectum as $4a$ is of order

38. The degree of the differential equation whose primitive is $c^2 + 2cy + a^2 - x^2 = 0$, where c is an arbitrary and a is definite constant is

39. The largest value of $\frac{2}{\pi}c$ such that there exists a differentiable function $h(x)$ for $-c < x < c$ that is a solution of $y_1 = 1 + y^2$ with $h(0) = 0$ is

40. If the solution of $\frac{x}{x^2 + y^2}\ dy = \left(\frac{y}{x^2 + y^2} - 1 \right) dx$ satisfies $y(0) = 1$, then the value of $\frac{16}{\pi}\ y(\pi/4)$ is

41. If the curve satisfying $(xy^4 + y)\,dx - xdy = 0$ passes through (1, 1) then the value of $-41\ (y(2))^3$ is

42. If the curve satisfying $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$ passes through (1, 1) then $9 + y(2)\,e^{2/y(2)} - e$ is equal to

43. If the curve satisfying $ydx + (x + x^2y)dy = 0$ passes through (1, 1) then $4\log y(4) - \frac{1}{y(4)}$ is equal to

SECTION V

MATRIX MATCH TYPE

44. The differential equation of

(a) Parabolas having their vertices at origin and foci on	(p) is a equation whose variables are separable

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SECTION V

MATRIX MATCH TYPE

44. The solution of the differential equations

(a) $\dfrac{x\,dx - y\,dy}{x\,dy - y\,dx} = \dfrac{\sqrt{1 + x^2 - y^2}}{\sqrt{x^2 - y^2}}$

(b) $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \dfrac{\sqrt{1 - (x^2 + y^2)}}{\sqrt{x^2 + y^2}}$

(c) $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\dfrac{dy}{dx} = 0$

(d) $xy_1 - y = \sqrt{x^2 + y^2}$

(p) $\sqrt{1 + x^2} + \dfrac{1}{2}\log\left|\dfrac{\sqrt{1 + x^2} - 1}{\sqrt{1 + x^2} + 1}\right| + \sqrt{1 + y^2} = C$

(q) $\sin^{-1}\sqrt{x^2 + y^2} = \tan^{-1} y/x + C$

(r) $x^2 = C^2 + 2Cy$

(s) $\sqrt{x^2 - y^2} + \sqrt{1 + x^2 - y^2} = C\left(\dfrac{x}{\sqrt{x^2 - y^2}} + \dfrac{y}{\sqrt{x^2 - y^2}}\right)$

45. The solution of the differential equations

(a) $(x^2 + y)dx - xdy = 0$

(b) $y\,(1 + xy)dx - xdy = 0$

(c) $y_1\,(y - x - 4) = x + y - 2$

(d) $xdy + ydx + y^2\,(xdy - ydx) = 0$

(p) $x^2 + 2xy - y^2 - 4x + 8y = C$

(q) $x - y/x = C$

(r) $y^2 - 1 + Cxy = 0$

(s) $x^2 + \dfrac{2x}{y} = C$

46. The functions in column 1 satisfies the differential equations in column 2.

(a) $y = x \tan (x + 2)$

(b) $y = \dfrac{1}{x} \tan (\log 2x)$

(c) $\dfrac{1}{y^2} = 4e^{2x^2} + x^2 + \dfrac{1}{2}$

(d) $x^2 = y^2(2 - y^2)$

(p) $xdx = \left(\dfrac{x^2}{y} - y^3\right) dy$

(q) $y' + 2xy = 2x^3y^3$

(r) $(x^2 + y^2 + y)dx = xdy$

(s) $(1 - xy + x^2y^2)dx = x^2dy$

47. The curves in column 1 are represented by column 2

(a) The curve such that the length of the normal is double the square of the ordinate.

(b) The curve such that the area of the figure bounded by the x-axis, two ordinates and arc MM' of this curve is double to the arc length, MM' at any choice of points M and M'

(c) The curve such that the initial ordinate at any tangent is less than the abscissa of the point of tangency of two scale units

(d) The curve such that subtangent is double to the abscissas of the point of tangency

(p) $y = Cx - x \log |x| - 2$

(q) $4y = e^{2x + C} + e^{-(2x + C)}$

(r) Catenary

(s) $y^2 = Cx$

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so putting $-1/x^2 = u$, the last equation can be written as $\frac{du}{dy} + 2uy = 2y \sin y^2$

The integrating factor of this equation is e^{y^2}
So required solution is

$$ue^{y^2} = \int 2y \sin y^2 \cdot e^{y^2}\, dy + C$$

$$= \frac{1}{2} e^{y^2} (\sin y^2 - \cos y^2) + C$$

$$\Rightarrow 2u = (\sin y^2 - \cos y^2) + Ce^{-y^2}$$

$$\Rightarrow 2 = x^2 [\cos y^2 - \sin y^2 - 2Ce^{-y^2}].$$

13. $y_1 = e^x (-a \sin x + b \cos x) + e^x (a \cos x + b \sin x)$

$$\Rightarrow y_1 = e^x (-a \sin x + b \cos x) + y$$

$$\Rightarrow y_2 = y_1 + e^x (-a \cos x - b \sin x) + e^x (-a \sin x + b \cos x)$$

$$\Rightarrow y_2 - 2y_1 + 2y = 0.$$

14. The given differential equation can be written as $y^5 x dx + y dx - x dy = 0$. Multiplying by x^3/y^5, we have $x^4\, dx + \frac{x^3}{y^3}\left(\frac{y dx - x dy}{y^2}\right) = 0$. Integrating, we get $x^5/5 + (1/4)(x/y)^4 = C$.

15. The given differential equation has variable separable so integrating, we have $y = x^2/2 - 1/x + C$. This will pass through (3, 9) if $9 = 9/2 - 1/3 + C \Rightarrow C = 29/6$. Hence the required equation is $6xy = 3x^3 + 29x - 6$.

16. The given equation can be written as

$$\frac{x dy - y dx}{x^2 + y^2} = -dx \Rightarrow \frac{x dy - y dx}{x^2} \times \frac{1}{1 + y^2/x^2} = -dx$$

$$\Rightarrow \frac{1}{1 + y^2/x^2} \frac{d}{dx} (y/x) = -dx. \text{ Integrating we have}$$

$$\tan^{-1} (y/x) = -x + c$$

$$\Rightarrow y = x \tan (c - x).$$

17. The given equation can be written as

$$y(1 + x^{-1})dx + (x + \log x)dy + \sin y\, dx + x \cos y\, dy = 0$$

$$\Rightarrow d(y(x + \log x)) + d(x \sin y) = 0$$

$$\Rightarrow y(x + \log x) + x \sin y = C \Rightarrow x(y + \sin y) + y \log x = C$$

18. The given equation can be written in the linear form as follows:

$$\frac{dy}{dx} + y\phi'(x) = \phi(x)\, \phi'(x)$$

The integrating factor of this equation is $e^{\int \phi'(x) dx} = e^{\phi(x)}$. Integrating, we have $ye^{\phi(x)} = \int te^t\, dt + C$, where $t = \phi(x) = te^t - e^t + C$. Hence $y = (\phi(x) - 1) + Ce^{-\phi(x)}$

19. The degree of the equation in (a) is clearly 1. The equation in (b) can be written as $(y - 2y_1)^2 = 9(1 - y_1^2)$ which is of degree 2. The equations in (c) and (d) are also of degree 1.

20. Put $2x + y = X \Rightarrow 2 + \frac{dy}{dx} = \frac{dX}{dx}$. Therefore, the given equation is reduced to

$$\frac{dX}{dx} - 2 = -\frac{X + 1}{2X - 1} = \Rightarrow \frac{dX}{dx} = \frac{3(X - 1)}{2X - 1}$$

21. Taking, $x = r \cos \theta$ and $y = r \sin \theta$, so that $x^2 + y^2 = r^2$ and $y/x = \tan \theta$, we have $x\, dx + y\, dy = r\, dr$ and $x\, dy - y\, dx = x^2 \sec^2 \theta\, d\theta = r^2\, d\theta$.
The given equation can be transformed into

$$\frac{r\, dr}{r^2 d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}} \Rightarrow \frac{dr}{d\theta} = \sqrt{a^2 - r^2}$$

$$\Rightarrow C + \sin^{-1} r/a = \theta = \tan^{-1} y/x$$

$$\sqrt{x^2 + y^2} = a \sin (\text{const} + \tan^{-1} y/x)$$

22. The given equation can be rewritten as

$$\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} - 2\right) y = 0 \qquad (1)$$

Let $\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} - 2\right) y = U$, then (i) reduces to

$$\frac{dU}{dx} - U = 0 \Rightarrow U = C_1' e^x. \text{ Therefore, we have}$$

$$\left(\frac{d}{dx} - 1\right)\left(\frac{dy}{dx} - 2y\right) = C_1' e^x. \text{ Putting } V = \frac{dy}{dx} - 2y$$

we have $\frac{dV}{dx} - V = C_1' e^x$ whose I.F. is e^{-x} so

$$Ve^{-x} = C_1' x + C_2' \Rightarrow V = (C_1' x + C_2') e^x. \text{ Hence } \frac{dy}{dx}$$

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Integrating both the sides we have

$$x - \frac{y^2}{2} + \frac{1}{2}\int \sqrt{u}\,du = C \qquad \text{where } u = x^2 + y^2$$

$$x - \frac{y^2}{2} + \frac{1}{3}(x^2 + y^2)^{3/2} = C$$

For (b), divide both the sides by $(x-1)y^{1/3}$ the given equation can be written as

$$\frac{1}{y^{1/3}}\frac{dy}{dx} + \frac{y^{2/3}}{x-1} = x \qquad (1)$$

Put $y^{2/3} = u$ so that $\frac{2}{3}\,y^{-1/3}\,\frac{dy}{dx} = \frac{du}{dx}$. Then (1) becomes

$$\frac{3}{2}\frac{du}{dx} + \frac{u}{x-1} = x \Rightarrow \frac{du}{dx} + \frac{2u}{3(x-1)} = \frac{2}{3}x$$

which is a liner equation in u and x whose integrating factor is $(x-1)^{2/3}$. Therefore,

$$\frac{d}{dx}(u(x-1)^{2/3}) = \frac{2}{3}\,x(x-1)^{2/3}$$

$$\Rightarrow \quad u(x-1)^{2/3} = \frac{2}{3}\int x(x-1)^{2/3}\,dx + C \qquad ...(i)$$

Putting $x - 1 = t^3$ in the last integral, it reduces to

$$2\int (t^3+1)t^4\,dt = 2\left(\frac{t^8}{8} + \frac{t^5}{5}\right)$$

Thus (1) can be written as

$$y^{2/3}(x-1)^{2/3} = \frac{(x-1)^{8/3}}{4} + \frac{2}{5}\,(x-1)^{5/3} + C$$

For (c), we can write the given equation as

$$(y + yx^{-1} + \sin y)dx + (x + \log x + x\cos y)dy = 0$$

$$\Rightarrow (y\,dx + x\,dy) + (yx^{-1} + dx + \log x\,dy) + (\sin y\,dx + x\cos y\,dy) = 0$$

$$\Rightarrow d(xy) + d(y\log x) + d(x\sin y) = 0$$

Integrating both the sides, we have

$$xy + y\log x + x\sin y = C.$$

For (d), put $y/x = V$.

48. Put $y/x = V$ in (a); put $\sin y = u$ in (b) and $1/y = z$ in (c) and $x = r\cos\theta$, $y = r\sin\theta$ in (d).

49. Separating the variable of given equation and then integrating, we have

$$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dy}{y\sqrt{y^2-1}}$$

$$\Rightarrow \quad \sec^{-1}x = \sec^{-1}y + C. \text{ Putting } x = 2, \text{ we have}$$

$$C = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}. \text{ So } \sec^{-1}x = \sec^{-1}y + \frac{\pi}{6}$$

$$\Rightarrow \quad y = \sec\left(\sec^{-1}x - \frac{\pi}{6}\right). \text{ Also } \cos^{-1}\frac{1}{x}$$

$$= \cos^{-1}\frac{1}{y} + \frac{\pi}{6} = \cos^{-1}\frac{1}{y} + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad \frac{2}{y} = \frac{\sqrt{3}}{x} - \sqrt{1 - \frac{1}{x^2}}$$

50. It is linear equation with I.F. $e^{\int \tan x\,dx} = \sec x$. Required solution is $y = \sin x + \cos x$.

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Example 27 A vector equation of the line of intersection of the planes

$$\mathbf{r} = \mathbf{b} + \lambda_1 (\mathbf{b} - \mathbf{a}) + \mu_1 (\mathbf{a} + \mathbf{c})$$
$$\mathbf{r} = \mathbf{c} + \lambda_2 (\mathbf{b} - \mathbf{c}) + \mu_2 (\mathbf{a} + \mathbf{b})$$

a, **b**, **c** being non-coplanar vectors is.

(a) $\mathbf{r} = \mathbf{a} + \mu_1 (\mathbf{b} + \mathbf{c})$ (b) $\mathbf{r} = \mathbf{b} + \mu_1 (\mathbf{a} + 2\mathbf{c})$
(c) $\mathbf{r} = a + \mu_1 (\mathbf{b} + 2\mathbf{c})$ (d) $\mathbf{r} = \mathbf{b} + \mu \mathbf{l} (\mathbf{a} + \mathbf{c})$

Ans. (d)

Solution At points of intersection of the two planes, we have

$$\mathbf{b} + \lambda_1 (\mathbf{b} - \mathbf{a}) + \mu_1 (\mathbf{a} + \mathbf{c}) = \mathbf{c} + \lambda_2 (\mathbf{b} - \mathbf{c}) + \mu_2 (\mathbf{a} + \mathbf{b})$$
$$\Rightarrow \quad (-\lambda_1 + \mu_1 - \mu_2)\,\mathbf{a} + (1 + \lambda_1 - \lambda_2 - \mu_2)\,\mathbf{b} + (\mu_1 - 1 + \lambda_2)\,\mathbf{c} = 0.$$

As **a**, **b**, **c** are non-coplanar, we have

$$-\lambda_1 + \mu_1 - \mu_2 = 0,\ 1 + \lambda_1 - \lambda_2 - \mu_2 = 0,\ \mu_1 - 1 + \lambda_2 = 0.$$

Eliminating λ_2, μ_2, i.e., writing $\lambda_2 = 1 - \mu_1$ from the last equation, in the second equation we have $\lambda_1 = 0$, $\mu_2 = \mu_1$. So the required line is

$$\mathbf{r} = \mathbf{b} + \mu_1 (\mathbf{a} + \mathbf{c})$$

Example 28 An equation of a plane containing the lines $r = \mathbf{a}_1 + t\,\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + t\,\mathbf{b}_2$ where $[\mathbf{a}_2 - \mathbf{a}_1, \mathbf{b}_1, \mathbf{b}_2] = 0$ is

(a) $[\mathbf{r} - \mathbf{a}_1, \mathbf{b}_1, \mathbf{b}_2] = 0$ (b) $[\mathbf{r} - \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2] = 0$
(c) $[\mathbf{r} - \mathbf{a}_2, \mathbf{a}_1, \mathbf{b}_2] = 0$ (d) $[r - \mathbf{a}, \mathbf{a}_2, \mathbf{b}_2] = 0$

Ans. (a)

Solution (i) The given lines passes through the points with position $\mathbf{a}_1$ and $\mathbf{a}_2$ and a normal to both the lines is $\mathbf{b}_1 \times \mathbf{b}_2$ (the given condition $[\mathbf{a}_2 - \mathbf{a}_1, \mathbf{b}_1, \mathbf{b}_2] = 0$ ensures that lines are either parallel or intersecting). Hence the required equation of the plane is

$$\mathbf{r} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = \mathbf{a}_1 \cdot (\mathbf{b}_1 \times \mathbf{b}_2)$$

or $\quad [\mathbf{r} - \mathbf{a}_1\ \mathbf{b}_1\ \mathbf{b}_2] = 0.$

Example 29 The equation of the plane which passes through the line of intersection of the planes $\mathbf{r} \cdot \mathbf{n}_1 = q_1$, $\mathbf{r} \cdot \mathbf{n}_2 = q_2$ and is parallel to the line of intersection of the line of intersection of the planes $\mathbf{r} \cdot \mathbf{n}_3 = q_2$ and $\mathbf{r} \cdot \mathbf{n}_4 = q_4$ given that $[\mathbf{n}_1\ \mathbf{n}_3\ \mathbf{n}_4\} = \{\mathbf{n}_2\ \mathbf{n}_3\ \mathbf{n}_4]$ is

(a) $\mathbf{r} \cdot \mathbf{n}_2 - q_2 = \mathbf{r} \cdot \mathbf{n}_3 - q_3$
(b) $\mathbf{r} \cdot \mathbf{n}_1 - q_1 = [\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3]\ (\mathbf{r} \cdot \mathbf{n}_2 - q_2)$
(c) $\mathbf{r} \cdot \mathbf{n}_1 - q_1 = \mathbf{r} \cdot \mathbf{n}_2 - q_2$
(d) $\mathbf{r} \cdot \mathbf{n}_3 - q_3 = \mathbf{r} \cdot \mathbf{n}_1 - q_1$,

Ans. (c)

Solution Equation of any plane through the intersection of $\mathbf{r} \cdot \mathbf{n}_1 = q_1$, $\mathbf{r}_2 \cdot \mathbf{n}_2 = q_2$ is of the form

$$\mathbf{r} \cdot \mathbf{n}_1 + \lambda\, \mathbf{r} \cdot \mathbf{n}_2 = q_1 + \lambda q_2 \qquad (1)$$

where λ is a parameter.

So $\mathbf{n}_1 + \lambda\ \mathbf{n}_2$ is normal to the plane (1). Any plane containing planes $\mathbf{r} \cdot \mathbf{n}_3 = q_3$, $\mathbf{r} \cdot \mathbf{n}_4 = q_4$ is of the form $\mathbf{r} \cdot (\mathbf{n}_3 + \mu \mathbf{n}_4) = q_3 + \mu q_4$. Hence we must have $\mathbf{n}_1 + \lambda\, \mathbf{n}_2 = k\,(\mathbf{n}_3 + \mu\, \mathbf{n}_4)$ for some k

$$\Rightarrow \quad [\mathbf{n}_1 + \lambda\, \mathbf{n}_2] \cdot [\mathbf{n}_3 \times \mathbf{n}_4] = 0$$
$$\Rightarrow \quad [\mathbf{n}_1, \mathbf{n}_3, \mathbf{n}_4] + \lambda\, [\mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_4] = 0$$
$$\Rightarrow \quad \lambda = -1.$$

Putting this value in (1), we have equation of required plane as

$$\mathbf{r} \cdot \mathbf{n}_1 - q_1 = (\mathbf{r} \cdot \mathbf{n}_2 - q_2)$$

or $\quad [\mathbf{n}_2\ \mathbf{n}_3\ \mathbf{n}_4]\ (\mathbf{r} \cdot \mathbf{n}_1 - q_1) = [\mathbf{n}_1\ \mathbf{n}_3\ \mathbf{n}_4]\ (\mathbf{r} \cdot \mathbf{n}_2 - q_2)$

Example 30 If $\mathbf{r} \cdot \mathbf{a} = 0$, $\mathbf{r} \cdot \mathbf{b} = 1$ and $[\mathbf{r}\ \mathbf{a}\ \mathbf{b}] = 1$, $\mathbf{a} \cdot \mathbf{b} \neq 0$, $(\mathbf{a} \cdot \mathbf{b})^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 = 1$ the value of **r** in terms of **a** and **b** is

(a) $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \times \mathbf{b}$ (b) $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|^2}$

(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{a}) + \dfrac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b} \times \mathbf{a}|^2}$ (d) $\dfrac{\mathbf{a} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times (\mathbf{a} \times \mathbf{b})|} + \mathbf{a} \times \mathbf{b}$

Ans. (b)

Solution Writing **r** as linear combination of **a**, **b** and $\mathbf{a} \times \mathbf{b}$, we have

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\,(\mathbf{a} \times \mathbf{b})$$

for scalars x, y, z

$$0 = \mathbf{r} \cdot \mathbf{a} = x\,|\mathbf{a}|^2 + y\mathbf{a} \cdot \mathbf{b}$$
$$1 = \mathbf{r} \cdot \mathbf{b} = x\mathbf{a} \cdot \mathbf{b} + \mathbf{y}\,|\mathbf{b}|^2$$

Solving we get $\quad y = \dfrac{|\mathbf{a}|^2}{|\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2} = |\mathbf{a}|^2$

and $\quad x = \dfrac{\mathbf{a} \cdot \mathbf{b}}{(\mathbf{a} \cdot \mathbf{b})^2 - |\mathbf{a}|^2\,|\mathbf{b}|^2} = \mathbf{a} \cdot \mathbf{b}$

Also $1 = [\mathbf{r}\ \mathbf{a}\ \mathbf{b}] = z\,|\mathbf{a} \times \mathbf{b}|^2$

$$\Rightarrow \quad z = \frac{1}{|a \times b|^2}$$

Thus $\mathbf{r} = ((\mathbf{a} \cdot \mathbf{b})\,\mathbf{a} - |\mathbf{a}|^2\,\mathbf{b}) + \dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|^2}$

$$= \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|^2}.$$

Example 31. If $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are perpendicular satisfying $\mathbf{a} \cdot \mathbf{c} = \lambda$, $\mathbf{b} \cdot \mathbf{d} = \lambda$, $(\lambda > 0)$ $\mathbf{a} \cdot \mathbf{d} = 4$, $\mathbf{b} \cdot \mathbf{c} = 9$ then λ is equal to

(a) 2 (b) 3 (c) 6 (d) 36

Ans. (c)

Solution. We know that

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$$

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$$= \frac{1}{4}[2 + 2\cos\alpha]$$

$$= \cos^2 \alpha/2$$

$\Rightarrow$ Unit vector along $\mathbf{AD} = \dfrac{1}{2\cos(\alpha/2)}(\mathbf{u} + \mathbf{v})$

EXERCISE

LEVEL 1

SECTION I

SINGLE CORRECT CHOICE TYPE

1. The volume of the parallelopiped whose sides are given by $\mathbf{OA} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{OB} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{OC} = 3\mathbf{i} - \mathbf{k}$ is
   (a) 4/13 (b) 4
   (c) 2/7 (d) none of these
2. The points with position vectors $60\mathbf{i} + 3\mathbf{j}$, $40\mathbf{i} - 8\mathbf{j}$, $a\mathbf{i} - 52\mathbf{j}$ are collinear if
   (a) $a = -40$ (b) $a = 40$
   (c) $a = 20$ (d) none of these
3. Let $\mathbf{p}, \mathbf{q}, \mathbf{r}$ be three mutually perpendicular vectors of the same magnitude. If a vector x satisfies the equation
   $\mathbf{p} \times ((\mathbf{x} - \mathbf{q}) \times \mathbf{p}) + \mathbf{q} \times ((\mathbf{x} - \mathbf{r}) \times \mathbf{q}) + \mathbf{r} \times ((\mathbf{x} - \mathbf{p}) \times \mathbf{r}) = \mathbf{O}$ then $\mathbf{x}$ is given by
   (a) $\frac{1}{2}(\mathbf{p} + \mathbf{q} - 2\mathbf{r})$ (b) $\frac{1}{2}(\mathbf{p} + \mathbf{q} + \mathbf{r})$
   (c) $\frac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r})$ (d) $\frac{1}{3}(2\mathbf{p} + \mathbf{q} - \mathbf{r})$
4. If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ $|\mathbf{c}| = 4$ and $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ then the value of $\mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b}$ is equal to
   (a) 19/2 (b) − 19/2
   (c) 29/2 (d) − 29/2
5. If A, B, C, D are four points in a space and $|\mathbf{AB} \times \mathbf{CD} + \mathbf{BC} \times \mathbf{AD} + \mathbf{CA} \times \mathbf{BD}| = \lambda$ (area of the triangle ABC). Then the value of λ is
   (a) 1 (b) 2
   (c) 3 (d) 4
6. Given $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. A unit vector perpendicular to both $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{c}$ is
   (a) $\dfrac{2\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{6}}$ (b) $\mathbf{j}$
   (c) $\mathbf{k}$ (d) $\dfrac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$
7. Let P, Q, R be points with position vectors $\mathbf{r}_1 = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{r}_2 = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{r}_3 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ relative to an origin 0. The distance of P from the plane OQR is (magnitude)
   (a) 2 (b) 3
   (c) 1 (d) $11/\sqrt{3}$
8. A vector $\mathbf{a}$ which is collinear with the vector $6\mathbf{i} - 8\mathbf{j} - \frac{15}{2}\mathbf{k}$ of magnitude 50 making on obtuse angle with z-axis is
   (a) $24\mathbf{i} - 32\mathbf{j} - 30\mathbf{k}$ (b) $-24\mathbf{i} + 32\mathbf{j} + 30\mathbf{k}$
   (c) $24\mathbf{i} + 32\mathbf{j} - 30\mathbf{k}$ (d) none of these
9. Let there be two points A, B on the curve $y = x^2$ in the plane OXY satisfying $\mathbf{OA} \cdot \hat{i} = 1$ and $\mathbf{OB} \cdot \hat{i} = -2$ then the length of the vector $2\mathbf{OA} - 3\mathbf{OB}$ is
   (a) $\sqrt{14}$ (b) $2\sqrt{51}$
   (c) $3\sqrt{41}$ (d) none of these
10. If A, B, C, D are four points in space satisfying $\mathbf{AB} \cdot \mathbf{CD} = K[|\mathbf{AD}|^2 + |\mathbf{BC}|^2 - |\mathbf{AC}|^2 - |\mathbf{BD}|^2]$ then the value of K is
    (a) 2 (b) 1/3
    (c) 1/2 (d) 1
11. For unit vectors $\mathbf{b}$ and $\mathbf{c}$ and any vector $\mathbf{a}$, the value of $\{\{(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})\} \cdot (\mathbf{b} \times \mathbf{c})\} \cdot (\mathbf{b} + \mathbf{c})$ is
    (a) $|\mathbf{a}|^2$ (b) $2|\mathbf{a}|^2$
    (c) $3|\mathbf{a}|^2$ (d) none of these
12. The distance of the point B with position vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ from the line passing through the point A with position vector $4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and parallel to the vector $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ is
    (a) $\sqrt{10}$ (b) $\sqrt{5}$
    (c) $\sqrt{6}$ (d) none of these
13. Three non-coplanar vector $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are drawn from a common initial point. The angle between the plane passing through the terminal points of these vectors and the vector $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is
    (a) $\pi/4$ (b) $\pi/2$
    (c) $\pi/3$ (d) none of these
14. If p^{th}, q^{th}, r^{th} term of a G.P. are the positive numbers a, b, c, then the angle between the vectors $\log a^2\mathbf{i} + \log b^2\mathbf{j} + \log c^2\mathbf{k}$ and $(q - r)\mathbf{i} + (r - p)\mathbf{j} + (p - q)\mathbf{k}$ is
    (a) $\pi/3$ (b) $\pi/2$
    (c) $\sin^{-1} 1/\sqrt{a^2 + b^2 + c^2}$ (d) none of these

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45. A parametrized tangent vector to $r(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at (2, 4, 8) is
    (a) $R(u) = 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k} + u(\mathbf{i} + \mathbf{j} + 4\mathbf{k})$
    (b) $R(u) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + u(\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$
    (c) $R(u) = \mathbf{i} + 4\mathbf{j} + 12\mathbf{k} + u(2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k})$
    (d) $R(u) = 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k} + u(\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$
46. The tangent vector to $r(t) = 2t^2\,\mathbf{i} + (1 - t)\,\mathbf{j} + (3t^2 + 2)\,\mathbf{k}$ at (2, 0, 5) is
    (a) $4\mathbf{i} + \mathbf{j} - 6\mathbf{k}$ (b) $4\mathbf{i} - \mathbf{j} + 6\mathbf{k}$
    (c) $2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$ (d) $2\mathbf{i} + \mathbf{j} - 6\mathbf{k}$

SECTION IV

INTEGER ANSWER TYPE

47. If Δ is the area of the triangle whose vertices A, B and C have position vectors $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ w.r.t. origin of reference O find Δ^2.
48. If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are the non-coplanar vectors, find
$$\frac{\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}}{\mathbf{C}\times\mathbf{A}\cdot\mathbf{B}} + \frac{\mathbf{B}\cdot\mathbf{A}\times\mathbf{C}}{\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}}.$$
49. If $\mathbf{A} = (1, 1, 1)$, $\mathbf{C} = (0, 1, -1)$ are given vector and $\mathbf{B}$ is a vector satisfying the equations $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ and $\mathbf{A} \cdot \mathbf{B} = 3$ find $9|\mathbf{B}|^2$.
50. If the vectors $a\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + b\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + c\mathbf{k}$, $(a \neq b \neq c \neq 1)$ are coplanar, find the value of
$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}.$$
51. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\mathbf{A} = (1, a, a^2)$ $\mathbf{B} = (1, b, b^2)$, $\mathbf{C} = (1, c, c^2)$ are non-coplanar find the product abc.
52. Find the value of $|\mathbf{a} \times \mathbf{i}|^2 + |\mathbf{a} \times \mathbf{j}|^2 + |\mathbf{a} \times \mathbf{k}|^2$ if $|\mathbf{a}| = 1$.
53. A nonzero vector $\mathbf{a}$ is parallel to the line of intersection of the plane determined by the vectors $\mathbf{i}$, $\mathbf{i} + \mathbf{j}$ and the plane determined by the vectors $\mathbf{i} - \mathbf{j}$, $\mathbf{i} + \mathbf{k}$. If the acute angle between $\mathbf{a}$ and the vector $\mathbf{i} - 2\mathbf{i} + 2\mathbf{k}$ is θ find $\sqrt{2}\cos\theta$.
54. Let $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = 10\mathbf{a} + 2\mathbf{b}$ and $\mathbf{OC} = \mathbf{b}$, where O, A and C are non-collinear points. Let p denote the area of the quadrilateral $OABC$, and let q denote the area of the parallelogram with OA and OC as adjacent sides. If $p = Kq$ find K.
55. Let $\mathbf{a} = \mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{d} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$. If p is the shortest distance between the lines $\mathbf{r} = \mathbf{a} + t\,\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + p\,\mathbf{d}$ find $2p^2$.
56. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. If the equation of the plane through points with position vectors $\mathbf{a}$ and $\mathbf{b}$ and parallel to the vector $\mathbf{c}$ is find $\mathbf{r} \cdot (-3\mathbf{i} + 2\mathbf{j} + 2\mathbf{R}) = \lambda$, find λ.
57. If $\mathbf{a}$, $\mathbf{c}$, $\mathbf{d}$ are non-coplanar vectors satisfying $\mathbf{d} \cdot (\mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{d}))) = k[\mathbf{a}\,\mathbf{c}\,\mathbf{d}]$ and $\mathbf{b} \cdot \mathbf{d} = 14$ find k.

SECTION V

MATRIX MATCH TYPE

58. An equation of the plane

(a) containing the line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and passing through a point $\mathbf{c}$	(p) $\mathbf{r} = (1 - t - p)\,\mathbf{a} + t\mathbf{b} + p\mathbf{c}$
(b) containing the line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and perpendicular to the plane $\mathbf{r} \cdot \mathbf{c} = q$	(q) $\mathbf{r} = \mathbf{a} + t(\mathbf{c} - \mathbf{a}) + p\mathbf{b}$
(c) through points with position vector $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$	(r) $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) + p\mathbf{b}$
(d) through two point whose position vectors are $\mathbf{a}$, $\mathbf{b}$ and parallel to $\mathbf{c}$	(s) $\mathbf{r} = (1 - t)\mathbf{a} + t\mathbf{a} + p\mathbf{c}$

59. If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$; $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ then

(a) $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}$	(p) 8
(b) $((\mathbf{a} \cdot \mathbf{c})\mathbf{c} + (\mathbf{c} \cdot \mathbf{b})\mathbf{a}) \cdot \mathbf{b}$	(q) -2
(c) $(\mathbf{a} + 2\mathbf{b}) \cdot (\mathbf{a} + (\mathbf{a} \cdot \mathbf{c})\mathbf{b})$	(r) 7
(d) $(\mathbf{a} + \mathbf{b})\,(\mathbf{a} + \mathbf{b} + \mathbf{c})$	(s) 1

60. The area / volume of

(a) triangle with vertices whose position vector w.r.t O is $-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{i} + \mathbf{j} - \mathbf{k}$	(p) $\sqrt{3}/2$
(b) tetrahedra with vertices O, $\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{i} - \mathbf{j} + \mathbf{k}$, $-\mathbf{i} + \mathbf{j} + \mathbf{k}$	(q) 2/3
(c) tetrahedra with vertices $-\mathbf{i} + \mathbf{k}$, $2\mathbf{i} - \mathbf{j}$, $\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$	(r) $\sqrt{89}/2$
(d) triangle with vertices $\mathbf{i}$, $\mathbf{j}$, $\mathbf{i} + \mathbf{j} + \mathbf{k}$	(s) 6

SECTION VI

REASONING TYPE

61. In each of the three planes determined by two of the lines OA, OB, OC, (O being origin) a straight line is drawn through O perpendicular to the third line.

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Statement-1: The three lines so determined are co-planar

Statement-2: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b}$ where $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c} = \mathbf{O}$.

62. If $\mathbf{a} \cdot \mathbf{c} = 3/2$, $\mathbf{b} \cdot \mathbf{d} = 2$, $\mathbf{a} \cdot \mathbf{d} = 3$ and $\mathbf{b} \cdot \mathbf{c} = 1/2$

Statement-1: $\mathbf{a} \times \mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are non-coplanar

Statement-2: $(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$

63. Let the vectors **PQ**, **QR**, **RS**, **ST**, **TU** and **UP** represent the sides the regular hexagon

Statement-1: $\mathbf{PQ} \times (\mathbf{RS} + \mathbf{ST}) \neq \mathbf{0}$

Statement-2: $\mathbf{PQ} \times \mathbf{RS} = \mathbf{0}$ and $\mathbf{PQ} \times \mathbf{ST} \neq \mathbf{0}$

64. Consider the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$

Statement-1: The parametric equation of the line of intersection of the given planes are $x = 3 + 14t$, $y = 1 + 2t$, $z = 15t$

Statement-2: The vector $14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ is parallel to the line of intersection of given planes.

LEVEL 2

SECTION I

SINGLE CORRECT CHOICE TYPE

1. The position vectors of the points A, B and C are $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, respectively. The greatest angle of the triangle ABC is
   (a) 135° (b) 90°
   (c) $\cos^{-1}\frac{2}{3}$ (d) $\cos^{-1}\frac{5}{7}$.

2. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$, $|\mathbf{c}| = 7$. Then an angle between $\mathbf{a}$ and $\mathbf{b}$ is
   (a) 15° (b) $\cos^{-1}\frac{2}{3}$
   (c) 30° (d) 60°.

3. If $\mathbf{r} \cdot \mathbf{a} = 0$, $\mathbf{r} \cdot \mathbf{b} = 0$ and $\mathbf{r} \cdot \mathbf{c} = 0$ for some non-zero vector $\mathbf{r}$, then the value of $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ is
   (a) 0 (b) 1/2
   (c) 1 (d) 2.

4. The position vectors of three consecutive vertices of a parallelogram are $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $7\mathbf{i} + 9\mathbf{j} + 11\mathbf{k}$. The position vector of the fourth vertex is
   (a) $6(\mathbf{i} + \mathbf{j} + \mathbf{k})$ (b) $7(\mathbf{i} + \mathbf{j} + \mathbf{k})$
   (c) $2\mathbf{j} - 4\mathbf{k}$ (d) $6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}$.

5. The volume of the parallelopiped whose sides are given by $\mathbf{OA} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{OB} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{OC} = 3\mathbf{i} - \mathbf{k}$ is
   (a) 4/13 (b) 4
   (c) 2/7 (d) none of these.

6. The value of $|\mathbf{a} \times \mathbf{i}|^2 + |\mathbf{a} \times \mathbf{j}|^2 + |\mathbf{a} \times \mathbf{k}|^2$ is
   (a) $\mathbf{a}^2$ (b) $2\mathbf{a}^2$
   (c) $3\mathbf{a}^2$ (d) none of these.

7. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ if and only if
   (a) $\mathbf{b}$ and $\mathbf{c}$ are collinear
   (b) $\mathbf{a}$ and $\mathbf{c}$ are collinear
   (c) $\mathbf{a}$ and $\mathbf{b}$ are collinear
   (d) none of these.

8. The value of α for which the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} + \alpha\mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ are coplanar is
   (a) 3 (b) –3
   (c) 2 (d) none of these.

9. The area of a parallelogram having diagonals $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ is
   (a) 4 (b) $2\sqrt{3}$
   (c) $4\sqrt{3}$ (d) none of these.

10. If $\mathbf{r}$ satisfies the equation $\mathbf{r} \times (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \mathbf{i} - \mathbf{k}$, then for any scalar t, $\mathbf{r}$ is equal to
    (a) $\mathbf{i} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ (b) $\mathbf{j} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
    (c) $\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ (d) $\mathbf{i} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$.

11. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are equal in length and taken pairwise, make equal angles. If $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$ and $\mathbf{c}$ make an obtuse angle with the base vector $\mathbf{i}$, then $\mathbf{c}$ is equal to
    (a) $\mathbf{i} + \mathbf{k}$ (b) $-\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
    (c) $-\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ (d) $\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$.

12. A line makes angles α, β, γ and δ with diagonals of a cube. The value of $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is
    (a) 1 (b) 1/3
    (c) 8/3 (d) 4/3

13. If $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} + k\mathbf{d}$ then the value of k is
    (a) $[\mathbf{b}\ \mathbf{a}\ \mathbf{c}]$ (b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
    (c) $[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$ (d) $[\mathbf{c}\ \mathbf{b}\ \mathbf{d}]$

14. The one of the value of x for which the angle between $\mathbf{c} = x\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{d} = \mathbf{i} + x\mathbf{j} + \mathbf{k}$ is $\pi/3$ is
    (a) $1 + \sqrt{2}$ (b) 4
    (c) $3 + \sqrt{2}$ (d) none of these

15. The line $x = -2$, $y = 4 + 2t$, $z = -3 + t$ intersect
    (a) the xy-plane
    (b) the xz-plane in $(-2, 0, -4)$
    (c) the yz-plane
    (d) none of these

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66. The value of a for which the volume of parallelopiped formed by the vectors $\mathbf{i} + a\mathbf{j} + \mathbf{k}$, $\mathbf{j} + a\mathbf{k}$ and $a\mathbf{i} + \mathbf{k}$ is minimum is
(a) -3 (b) 3
(c) $1/\sqrt{3}$ (d) $-\sqrt{3}$ [2003]

67. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three unit non-coplanar vectors and α, β, γ be the angles between $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{c}$ and $\mathbf{a}$ respectively. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be unit vectors along the bisectors of angles α, β, γ respectively. Prove that

$$[\mathbf{x}\times\mathbf{y}\ \ \mathbf{y}\times\mathbf{z}\ \ \mathbf{z}\times\mathbf{x}] = \frac{1}{16}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$$

[2003]

68. If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{a}\cdot\mathbf{b} = 1$, $\mathbf{a}\times\mathbf{b} = \mathbf{j} - \mathbf{k}$, then $\mathbf{b}$ is equal to
(a) $2\mathbf{i}$ (b) $\mathbf{i}$
(c) $2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ (d) $2\mathbf{i} - \mathbf{k}$ [2004]

69. A unit vector coplanar with $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ and orthogonal to $5\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, is
(a) $\frac{1}{\sqrt{10}}(3\mathbf{i} - \mathbf{k})$
(b) $\frac{1}{\sqrt{37}}(6\mathbf{i} - \mathbf{k})$
(c) $\frac{1}{\sqrt{57}}(5\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$
(d) $\frac{1}{3}(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ [2004]

70. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are four distinct vectors satisfying the conditions

$$\mathbf{a}\times\mathbf{b} = \mathbf{c}\times\mathbf{d}$$
and $$\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{d}$$

then show that

$$\mathbf{a}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{d} \neq \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}$$

[2004]

71. A plane π passes through the point $(1, 1, 1)$ and is a parallel to the vectors $\mathbf{b} = (1, 0, -1)$ and $\mathbf{c} = (-1, 1, 0)$. If π meets the axes in A, B and C, find the volume of the tetrahedron $OABC$. [2004]

72. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors and

$$\mathbf{b}_1 = \mathbf{b} - \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a},\ \mathbf{b}_2 = \mathbf{b} + \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a},$$

$$\mathbf{c}_1 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} + \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{b}_1,$$

$$\mathbf{c}_2 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} - \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|^2}\mathbf{c}$$

$$\mathbf{c}_3 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{c}|^2}\mathbf{a} + \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{b}_1,$$

$$\mathbf{c}_4 = \mathbf{c} - \frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{c}|^2}\mathbf{a} - \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}_1|^2}\mathbf{b}_1$$

then a set of orthogonal vectors is given by
(a) $(\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1)$ (b) $(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_2)$
(c) $(\mathbf{a}, \mathbf{b}_2, \mathbf{c}_3)$ (d) $(\mathbf{a}, \mathbf{b}_1, \mathbf{c}_4)$ [2005]

73. Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. A vector in the plane of $\mathbf{a}$ and $\mathbf{b}$ whose projection on $\mathbf{c}$ is $1/\sqrt{3}$, is
(a) $4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ (b) $3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$
(c) $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (d) $4\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ [2006]

74. Let $\mathbf{A}$ be vector parallel to the line of intersection of planes P_1 and P_2 through origin where P_1 is parallel to the vector $2\mathbf{j} + 3\mathbf{k}$ and $4\mathbf{j} - 3\mathbf{k}$ and P_2 is parallel to $\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + 3\mathbf{j}$, then the angle between $\mathbf{A}$ and $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is
(a) $\pi/2$ (b) $\pi/4$
(c) $\pi/6$ (d) $3\pi/4$ [2006]

75. The number of distinct real values of λ for which the vectors $-\lambda^2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - \lambda^2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} - \lambda^2\mathbf{k}$ are coplanar, is
(a) zero (b) one (c) two (d) three [2007]

REASONING TYPE

76. Let the vectors **PQ**, **QR**, **RS**, **ST**, **TU** and **UP** represent the sides of the regular hexagon
Statement-1: $\mathbf{PQ}\times(\mathbf{RS} + \mathbf{ST}) \neq \mathbf{0}$.
Statement-2: $\mathbf{PQ}\times\mathbf{RS} = \mathbf{O}$ and $\mathbf{PQ}\times\mathbf{ST} \neq \mathbf{0}$ [2007]

77. Consider the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.
Statement-1: The parametric equation of the line of intersection of the given planes are $x = 3 + 14t$, $y = 1 + 2t$, $z = 15t$
Statement-2: The vector $14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ is parallel to the line of intersection of given planes. [2007]

SINGLE CORRECT CHOICE TYPE

78. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Which one of the following is correct?
(a) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a} = \mathbf{0}$
(b) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a} \neq \mathbf{0}$
(c) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{c} \neq \mathbf{0}$
(d) $\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}$ are mutually perpendicular. [2007]

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49. Let $\mathbf{B} = (b_1, b_2, b_3)$. Since $\mathbf{A} \cdot \mathbf{B} = 3$ so we get $b_1 + b_2 + b_3 = 3$.(i) Also $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ so $(1, 1, 1) \times (b_1, b_2, b_3) = (0, 1, -1)$

$\Rightarrow \quad b_3 - b_2 = 0, b_1 - b_3 = 1, b_2 - b_1 = -1$

$\Rightarrow \quad b_3 = b_2, b_1 = 1 + b_3, b_1 = 1 + b_2 = 1 + b_3$

Putting these values in (i), we get $b_3 = b_2 = 2/3$ and $b_1 = 5/3$. Hence $\mathbf{B} = (5/3, 2/3, 2/3)$.

50. Since $a\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + b\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + c\mathbf{k}$ are coplanar, there exist scalars x, y, z (not all zero) such that

$$x(a\mathbf{i} + \mathbf{j} + \mathbf{k}) + y(\mathbf{i} + b\mathbf{j} + \mathbf{k}) + z(\mathbf{i} + \mathbf{j} + c\mathbf{k}) = \mathbf{0}$$

$$\Rightarrow \quad (ax + y + z)\mathbf{i} + (x + by + z)\mathbf{j} + (x + y + cz)\mathbf{k} = \mathbf{0}$$

$$\Rightarrow \quad ax + y + z = 0, x + by + z = 0, x + y + cz = \mathbf{0}$$

Since at least one of x, y, z is different from zero, the above system must have a non-zero solution.

$$\therefore \quad \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1-2 & b-1 & 0 \\ 1-a & 0 & c-1 \end{vmatrix} = 0$$

$$\Rightarrow \quad a(b-1)(c-1) - (1-a)(b-1) - (c-1)(1-a) = 0$$

Dividing by $(1 - a)(1 - b)(1 - c)$, we get

$$\frac{a}{1-a} + \frac{1}{1-c} + \frac{1}{1-b} = 0$$

$$\Rightarrow \quad \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = \frac{1}{1-a} - \frac{a}{1-a} = 1.$$

51. Since $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are non-coplanar,

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0. \text{ We have}$$

$$0 = \begin{vmatrix} a & a^2 & 1+a^2 \\ b & b^2 & 1+b^2 \\ c & c^2 & 1+c^2 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

$$\Rightarrow \quad 0 = (1 + abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (1 + abc)\,\Delta.$$

As $\Delta \neq 0$, we get $1 + abc = 0$ or $abc = -1$.

52. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{a} \times \mathbf{i} = a_2\mathbf{j} \times \mathbf{i} + a_3\mathbf{k} \times \mathbf{i} = -a_2\mathbf{k} + a_3\mathbf{j} \Rightarrow |\mathbf{a} \times \mathbf{i}|^2 = a_2^2 + a_3^2$.

Similarly $|\mathbf{a} \times \mathbf{j}|^2 = a_1^2 + a_3^2$ and $|\mathbf{a} \times \mathbf{k}|^2 = a_1^2 + a_2^2$.

So $|\mathbf{a} \times \mathbf{i}|^2 + |\mathbf{a} \times \mathbf{j}|^2 + |\mathbf{a} \times \mathbf{k}|^2 = 2(a_1^2 + a_2^2 + a_3^2) = 2$.

53. Equation of the plane containing $\mathbf{i}$ and $\mathbf{i} + \mathbf{j}$ is

$[\mathbf{r} - \mathbf{i}, \mathbf{i}, \mathbf{i} + \mathbf{j}] = 0$

$(\mathbf{r} - \mathbf{i}) \cdot (\mathbf{i} \times (\mathbf{i} + \mathbf{j})) = 0$

$\Rightarrow \quad ((x - 1)\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{k} = 0$

$\Rightarrow \quad z = 0$ (i)

Equation of the plane containing $\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$ is

$[\mathbf{r} - (\mathbf{i} - \mathbf{j}), \mathbf{i} - \mathbf{j}, \mathbf{i} + \mathbf{k}] = 0$

$(\mathbf{r} - \mathbf{i} + \mathbf{j}) \cdot ((\mathbf{i} - \mathbf{j}) \times (\mathbf{i} + \mathbf{k})) = 0$

$(x - 1)\mathbf{i} + (y + 1)\mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$

$\Rightarrow \quad -(x - 1) - (y + 1) + z = 0$

$\Rightarrow \quad x + y - z = 0$ (ii)

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Since $\mathbf{a}$ is parallel to (i) and (ii) $a_3 = 0$ and $a_1 + a_2 - a_3 = 0 \Rightarrow a_1 = -a_2, a_3 = 0$. Thus a unit vector in the direction of

$\mathbf{a} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$. So $\cos\theta = \frac{1}{\sqrt{2}} \frac{1 \times 1 + (-2) \times (-1)}{1 \cdot \sqrt{1 + 4 + 4}}$

$= \frac{1}{\sqrt{2}}$

54. q = area of parallelogram with OA and OC as adjacent sided $= |\mathbf{OA} \times \mathbf{OC}|$

$= |\mathbf{a} \times \mathbf{b}|$ and p = area of quadrilateral $OABC$

$= (1/2)\,|\mathbf{OA} \times \mathbf{OB}| + (1/2)\,|\mathbf{OB} \times \mathbf{OC}|$

$= (1/2)\,|\mathbf{a} \times (10\mathbf{a} + 2\mathbf{b})| + (1/2)\,|(10\mathbf{a} + 2\mathbf{b}) \times \mathbf{b}|$

$= |\mathbf{a} \times \mathbf{b}| + 5\,|\mathbf{a} \times \mathbf{b}| = 6\,|\mathbf{a} \times \mathbf{b}| = 6q$.

Thus $K = 6$.

55. $[\mathbf{c}\,\mathbf{b}\,\mathbf{d}] = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ 2 & -1 & 1 \end{vmatrix} = -4\,[\mathbf{a}\,\mathbf{b}\,\mathbf{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 2 \\ 2 & -1 & 1 \end{vmatrix} = 1$

and $\mathbf{b} \times \mathbf{d} = 4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ so $|\mathbf{b} \times \mathbf{d}| = 5\sqrt{2}$.

Hence the shortest distance $= \frac{|[\mathbf{c}\,\mathbf{b}\,\mathbf{d}] - [\mathbf{a}\,\mathbf{b}\,\mathbf{d}]|}{|\mathbf{b} \times \mathbf{d}|}$

$= \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$

56. $\mathbf{b} \times \mathbf{c} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{c} \times \mathbf{a} = -\mathbf{i} + \mathbf{k}$ and $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \end{vmatrix} = 1$.

The required equation of the plane is

$\mathbf{r} \cdot [\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$

$\Rightarrow \quad \mathbf{r} \cdot (-3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 1$

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$$A = \{z : \text{Im} z \geq 1\}$$

$$B = \{z : |z - 2 - i| = 3\}$$

$$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}.$$

15. The number of elements in the set $A \cap B \cap C$ is
(a) 0 (b) 1
(c) 2 (d) ∞

16. Let z be any point in $A \cap B \cap C$. Then $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between
(a) 25 and 29 (b) 30 and 34
(c) 35 and 39 (d) 40 and 44

17. Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between
(a) -6 and 3 (b) -3 and 6
(c) -6 and 6 (d) -3 and 9

Paragraph for Question Nos. 18 to 20

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line.

If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$.

If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

18. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$

(a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$

(c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

19. The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

(a) $\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)}\,dx + bf(b) - af(a)$

(b) $-\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)}\,dx + bf(b) - af(a)$

(c) $\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)}\,dx - bf(b) + af(a)$

(d) $-\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)}\,dx - bf(b) + af(a)$

20. $\int_{-1}^{1} g'(x)\,dx =$

(a) $2g(-1)$ (b) 0
(c) $-2g(1)$ (d) $2g(1)$

Paragraph for Question Nos. 21 to 23

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

Further, it is given that the origin and the centre of C are on the same side of the line PQ.

21. The equation of circle C is

(a) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$

(b) $(x - 2\sqrt{3})^2 + (y + \frac{1}{2})^2 = 1$

(c) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$

(d) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

22. Points E and F are given by

(a) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$ (b) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$

(c) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

23. Equations of the sides QR, RP are

(a) $y = \frac{2}{\sqrt{3}}x + 1,\ y = -\frac{2}{\sqrt{3}}x - 1$

(b) $y = \frac{1}{\sqrt{3}}x,\ y = 0$

(c) $y = \frac{\sqrt{3}}{2}x + 1,\ y = -\frac{\sqrt{3}}{2}x - 1$

(d) $y = \sqrt{3}x,\ y = 0$

Answers

Section-I

Straight Objective Type

1. (c)	2. (a)	3. (b)
4. (c)	5. (b)	6. (c)

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Column I	**Column II**
(a) L_1, L_2, L_3 are concurrent, if	(p) $k = -9$
(b) One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(q) $k = -\frac{6}{5}$
(c) L_1, L_2, L_3 form a triangle, if	(r) $k = \frac{5}{6}$
(d) L_1, L_2, L_3 do not form a triangle, if	(s) $k = 5$

21. Consider all possible permutations of the letters of the word ENDEANOEL.
Match the Statements/Expressions in **Column I** with the Statements/Expressions in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column I	**Column II**
(a) The number of permutations containing the word ENDEA is	(p) 5!
(b) The number of permutations in which the letter E occurs in the first and the last positions is	(q) $2 \times 5!$
(c) The number of permutations in which none of the letters D, L, N occurs in the last five position is	(r) $7 \times 5!$
(d) The number of permutations in which the letters A, E, O occur only in odd positions is	(s) $21 \times 5!$

22. Match the Statements/Expressions in **Column I** with the Statements/Expressions in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column I	**Column II**
(a) The minimum value of $\frac{x^2+2x+4}{x+2}$ is	(p) 0
(b) Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and $(A + B)(A - B) = (A - B)(A + B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible values of k are	(q) 1
(c) Let $a = \log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3-a)} < 2$, must be less than	(r) 2
(d) If $\sin\theta = \cos\varphi$, then the possible values of $\frac{1}{\pi}\left(\theta \pm \varphi - \frac{\pi}{2}\right)$ are	(s) 3

Answers

Section-I

Straight Objective Type

1. (b)	2. (d)	3. (c)
4. (d)	5. (b)	6. (d)
7. (c)	8. (a)	9. (a)

Section II

Reasoning Type

10. (c)	11. (c)	12. (c)
13. (b)		

Section III

Linked Comprehension Type

14. (a)	15. (a)	16. (b)
17. (b)	18. (d)	19. (c)

Section IV

Column Matching Type

20.

	p	q	r	s
a	p	q	r	**s**
b	**p**	**q**	r	s
c	p	q	**r**	s
d	**p**	**q**	r	**s**

21.

	p	q	r	s
a	**p**	q	r	s
b	p	q	r	**s**
c	p	**q**	r	s
d	p	**q**	r	s

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$$\Rightarrow \qquad \mathbf{a}_2 - \mathbf{a}_1 = 3\mathbf{i} + 4\mathbf{k}$$

Thus,

$$s.d. = \frac{|-3+20|}{\sqrt{75}} = \frac{17}{5\sqrt{3}}$$

19. Equation of plane through $(-1, -2, -1)$ and perpendicular to L_1 and L_2 is

$$-1(x+1) - 7(y+2) + 5(z+1) = 0$$

$$\Rightarrow \qquad x + 7y - 5z + 10 = 0 \qquad (1)$$

Distance of (1, 1, 1) from plane (1) is given by

$$\frac{|1+7-5+10|}{\sqrt{1+49+25}} = \frac{13}{\sqrt{75}}$$

Section-IV

20. Given lines are

$$x + 3y - 5 = 0 \qquad (1)$$
$$3x - ky - 1 = 0 \qquad (2)$$
$$5x + 2y - 12 = 0 \qquad (3)$$

Solving (1) and (3) we obtain $x = 2$, $y = 1$.
L_1, L_2, L_3 will be concurrent if the line L_2 also passes through (2, 1), that is, if

$$3(2) - k(1) - 1 = 0 \Rightarrow k = 5.$$

Note that L_1 is not parallel to L_3;
For L_2 to be parallel to L_1,

$$\frac{1}{3} = \frac{3}{-k} \Rightarrow k = -9.$$

For L_2 to be parallel to L_3

$$\frac{3}{5} = \frac{-k}{2} \Rightarrow k = -\frac{6}{5}.$$

The lines L_1, L_2 and L_3 will form a triangle if no two of L_1, L_2, L_3 are parallel and L_1, L_2, L_3 are not concurrent. Thus, $k = 5/6$
The lines L_1, L_2 and L_3 will not form a triangle if two of L_1, L_2, L_3 are parallel or all three of them are concurrent.

21. The number of permutations of

E N D E A N, O, E, L is 5!

When E occurs at the first and last place, we need to permute

2N's, 1E, 1A, 1D, 1O and 1L.

This can be done in $\frac{7!}{2!} = \frac{7\times6}{2}\times 5! = (21)(5!)$ ways.

Letters D, L, 2 N can occur at the first four positions in ${}^5P_3$ ways.
The remaining letters viz.

3E's, 1A, 1O

can be permuted in $\frac{5!}{3!}$

Required number of

$$\text{permutations} = \frac{4!}{2!}\times\frac{5!}{3!}$$
$$= (2)(5!)$$

A, E, E, E, O can occur at odd places in $\frac{5!}{3!}$ ways and the remaining letters can be arranged in $\frac{4!}{2!}$ ways.
$\therefore$ required number of ways

$$= \left(\frac{5!}{3!}\right)\left(\frac{4!}{2!}\right) = 2(5!).$$

22. (a) Let $y = \frac{x^2+2x+4}{x+2}$

$$\Rightarrow \qquad (x+2)y = x^2 + 2x + 4$$
$$\Rightarrow \qquad x^2 + x(2-y) + 4 - 2y = 0$$

As x is real,

$$(2-y)^2 - 4(4-2y) \geq 0$$
$$\Rightarrow \qquad y^2 + 4y - 12 \geq 0$$
$$\Rightarrow \qquad (y+6)(y-2) \geq 0$$
$$\Rightarrow \qquad y \in (-\infty, -6] \cup [2, \infty)$$

Thus, local minimum value of y is 2 which is attained at $x = 0$.

(b) $A' = A$, $B' = -B$ (given)

$$(A+B)(A-B) = (A-B)(A+B)$$
$$\Rightarrow \quad AA + BA - AB - BB = AA - BA + AB - BB$$
$$\Rightarrow \qquad BA = AB$$

Now, $(AB)' = B'A' = (-B)A$

$$\Rightarrow \qquad (-1)^k AB = -(BA) = -(AB)$$
$$\therefore \qquad (-1)^k = -1 \Rightarrow k = 1, 3$$

(c) $a = \log_3(\log_3 2) \Rightarrow 3^a = \log_3 2$

$$\Rightarrow \qquad 3^{-a} = \log_2 3 \Rightarrow 2^{3^{-a}} = 3.$$

Now, $1 < 2^{-k+3^{-a}} < 2$

$$\Rightarrow \qquad 1 < (2^{-k})(3) < 2 \Rightarrow \frac{1}{3} < 2^{-k} < \frac{2}{3}$$

$$\Rightarrow \qquad \frac{3}{2} < 2^k < 3$$

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In this case exactly three of x_i's is 2 and the remaining four x_i's are 1.
In this case, the number of 7 digit numbers is
$\frac{7!}{3!4!} = 35.$
Hence, the required number of 7 digit numbers is $42 + 35 = 77.$

7. $z\bar{z}^3 + \bar{z}\bar{z}^3 = 350 \Rightarrow z\bar{z}(\bar{z}^2 + z^2) = 350$
$\Rightarrow (x^2 + y^2)\,[(x - iy)^2 + (x + iy)^2] = 350$
$\Rightarrow (x^2 + y^2)\,(2x^2 - 2y^2) = 350$
$\Rightarrow x^4 - y^4 = 175$
$\Rightarrow x^4 \geq 175 \Rightarrow x^4 \geq 256$
Let us try $x = \pm 4$
Thus, $y^4 = 256 - 175 = 81 \Rightarrow y = \pm 3$
$\therefore$ roots of $z\bar{z}^3 + \bar{z}\bar{z}^3 = 350$ are $z = \pm 4 \pm 3i$
Area of rectangle whose vertices are $\pm 4 \pm 3i$ is $(8)\,(6) = 48.$

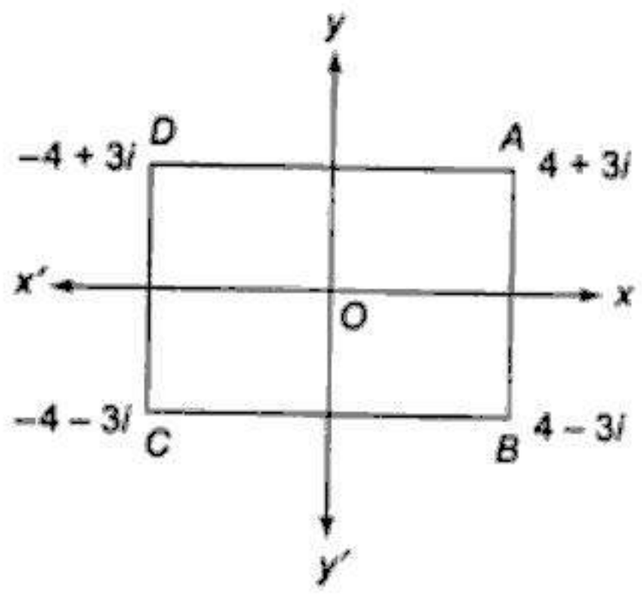

Fig. M.3

8. Equation of chord AB is $T = 0$
$\Rightarrow 1x + 8y - 3(x + 1) - 2(y + 8) - 11 = 0$
$\Rightarrow -2x + 6y - 30 = 0$
or $x - 3y + 15 = 0$

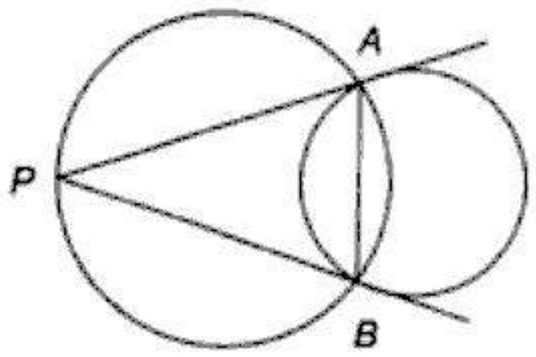

Fig. M.4

Equation of any circle through AB is
$$S + \lambda T = 0$$
$\Rightarrow x^2 + y^2 - 6x - 4y - 11 + \lambda(x - 3y + 15) = 0$
It will pass through $P\,(1, 8)$ if
$1 + 64 - 6(1) - 4(8) - 11 + \lambda\,(1 - 3(8) + 15) = 0$
$\Rightarrow 16 - 8\lambda = 0 \Rightarrow \lambda = 2.$
Thus, required circle is
$x^2 + y^2 - 6x - 4y - 11 + 2(x - 3y + 15) = 0$
$\Rightarrow x^2 + y^2 - 4x - 10y + 19 = 0$

Section-II

9. $\cos B + \cos C = 4 \sin^2 (A/2)$

$$\Rightarrow 2 \cos \frac{B+C}{2} \cos \frac{B-C}{2} = 4 \sin^2\left(\frac{A}{2}\right)$$

$$\Rightarrow 2 \cos \left(\frac{\pi}{2} - \frac{A}{2}\right) \cos \frac{B-C}{2} = 4 \sin\left(\frac{A}{2}\right) \sin \left(\frac{\pi}{2} - \frac{B+C}{2}\right)$$

$$\Rightarrow 2 \sin \frac{A}{2} \cos \frac{B-C}{2} = 4 \sin \left(\frac{A}{2}\right) \cos \frac{B+C}{2}$$

$$\Rightarrow \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 2 \left(\cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2}\right)$$

$\Rightarrow 3 \sin (B/2) \sin (C/2) = \cos (B/2) \cos (C/2)$

$$\Rightarrow \tan \frac{B}{2} \tan \frac{C}{2} = \frac{1}{3}$$

$$\Rightarrow \frac{\Delta}{s(s-b)} \cdot \frac{\Delta}{s(s-c)} = \frac{1}{3}$$

$\Rightarrow 3s(s - a)\,(s - b)\,(s - c) = s^2\,(s - b)\,(s - c)$
$\Rightarrow 3(s - a) = s \Rightarrow 2s = 3a$
$\Rightarrow a + b + c = 3a \Rightarrow b + c = 2a$

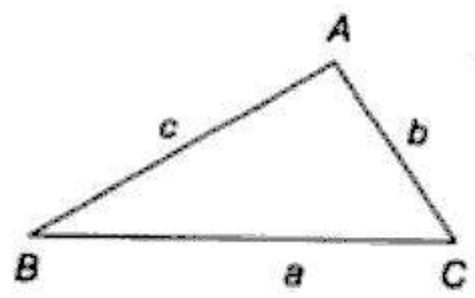

Fig. M.5

$\Rightarrow AB + AC = 2a$
Also, $AB + AC > BC = a$
Thus, locus of A is an ellipse with focii at B and C and length of major axis as $2a$.

10. $L = \lim\limits_{x \to 0} \dfrac{a - \sqrt{a^2 - x^2} - \frac{1}{4}x^2}{x^4}$

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